

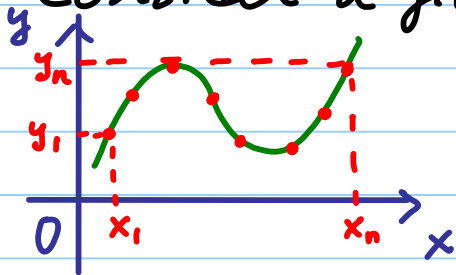
Lecture 6: Applications of Calc. Variations to Interpolation & Approximation

Note Title

1/6/2015

* Interpolation vs Approximation

Consider a given set of pts $\{(x_j, y_j)\}_{j=1:n} \subset \mathbb{R}^2$.

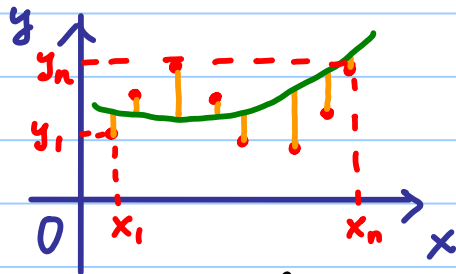


• Interpolation:

Find a fcn $y = f(x)$ s.t.

$$y_j = f(x_j), \quad j=1:n$$

$\Rightarrow$ Once you find $f(x)$, you can evaluate at **any** pt x ($\neq x_j$)



• Approximation (smoothing):

Find a fcn $y = f(x)$ s.t.

maximizing the fidelity to the data (i.e., minimizing the

residual error) subj. to some smoothness constraint on f . $\Rightarrow$ This includes the least squares fit of a polynomial to the data, e.g.,

$$\sum_{j=1}^n (y_j - f(x_j))^2 \rightarrow \min!$$

subj. to $f \in \mathcal{P}_k$ (k th order polynomial.)
 $k \leq n-1$

We'll mainly focus on interpolation here.

Note: $\exists$ vast literature on both interpol.

& approx. Multidimensional cases are particularly important in geophysics, medicine, cartography, spatial statistics, and image processing, ...

As you can imagine, **Runge** mathematicians started analyzing these problems using **Gibbs** algebraic poly's, trigonometric poly's $\Rightarrow$ Chebyshev Poly's, wavelets, rational fcn's ...

★ Lagrange Polynomials / The Runge Phenomenon

Def. (Lagrange polynomial)

Given a set of pts $\{(x_j, y_j)\}_{j=0:k}$ ($k+1$ pts)

let $L_k(x) := \sum_{j=0}^k y_j l_j(x)$ $\rightarrow x_j \neq x_i$ if $j \neq i$

where $l_j(x) := \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$

$$= \frac{(x - x_0) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_k)}{(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_k)}$$

Hence, $l_j(x_m) = \delta_{jm} = \begin{cases} 1 & \text{if } m=j \\ 0 & \text{if } m \neq j \end{cases}$
 $\rightarrow$ Kronecker's delta

$\rightarrow$ This is a fundamental property for interpolation ($L_k(x_j) = y_j$)

Note: • $\deg(l_j) = k$.

• By defining $\Phi(x) := \prod_{j=0}^k (x - x_j)$,

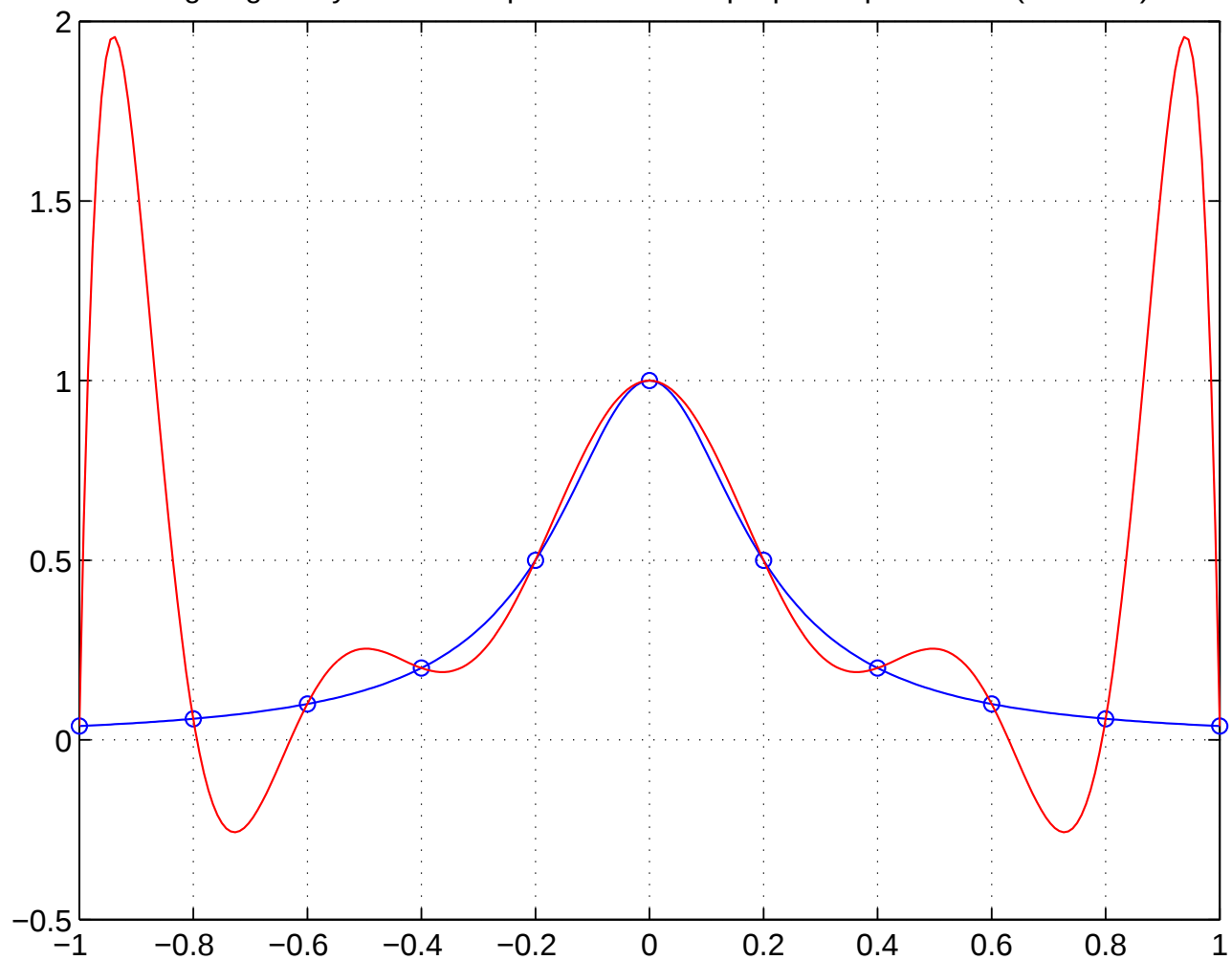
we can write $l_j(x) = \frac{\Phi(x)}{(x - x_j) \Phi'(x_j)}$

• The **Runge** Phenomenon (1901)

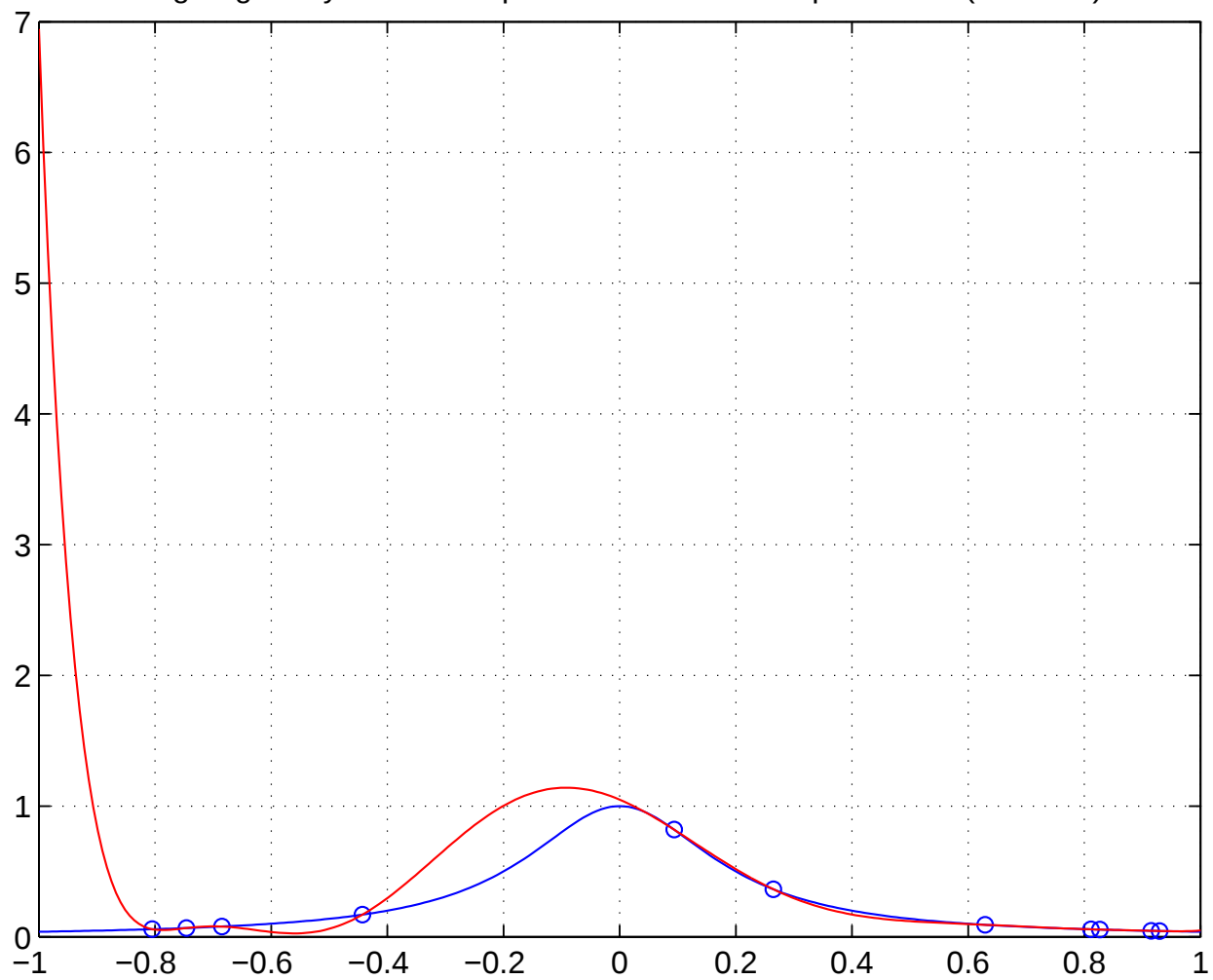
Carl Runge (1856-1927) reported the following observation: Consider $f(x) = \frac{1}{1+25x^2}$ over $[-1, 1]$, and an interpolation poly. L_k at the equidistant nodes $x_j = -1 + j \frac{2}{k}$, $j=0, 1, \dots, k$.

Then, the interpolating poly. L_k oscillates toward the edge of the interval $[-1, 1]$.
Moreover, $\lim_{k \rightarrow \infty} (\max_{-1 \leq x \leq 1} |f(x) - L_k(x)|) = +\infty!$

Lagrange Polynomial Interpolation at 11 equispaced points to $1/(1+25x^2)$



Lagrange Polynomial Interpolation at 11 random points to $1/(1+25x^2)$



(Proof) First, I leave the following as an exercise:

$$f(x) - L_k(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!} \underbrace{\prod_{j=0}^k (x - x_j)}_{= \Phi(x)}, \quad \exists \xi \in [-1, 1]$$

Then

$$\max_{x \in [-1, 1]} |f(x) - L_k(x)| \leq \max_{x \in [-1, 1]} \frac{|f^{(k+1)}(x)|}{(k+1)!} \cdot \max_{x \in [-1, 1]} \prod_{j=0}^k |x - x_j|$$

$\downarrow$ ask $k \rightarrow \infty$ $\downarrow$
 $+\infty$ $+\infty$
 again exercises.

$\Rightarrow$ Avoid global polynomial interpolation (at least for equidistant pts) !!

- Hence, the idea of using piecewise algebraic polynomials was conceived.
- We'll defer interpolation using trigonometric poly. until we learn Fourier Series.
- Here, our discussion is based on the paper by Micula (2002).

Notation: $I := [a, b] \subset \mathbb{R}$

$\mathcal{P}_m := \{ \text{a set of poly's of deg.} \leq m \}$

$\underline{H^m(I)} := \{ f: I \rightarrow \mathbb{R} \mid \underbrace{f^{(m-1)} \in AC(I)}_{\text{the space of abs. cont. fns on } I} \text{ \& } f^{(m)} \in L^2(I) \}$

= L^2 -Sobolev space

$W^{m,2}(I)$

$\Leftrightarrow f^{(m)}$ exists a.e. on I

Def. Let $\mathcal{P} := \{ a = x_0 < x_1 < \dots < x_n < x_{n+1} = b \}$
 be a partition of I . Then $s: I \rightarrow \mathbb{R}$
 is a polynomial spline of deg. m w.r.t. $\mathcal{P}$

if $s \in C^{m-1}(I)$ & $s|_{[x_j, x_{j+1}]} \in P_m, j=0:n$.

The interior pts $\{x_1, \dots, x_n\}$ are called **knots**.

★ Natural Cubic Splines

Suppose that the knots and the values at the knots $\{y_1, \dots, y_n\}$ are given.

Find a **nice** fcn $\varphi: I \rightarrow \mathbb{R}$ s.t. $\varphi(x_j) = y_j$
 $j=1:n$.

Thm (Holladay 1957)

Given $\{(x_j, y_j)\}_{j=1:n}, n \geq 2$,

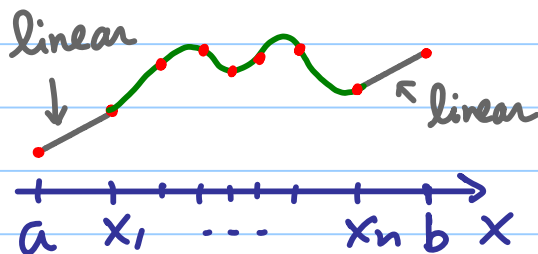
$\exists! \sigma \in X_n(I) := \{f \in H^2(I) \mid f(x_j) = y_j, j=1:n\}$

with $\int_I |\sigma''(x)|^2 dx = \min_{f \in X_n(I)} \int_I |f''(x)|^2 dx$

i.e., $\sigma = \arg \min_{f \in X_n(I)} \int_I |f''(x)|^2 dx$.

Furthermore,

$$\begin{cases} \cdot \sigma \in C^2(I) \\ \cdot \sigma|_{[x_j, x_{j+1}]} \in P_3, j=1:n-1 \\ \cdot \sigma|_{[a, x_1]}, \sigma|_{[x_n, b]} \in P_1 \end{cases}$$



"Natural" since outside of $[x_1, x_n]$, σ is linear.

why would one choose to minimize $\int_I |f''(x)|^2 dx$?
 (A1) The curvature of f is $\frac{f''}{(1+f'^2)^{3/2}}$, which implies

$\sigma \approx$ minimizes the total curvature if σ' is small!
 i.e., σ is smooth, no kinks.

Note: why the curvature of f is $\frac{f''}{(1+f'^2)^{3/2}}$?

Ans: For a plane curve $s(t) = (x(t), y(t))$, its curvature is $\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$, which is

well-known. Then set $x(t) = t$ to get $\ddot{y}/(1+\dot{y}^2)^{3/2}$

(A2) Nat. Cubic Spline $\approx$ the solution of a physical problem of deformation of a bar of uniform, thin, elastic material without external force.

$\Rightarrow$ min. elastic potential energy
 $\approx \min \int_I |f''(x)|^2 dx$

(A3) σ can be interpreted as a curve closest to a straight line yet passing through those given pts.

Note: For each subinterval, if we want $I_j = \int_{x_{j-1}}^{x_j} f(x, y, y', y'') dx \rightarrow \min!$

then the E-L eqn. becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0 \quad \text{Deriving this is an exercise!}$$

Now let $f = y''^2$. Then the E-L eqn. becomes $y^{(4)} = 0$ with an appropriate B.C.
 $\rightarrow$ Bi-harmonic eqn. (1D)

⇒ All sorts of generalization have been developed.
See Micula and the references therein.

⇒ For computational algorithm, see **spline** fcn in MATLAB, and Algorithm 472 (ACM) by Herriot & Reinsch.

(Proof of the Holladay Thm)

It suffices to prove for any $f \in X_n(I)$,

$$\int_I |f''(x)|^2 dx > \int_I |\sigma''(x)|^2 dx.$$

Let $\eta(x) := f(x) - \sigma(x)$, i.e., $f(x) = \eta(x) + \sigma(x)$

Since $\sigma''' \equiv \text{const.} =: c_j$ in each $[x_j, x_{j+1}]$, we have

$$\int_I |f''(x)|^2 dx - \int_I |\sigma''(x)|^2 dx$$

$$= \int_I |\eta''(x)|^2 dx + 2 \int_I \eta''(x) \sigma''(x) dx$$

$$= \int_I |\eta''(x)|^2 dx + 2 \sum_{j=0}^n \int_{x_j}^{x_{j+1}} \eta''(x) \sigma''(x) dx$$

$$= \int_I |\eta''(x)|^2 dx + 2 \sum_{j=0}^n \left[\underbrace{\eta' \sigma''}_{\text{telescopic sum!}} \Big|_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \eta' \underbrace{\sigma'''}_{=c_j} dx \right]$$

$$= \int_I |\eta''(x)|^2 dx + 2 \left(\underbrace{\eta'(b) \sigma''(b) - \eta'(a) \sigma''(a)}_{\text{telescopic sum!}} \right)$$

$$- 2 \sum c_j (\underbrace{\eta(x_{j+1}) - \eta(x_j)}_{=0 \text{ since } f(x_j) = \sigma(x_j)})$$

$$= \int_I |\eta''(x)|^2 dx \text{ by setting}$$

$$\underbrace{\sigma''(a) = \sigma''(b) = 0}$$

$$> 0 //$$

hence, σ is linear in $[a, x_1], [x_n, b]$.