

Lecture 11 : Basics of PDEs V

Note Title

* The Dirichlet & Neumann Problems

Given fcn f on Ω , g on $\partial\Omega$,
find a fcn u on $\bar{\Omega}$ s.t.

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \Rightarrow \text{the Dirichlet problem}$$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \partial_\nu u = g & \text{on } \partial\Omega \end{cases} \Rightarrow \text{the Neumann problem}$$

Here Ω is a bdd. domain, not necessarily connected.

* For the Dirichlet problem, if the sol. exists, then it's unique as we already showed.

* For the Neumann problem, the uniqueness doesn't hold! why?
 $\Rightarrow u + \text{any const.}$ is also a sol.
Also, $\exists$ a necessary solvability cond. for the Neumann problem as follows.

Let Ω' be any connected component of Ω .
Then, using Green's 2nd id.

$$\int_{\Omega'} (v \Delta u - u \Delta v) dx = \int_{\partial\Omega'} (v \partial_\nu u - u \partial_\nu v) d\sigma$$

and setting $v \equiv 1$, we have

$$\int_{\Omega'} \underbrace{\Delta u}_{=f} dx = \int_{\partial\Omega'} \underbrace{\partial_n u}_{=g} d\sigma$$

So, f & g must satisfy

$$\int_{\Omega'} f dx = \int_{\partial\Omega'} g d\sigma \quad \text{on each connected component of } \Omega.$$

In particular, in the case of Laplace's eqn. (i.e., $f \equiv 0$), g must satisfy

$$\int_{\partial\Omega'} g d\sigma = 0 \quad \text{on each connected component of } \Omega.$$

* The reduction of the Dirichlet problem to $f = 0$ or $g = 0$.

If we can find fcn's v & w s.t.

$$\textcircled{1} \begin{cases} \Delta v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad \textcircled{2} \begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega, \end{cases}$$

Poisson with homog. Dirichlet B.C. Laplace with inhomog. Dirichlet B.C.

Then clearly $u = v + w$ satisfies the original Dirichlet problem!

* Equivalence of $\textcircled{1}$ & $\textcircled{2}$

Now, we will demonstrate that $\textcircled{1}$ & $\textcircled{2}$ are more or less equivalent. This is important since a numerical solver developed for $\textcircled{1}$ can also handle and solve $\textcircled{2}$!!

Suppose we can solve ① and wish to solve ②.
 Assume that g has an extension $\tilde{g} \in C^2(\bar{\Omega})$
 s.t. $\tilde{g} = g$ on $\partial\Omega$.

Then, we can find v satisfying

$$\begin{cases} \Delta v = \Delta \tilde{g} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \Rightarrow \text{This is a form of ①} \\ \text{so we can solve this.}$$

Now take $w = \tilde{g} - v$.

$$\Rightarrow \Delta w = \Delta \tilde{g} - \Delta v = 0 \quad \text{in } \Omega$$

$$w|_{\partial\Omega} = \tilde{g}|_{\partial\Omega} - v|_{\partial\Omega} = g - 0 = g$$

So, ② is solved! ✓

On the other hand, suppose we can solve ② and wish to solve ①.

First, let's extend f to the outside of Ω by zero, i.e., $\tilde{f} = \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega \end{cases}$

and set $v' = \tilde{f} * \Phi$ so that $\Delta v' = \tilde{f}$

Second, we solve $\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = v' & \text{on } \partial\Omega \end{cases} \Rightarrow \text{This is of ②}$

Now take $v = v' - w$.

Then we see

$$\begin{cases} \Delta v = \Delta v' - \Delta w = f - 0 = f & \text{in } \Omega \\ v|_{\partial\Omega} = v'|_{\partial\Omega} - w|_{\partial\Omega} = v'|_{\partial\Omega} - v'|_{\partial\Omega} = 0 \end{cases}$$

So, ① is solved!! ///

Exercise: Prove the Neumann case.

★ A Fast & Accurate Laplace/Poisson Solver on a Rectangle

(A. Averbuch, M. Israeli, and L. Vozovoi, SIAM J. Sci. Comput., vol. 19, no. 3, pp. 933-952, 1998)

Consider, for simplicity, $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

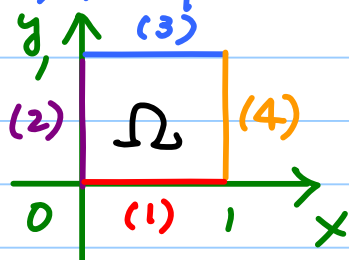
Then AIV showed that the sol. to this Dirichlet problem is

$$u(x, y) = p(x, y) + \sum_{k \geq 1} \{ b_k^{(1)} h_k(x, 1-y) + b_k^{(2)} h_k(y, 1-x) + b_k^{(3)} h_k(x, y) + b_k^{(4)} h_k(y, x) \}$$

$p(x, y) \rightarrow$ where $p(x, y) = a_3 xy + a_2 x + a_1 y + a_0$
is harmonic!
s.t. $p(x, y) = f(x, y)$ at the 4 corners
 $(x, y) = (0, 0), (1, 0), (0, 1), (1, 1)$,

$$h_k(x, y) := \sin(\pi k x) \frac{\sinh(\pi k y)}{\sinh(\pi k)},$$

and $\{b_k^{(1)}\}$, $\{b_k^{(2)}\}$, $\{b_k^{(3)}\}$, $\{b_k^{(4)}\}$ are the Fourier sine series coeff's of
 $f(x, 0) - p(x, 0)$, $f(0, y) - p(0, y)$,
 $f(x, 1) - p(x, 1)$, $f(1, y) - p(1, y)$, respectively.



$$\begin{aligned} \text{i.e., } \sum b_k^{(1)} \sin \pi k x &= f(x, 0) - p(x, 0) \\ \sum b_k^{(2)} \sin \pi k y &= f(0, y) - p(0, y) \\ \sum b_k^{(3)} \sin \pi k x &= f(x, 1) - p(x, 1) \\ \sum b_k^{(4)} \sin \pi k y &= f(1, y) - p(1, y) \end{aligned}$$

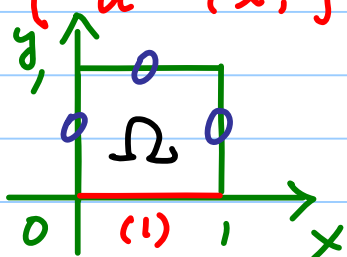
In practice, one needs to truncate these Fourier sine series. Also x, y must be discretized, then FFT-based Discrete Sine Transform should be used.

Note that

$$u^{(1)}(x, y) := \sum_{k \geq 1} b_k^{(1)} h_k(x, 1-y)$$

is the sol. to the following Dirichlet problem :

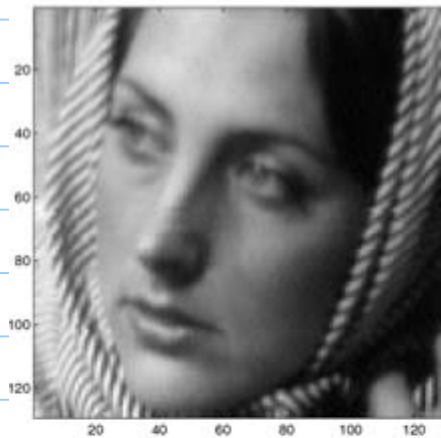
$$\begin{cases} \Delta u^{(1)} = 0 & \text{in } \Omega \\ u^{(1)}(x, 0) = f(x, 0) - p(x, 0) & \text{on } (1) \subset \partial\Omega \\ u^{(1)}(x, y) = 0 & \text{on } (2), (3), (4) \subset \partial\Omega \end{cases}$$



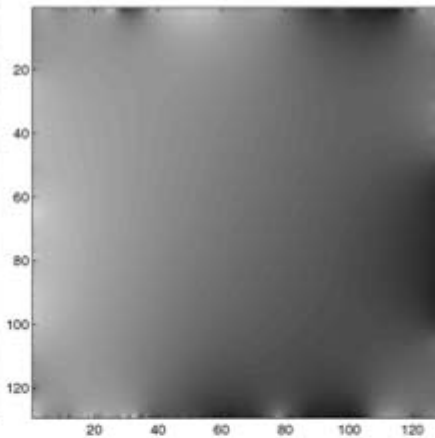
So as the other three sums.

$$\Rightarrow u(x, y) = p(x, y) + \boxed{u^{(1)}} + \boxed{u^{(2)}} + \boxed{u^{(3)}} + \boxed{u^{(4)}}$$

$\Rightarrow$ No finite difference approx. of Δ !



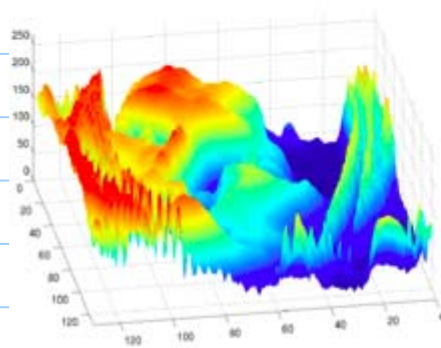
(a) Original



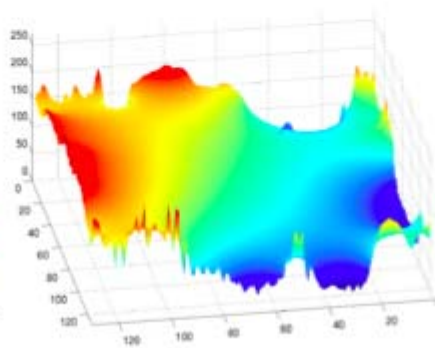
(b) u component



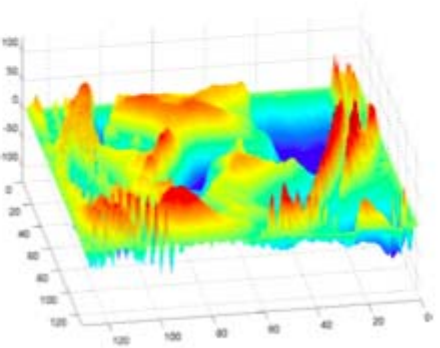
(c) v component



(d) Original



(e) u component



(f) v component

$\uparrow$
 u is the solution
 of the Dirichlet problem
 using the AIV method.

$\uparrow$
 $v = f - u.$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

where f is the original
 image intensities in (a), (d).