

Lecture 12: Fourier Series I

Note Title

1/22/2015

★ Fourier Series of a Periodic Function

Def. Suppose f is defined on the real axis s.t., $f(\theta + 2\pi) = f(\theta)$, $\forall \theta \in \mathbb{R}$.
Such f is called 2π -periodic.

We assume f : Riemann integrable on every bdd. interval, and $\mathbb{C}$ -valued.
Want to know if f can be expanded in a series:

$$(*) \quad f(\theta) = \underbrace{\left(\frac{1}{2}\right)}_{\text{for convenience}} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

corresp. to $n=0$ case

We can express $(*)$ as

$$(**) \quad f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$\text{Thanks to } \cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}, \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

$\Rightarrow$ An important relationship:

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \quad n \in \mathbb{N}.$$

Now, how can we compute c_n (or a_n & b_n)?

Formally,

$$f(\theta) \cdot e^{-ik\theta} = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \cdot e^{-ik\theta}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{i(n-k)\theta} d\theta$$

Need justification $\downarrow$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta \quad \text{--- } (*)$$

It's an easy exercise to derive:

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta = 2\pi \delta_{n,k} = \begin{cases} 2\pi & \text{if } n=k \\ 0 & \text{if } n \neq k \end{cases}$$

the Kronecker delta

So, $(\star) = \sum_{n=-\infty}^{\infty} c_n 2\pi \delta_{n,k} = 2\pi c_k$, i.e.,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}$$

Using the $c_n \leftrightarrow (a_n, b_n)$ formulas, we get

$(\star\star)$

$$a_0 = 2c_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \rightarrow \text{often called the DC component or mean val.}$$

$\uparrow_{n=0}$

$$a_n = c_n + c_{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta, \quad n \in \mathbb{N}.$$

$$b_n = i(c_n - c_{-n}) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

So, if the series converges nicely so that term-by-term integrals in $(\star)$ are justified, then the coeff's are given as $(\star\star)$.

But, if f : Riemann integrable & periodic, then the RHS of $(\star\star)$ make perfectly good sense and we can define these coeff's.

Def. Suppose f is 2π -periodic & Riemann-integrable on $[-\pi, \pi]$. Then the numbers c_n, a_n, b_n defined by $(\star\star)$ are called the **Fourier coef's** of f , and the corresponding series $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$ are called the **Fourier series** of f .

Remark: Thanks to the 2π -periodicity of f , we can replace $\int_{-\pi}^{\pi} \cdot d\theta$ by $\int_a^{2\pi+a} \cdot d\theta$, any $a \in \mathbb{R}$, e.g., $a=0$.

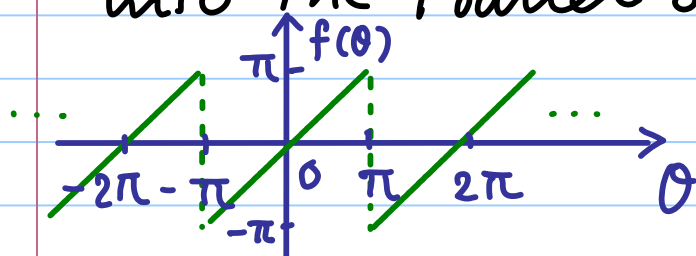
Convenient Facts:

If F is even, i.e., $F(-x) = F(x)$
then $\int_{-a}^a F(x) dx = 2 \int_0^a F(x) dx$.

If F is odd, i.e., $F(-x) = -F(x)$
then $\int_{-a}^a F(x) dx = 0$.

Hence, f : even $\Rightarrow a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos n\theta d\theta$, $b_n \equiv 0$.
 f : odd $\Rightarrow a_n \equiv 0$, $b_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta d\theta$.

Example 1. Expand $f(\theta) = \theta$, $-\pi \leq \theta \leq \pi$ into the Fourier series.

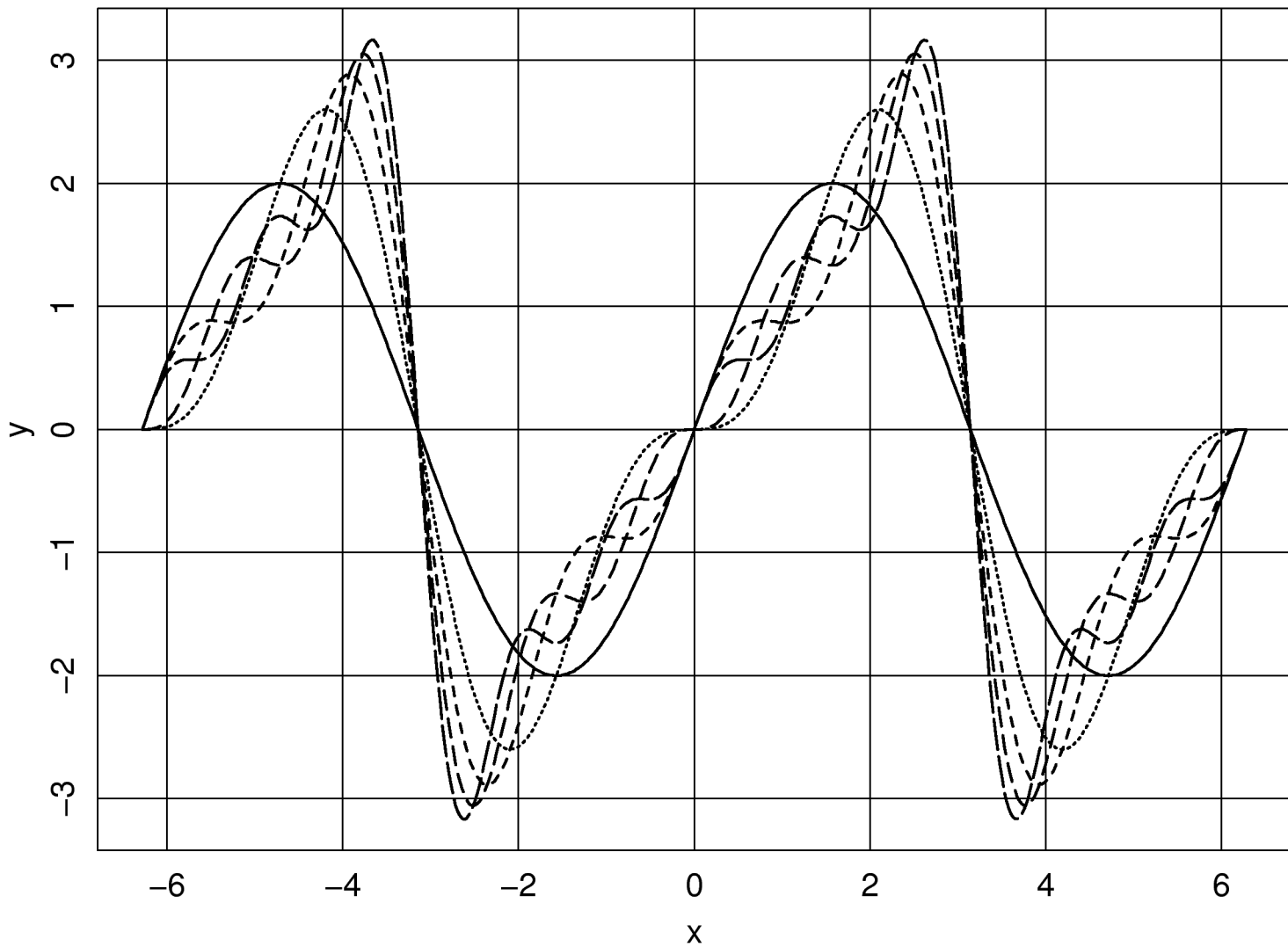


This is an **odd** fcn.
 $\Rightarrow$ Expand with only **sine** terms.

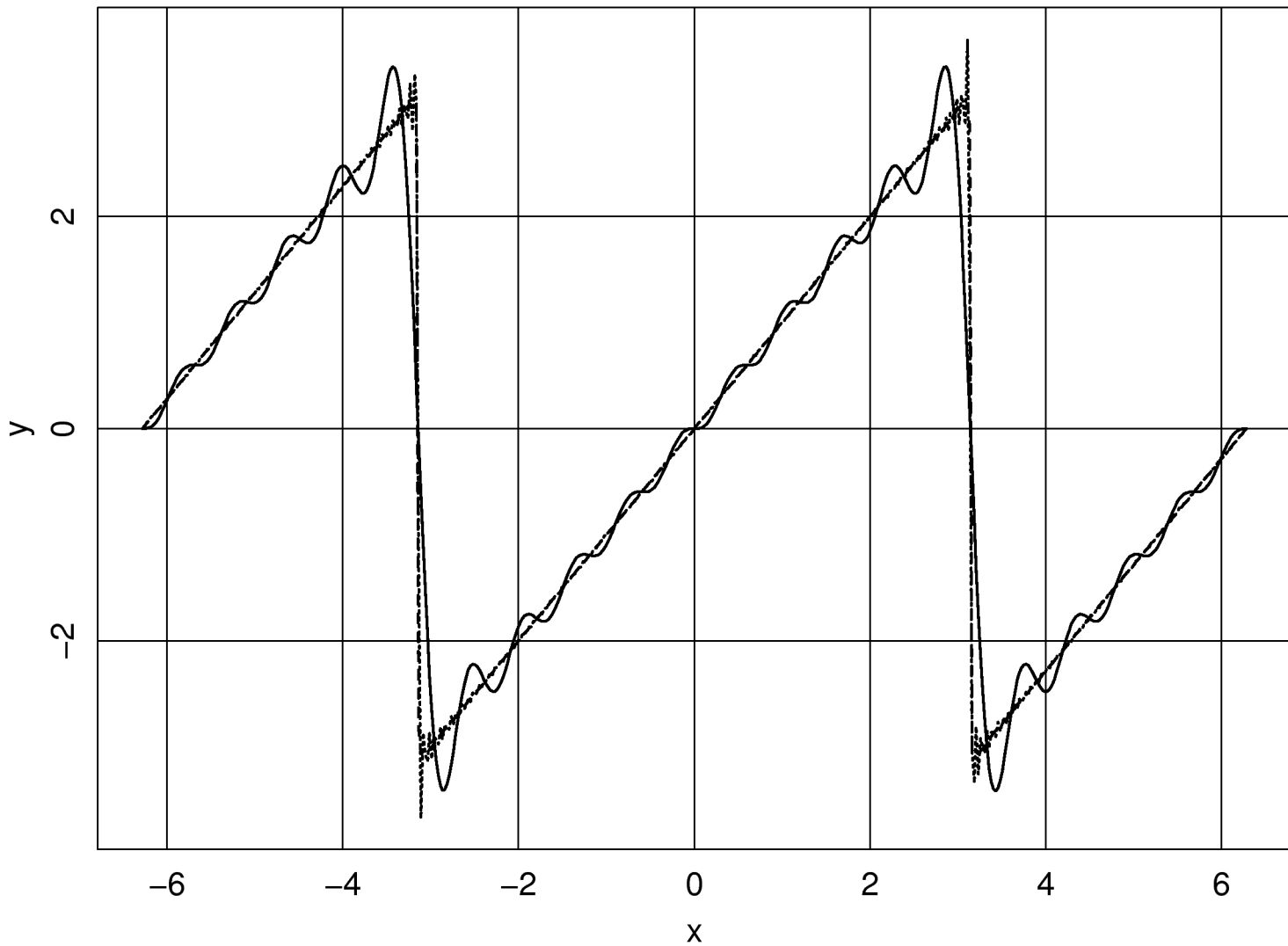
$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin n\theta d\theta = \frac{2}{\pi} \int_0^{\pi} \theta \sin n\theta d\theta, \quad n \geq 1. \\
 &\stackrel{\text{Int. by Parts}}{=} \frac{2}{\pi} \left\{ -\frac{\cos n\theta}{n} \theta \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos n\theta \cdot 1 \cdot d\theta \right\} \\
 &= \frac{2}{\pi} \left\{ -\frac{\pi \cos n\pi}{n} + \frac{1}{n^2} \sin n\theta \Big|_0^{\pi} \right\} = \frac{2}{n} (-1)^{n+1} \quad \left(\cos n\pi = (-1)^n \right)
 \end{aligned}$$

$\Rightarrow f(\theta) = \theta \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta$
 Use $\sim$ since for $\theta = k\pi$, RHS = 0 while LHS is undefined.

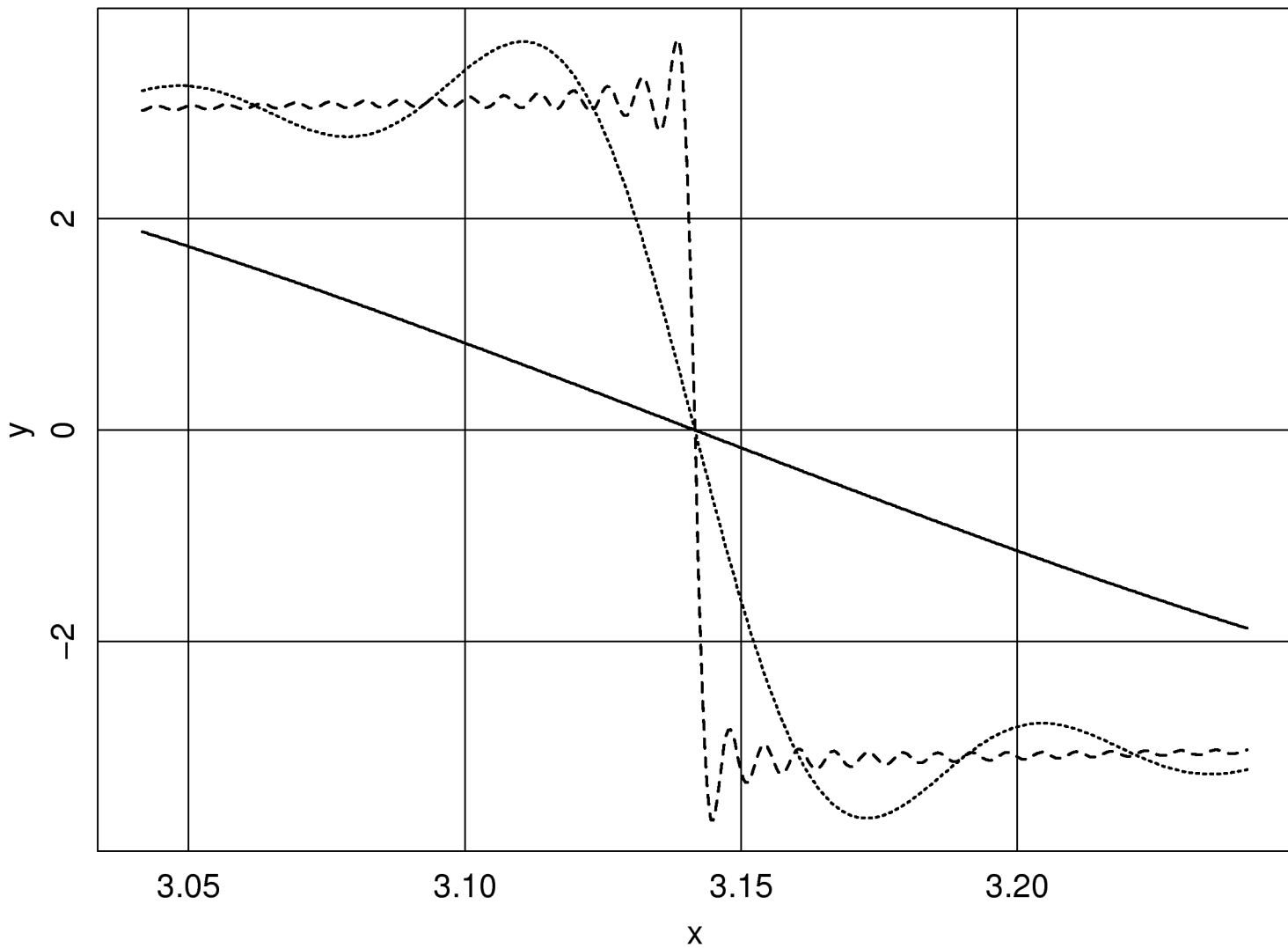
First 5 Partial Sums of the Fourier Series of $f(x)=x$ (2π period)



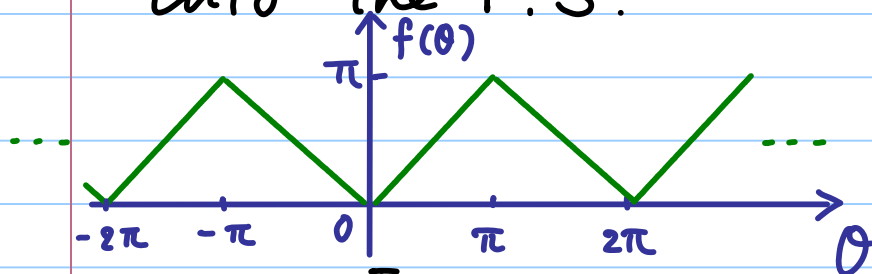
Partial Sums: N=10, 100, 1000



Zoom around $x=\pi$ of Partial Sums: $N=10$, 100 , 1000



Example 2. Expand $f(\theta) = |\theta|$, $-\pi \leq \theta \leq \pi$ into the F.S.



This is an **even** fcn.
 $\Rightarrow$ Expand with only **cosine** terms.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\theta| \cos n\theta d\theta \quad n \geq 1.$$

$$= \frac{2}{\pi} \int_0^{\pi} \theta \cos n\theta d\theta$$

$$= \frac{2}{\pi} \left\{ \frac{\sin n\theta}{n} \theta \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin n\theta \cdot 1 d\theta \right\}$$

$$= \frac{2}{\pi} \cdot \left\{ -\frac{1}{n} - \frac{\cos n\theta}{n} \Big|_0^{\pi} \right\} = \frac{2}{\pi} \cdot \frac{(-1)^n - 1}{n^2}$$

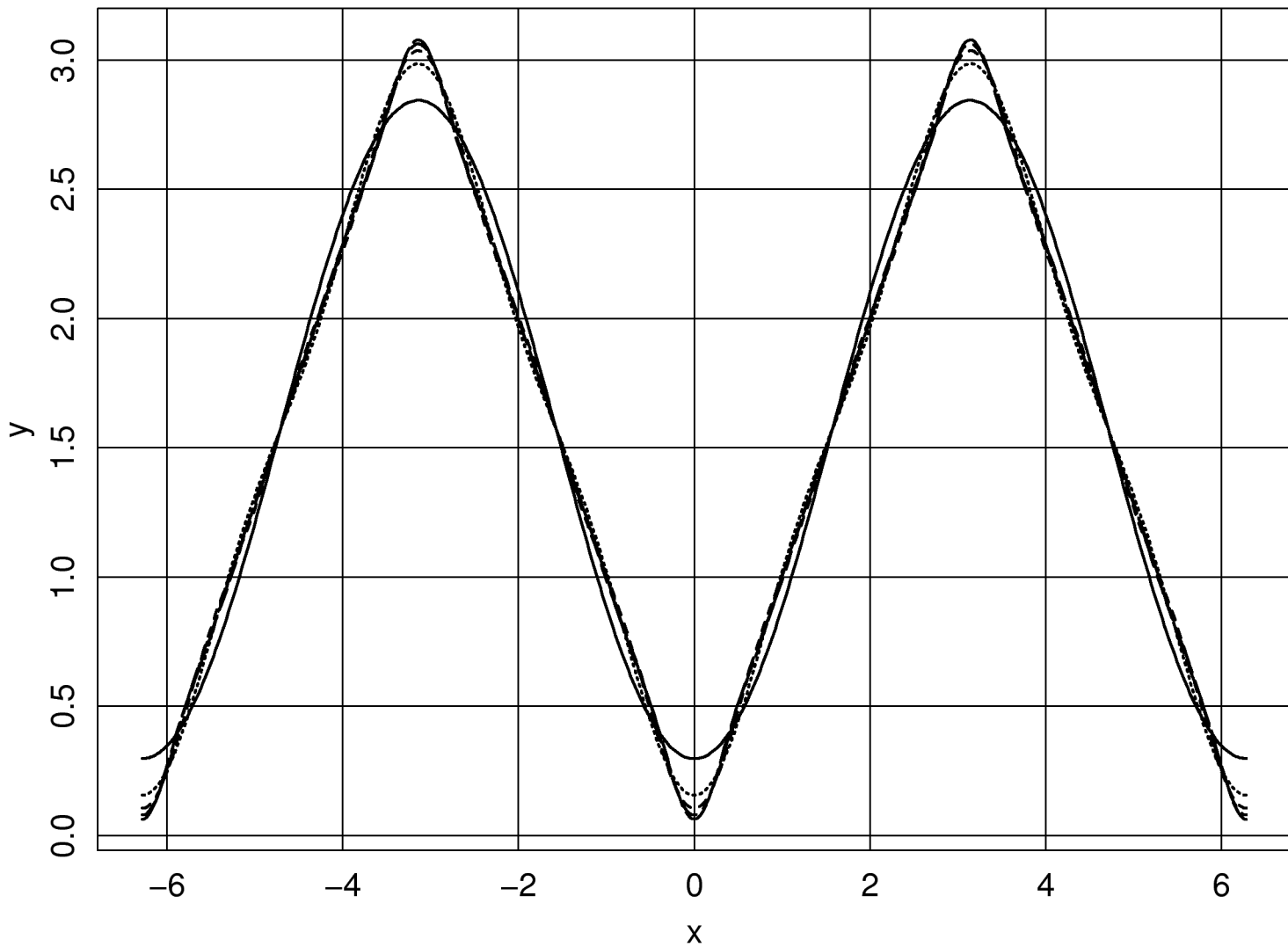
$$a_0 = \frac{2}{\pi} \int_0^{\pi} \theta d\theta = \frac{2}{\pi} \cdot \frac{\theta^2}{2} \Big|_0^{\pi} = \pi$$

$$\begin{aligned} \Rightarrow f(\theta) = |\theta| &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \\ &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\theta \\ &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{-2}{(2m-1)^2} \cos(2m-1)\theta \\ &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos(2m-1)\theta}{(2m-1)^2} \end{aligned}$$

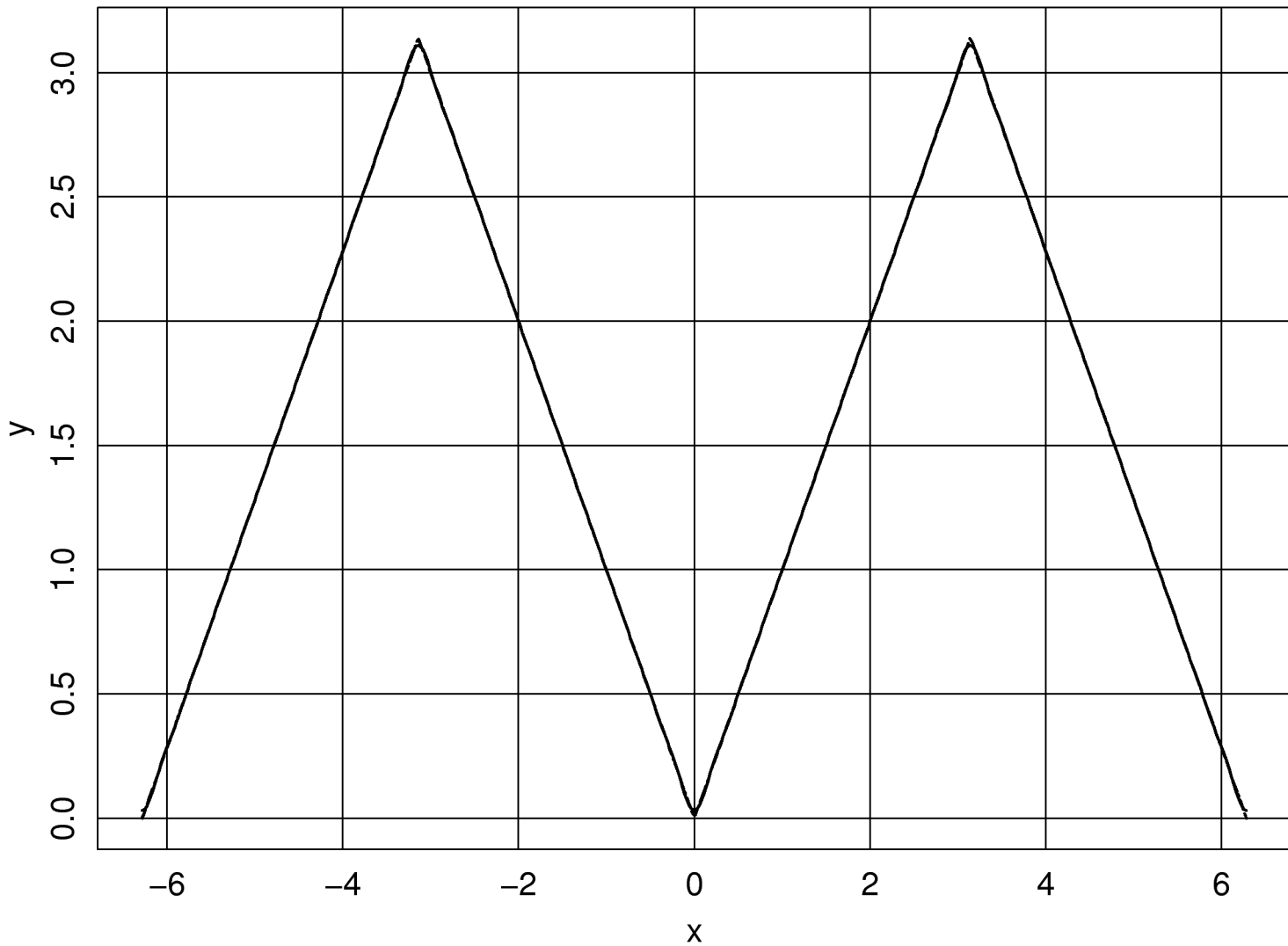
You can see that the convergence of this F.S. is faster than that of Example 1!

Exercise: Derive the same F.S. using the complex F.S. with C_n formula.

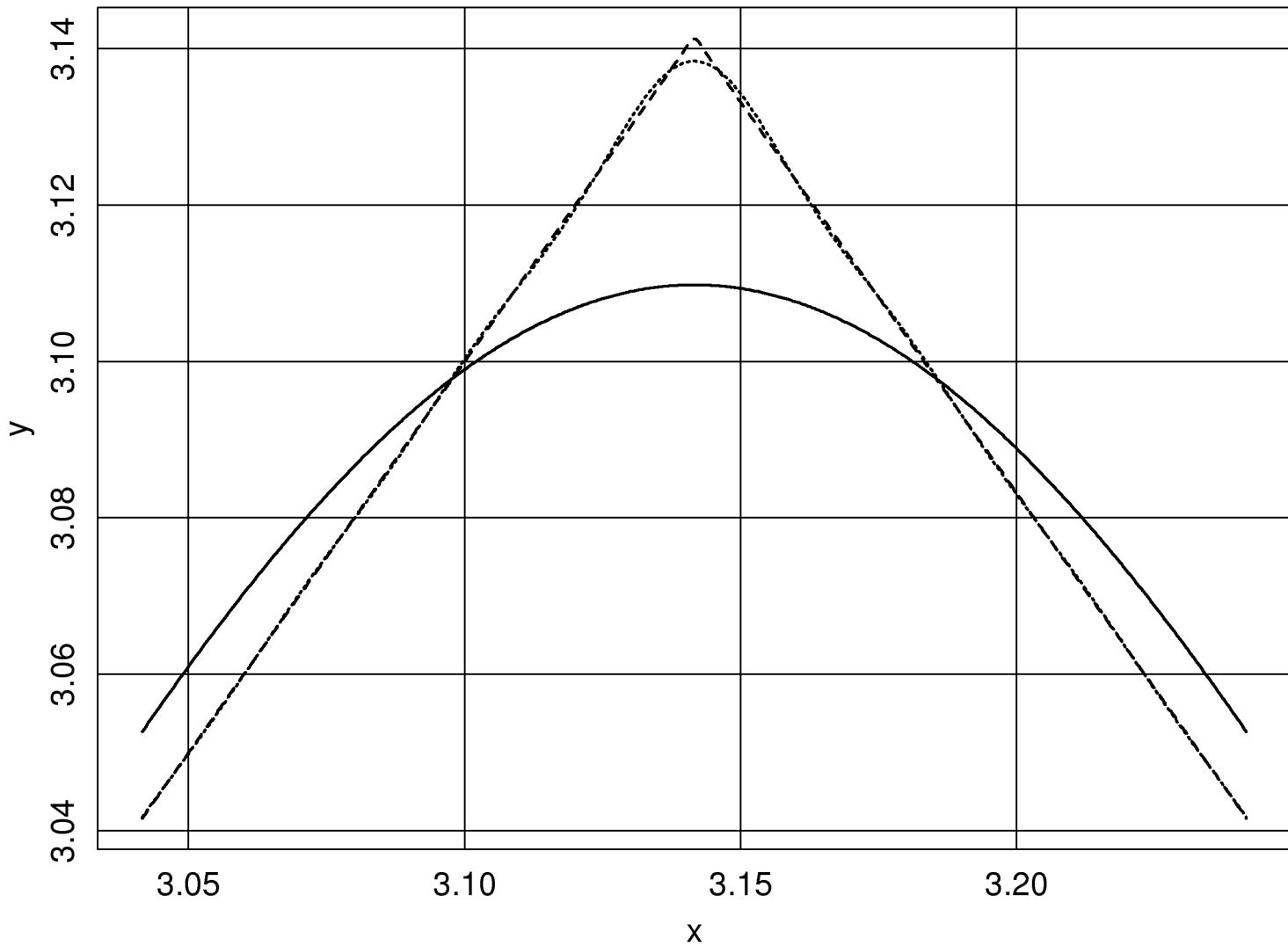
First 5 Partial Sums of the Fourier Series of $f(x)=|x|$ (2π periodic)



Partial Sums: $N=10, 100, 1000$



Zoom around $x=\pi$ of Partial Sums: $N=10$, 100 , 1000



★ Bessel's Inequality

Let f be 2π -periodic & Riemann integrable on $[-\pi, \pi]$.
Let C_n be the Fourier coef of f .

Then,
$$\sum_{-\infty}^{\infty} |C_n|^2 \leq \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta}_{\text{mean energy}}$$

(Proof)
$$\begin{aligned} & \left| f(\theta) - \underbrace{\sum_{-N}^N C_n e^{in\theta}}_{N\text{-th partial sum}} \right|^2 \\ &= \left(f(\theta) - \sum_{-N}^N C_n e^{in\theta} \right) \overline{\left(f(\theta) - \sum_{-N}^N C_n e^{in\theta} \right)} \\ &= |f(\theta)|^2 - 2 \operatorname{Re} \left\{ f(\theta) \sum_{-N}^N \bar{C}_n e^{-in\theta} \right\} + \sum_{m,n=-N}^N C_m \bar{C}_n e^{i(m-n)\theta} \end{aligned}$$

Integrate both sides:

$$\begin{aligned} 0 &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^N C_n e^{in\theta} \right|^2 d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \frac{2}{2\pi} \operatorname{Re} \left\{ \sum_{-N}^N \bar{C}_n \underbrace{\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta}_{= 2\pi C_n} \right\} \\ &\quad + \frac{1}{2\pi} \sum_{m,n=-N}^N C_m \bar{C}_n \underbrace{\int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta}_{= 2\pi \delta_{m,n}} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^N |C_n|^2 \end{aligned}$$

Then letting $N \rightarrow \infty$ proves Bessel's Ineq.!

Remark:

It turns out this is in fact an **equality**, which is called **Parseval's Equality**.

Orthonormal **sequence** in a Hilbert space $\Rightarrow$ Bessel **basis** $\Rightarrow$ Parseval

Bessel's Ineq.: f is of finite energy $\Rightarrow \sum_{-\infty}^{\infty} |C_n|^2$: convergent

Corollary: The Fourier coef's $a_n, b_n, C_n, C_{-n} \rightarrow 0$ as $n \rightarrow \infty$.