

Lecture 13: Fourier Series II

Note Title

1/24/2015

★ Convergence Thm's

A big question:

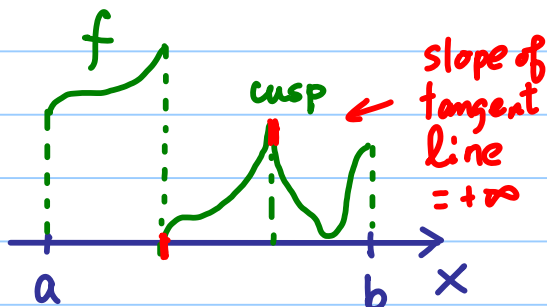
$$f(\theta) \stackrel{?}{=} \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \text{ at every pt } \theta \in \mathbb{R}?$$

$$\text{or } f(\theta) \stackrel{?}{=} \lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n e^{in\theta} =$$

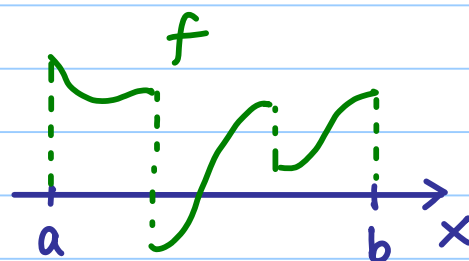
$\Rightarrow$ For a certain **class of fcn's**, this is Yes!

Def. $f \in PC[a, b]$ (**piecewise continuous on $[a, b]$**)
 if (i) f is continuous on $[a, b]$ except perhaps at finitely many pts, say, $x_1, \dots, x_k$; and
 (ii) $f(x_j^-) = \lim_{h \downarrow 0} f(x_j - h)$, $f(x_j^+) = \lim_{h \downarrow 0} f(x_j + h)$
 exist for each x_j , $j = 1, \dots, k$.

Def. $f \in PS[a, b]$ (**piecewise smooth on $[a, b]$**)
 if (i) $f \in PC[a, b]$; and (ii) $f' \in PC(a, b)$.



$f \in PC[a, b]$
 $\notin PS[a, b]$



$f \in PS[a, b]$

Remark: Classes of fcn's w.r.t. smoothness

(see, e.g., Davis & Rabinowitz: Methods of Numerical Integration, 2nd Ed., Dover, 2007)

Rougher $\leftarrow$ $\rightarrow$ Smoother

$R[a, b]$, $BV[a, b]$, $C[a, b]$, $C^\alpha[a, b]$, $C^k[a, b]$, $A(I)$, $E(\mathbb{C})$, P_n
 Riemann Integrable Bdd. Variation $0 < \alpha \leq 1$ Lip α $k \in \mathbb{N}$ $I \supset [a, b]$ Entire Analytic Polynom deg $\leq n$

We'll work with c_n and complex F.S. $\sum_{-\infty}^{\infty} c_n e^{in\theta}$.

Recall $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) e^{-in\psi} d\psi$

Consider the **N th partial sum** of f

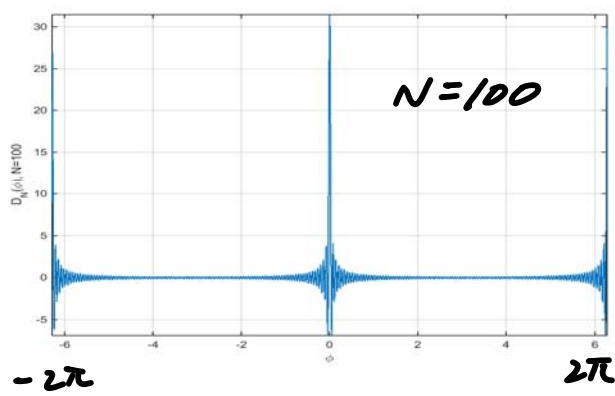
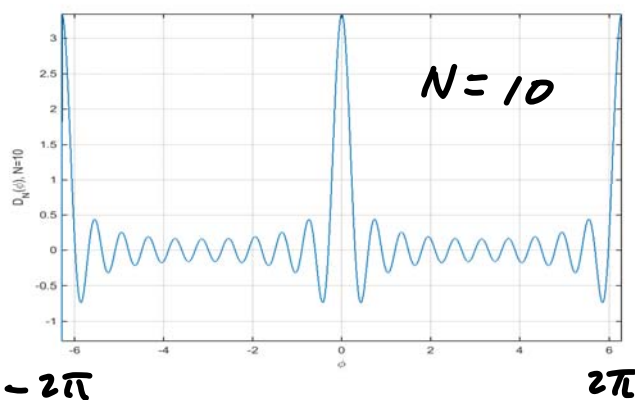
$$S_N[f](\theta) := \sum_{n=-N}^N c_n e^{in\theta}$$

Question: $S_N[f](\theta) \xrightarrow[N \rightarrow \infty]{?} f(\theta)$, $\forall \theta \in \mathbb{R}$

$$\begin{aligned} S_N[f](\theta) &= \sum_{n=-N}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) e^{in(\theta-\psi)} d\psi \\ &\stackrel{n \mapsto -n}{=} \frac{1}{2\pi} \sum_{n=-N}^N \int_{-\pi}^{\pi} f(\psi) e^{in(\psi-\theta)} d\psi \\ &\stackrel{\phi = \psi - \theta}{=} \frac{1}{2\pi} \sum_{n=-N}^N \int_{-\pi-\theta}^{\pi-\theta} f(\theta+\phi) e^{in\phi} d\phi \\ &\stackrel{f: 2\pi \text{ periodic}}{=} \frac{1}{2\pi} \sum_{n=-N}^N \int_{-\pi}^{\pi} f(\theta+\phi) e^{in\phi} d\phi \\ &= \int_{-\pi}^{\pi} f(\theta+\phi) \left\{ \frac{1}{2\pi} \sum_{n=-N}^N e^{in\phi} \right\} d\phi \end{aligned}$$

$=: D_N(\phi)$ the N th Dirichlet kernel

$$\begin{aligned} D_N(\phi) &= \frac{1}{2\pi} e^{-in\phi} (1 + e^{i\phi} + \dots + e^{i2N\phi}) \\ &= \frac{1}{2\pi} e^{-in\phi} \frac{1 - e^{i(2N+1)\phi}}{1 - e^{i\phi}} = \frac{1}{2\pi} \frac{e^{i(N+1)\phi} - e^{-in\phi}}{e^{i\phi} - 1} \\ &= \frac{1}{2\pi} \frac{e^{i(N+\frac{1}{2})\phi} - e^{-i(N+\frac{1}{2})\phi}}{e^{i\phi/2} - e^{-i\phi/2}} = \frac{1}{2\pi} \frac{\sin((N+\frac{1}{2})\phi)}{\sin \frac{\phi}{2}} \end{aligned}$$



Lemma $\int_0^{\pi} D_N(\theta) d\theta = \int_{-\pi}^0 D_N(\theta) d\theta = \frac{1}{2}$
 i.e., $\int_{-\pi}^{\pi} D_N(\theta) d\theta = 1$.

(Proof) By the def. of $D_N(\theta)$, we can see

$$D_N(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_1^N \cos n\theta \Rightarrow \int_{-\pi}^0 \left(\frac{1}{2\pi} + \frac{1}{\pi} \sum_1^N \cos n\theta \right) d\theta$$

$$= \frac{\theta}{2\pi} \Big|_{-\pi}^0 + \sum_1^N \frac{\sin n\theta}{\pi n} \Big|_{-\pi}^0$$

$$= \frac{1}{2} //$$

Thm If $f : 2\pi$ -periodic & $\in PS(\mathbb{R})$, then
 $\lim_{N \rightarrow \infty} S_N[f](\theta) = \frac{1}{2} [f(\theta-) + f(\theta+)]$, $\forall \theta \in \mathbb{R}$.
 In particular, $\lim_{N \rightarrow \infty} S_N[f](\theta) = f(\theta)$ at every θ where f is continuous.

(Proof) We'll use the Lemma as
 $\frac{1}{2} f(\theta-) = f(\theta-) \int_{-\pi}^0 D_N(\phi) d\phi$, $\frac{1}{2} f(\theta+) = f(\theta+) \int_0^{\pi} D_N(\phi) d\phi$
 Now, $S_N[f](\theta) - \frac{1}{2} [f(\theta-) + f(\theta+)]$

$$= \int_{-\pi}^{\pi} f(\theta+\phi) D_N(\phi) d\phi - \int_{-\pi}^0 f(\theta-) D_N(\phi) d\phi - \int_0^{\pi} f(\theta+) D_N(\phi) d\phi$$

$$= \int_{-\pi}^0 [f(\theta+\phi) - f(\theta-)] D_N(\phi) d\phi + \int_0^{\pi} [f(\theta+\phi) - f(\theta+)] D_N(\phi) d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 \frac{f(\theta+\phi) - f(\theta-)}{e^{i\phi} - 1} (e^{i(N+1)\phi} - e^{-iN\phi}) d\phi + \frac{1}{2\pi} \int_0^{\pi} \frac{f(\theta+\phi) - f(\theta+)}{e^{i\phi} - 1} (e^{i(N+1)\phi} - e^{-iN\phi}) d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi; \theta) (e^{i(N+1)\phi} - e^{-iN\phi}) d\phi \quad (*)$$

where $g(\phi; \theta) := \begin{cases} \frac{f(\theta+\phi) - f(\theta-)}{e^{i\phi} - 1} & -\pi \leq \phi \leq 0 \\ \frac{f(\theta+\phi) - f(\theta+)}{e^{i\phi} - 1} & 0 \leq \phi \leq \pi \end{cases}$

g is a well-behaved smooth fcn on $[-\pi, \pi]$ except $\phi = 0$. But by l'Hôpital's rule,

$$\begin{aligned} \lim_{\phi \downarrow 0} g(\phi; \theta) &= \lim_{\phi \downarrow 0} \frac{f'(\theta+\phi)}{ie^{i\phi}} = \frac{f'(\theta+)}{i} \\ \lim_{\phi \uparrow 0} g(\phi; \theta) &= \lim_{\phi \uparrow 0} \frac{f'(\theta+\phi)}{ie^{i\phi}} = \frac{f'(\theta-)}{i} \end{aligned} \quad \left. \begin{array}{l} \text{both exist} \\ \text{since} \\ f \in \text{PS}(\mathbb{R}), \\ \text{i.e., } |f'(\theta_{\pm})| < \infty \end{array} \right\}$$

$$\Rightarrow g \in \text{PC}[-\pi, \pi].$$

So, by Corollary to Bessel's ineq.,

$$\hat{g}_n = C_n[g] = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi; \theta) e^{-in\phi} d\phi \xrightarrow{|n| \rightarrow \infty} 0.$$

$$\begin{aligned} \text{Now } (*) &= \hat{g}_{-(N+1)} - \hat{g}_N = C_{-(N+1)}[g] - C_N[g] \\ &\longrightarrow 0 \text{ as } N \rightarrow \infty \end{aligned}$$

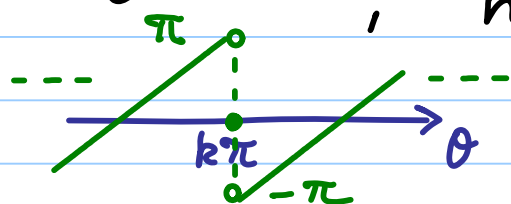
★ Some applications

Example 1 $f(\theta) = \theta$ $-\pi \leq \theta \leq \pi$, 2π periodic.

This f is clearly in $\text{PS}(\mathbb{R})$.

Recall its Fourier series.

$$\theta \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta = \frac{1}{2} [(\theta+) + (\theta-)]$$



$$= \begin{cases} \theta & \text{if } -\pi < \theta < \pi \\ 0 & \text{if } \theta = \pm\pi \end{cases}$$

Example 2 $f(\theta) = |\theta|$, $-\pi \leq \theta \leq \pi$, 2π per.
This f is in $PS(\mathbb{R}) \cap C(\mathbb{R})$.

$$\Rightarrow \frac{1}{2} [|\theta+1| + |\theta-1|] = |\theta| \quad \forall \theta \in \mathbb{R}.$$

$$\text{So, } |\theta| = \frac{\pi}{2} - \frac{4}{\pi} \sum_1^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2}, \quad \forall \theta \in [-\pi, \pi].$$

Example 3 $f(\theta) = \theta^2$, $-\pi \leq \theta \leq \pi$, 2π per.

Again this is in $PS(\mathbb{R}) \cap C(\mathbb{R})$.

Computing its F.S. (an exercise!), we have

$$\theta^2 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos n\theta, \quad \forall \theta \in [-\pi, \pi].$$

Set $\theta = \pi$.

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$$\Leftrightarrow \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{The Basel problem due to Euler!}$$

You can also derive this using Example 2.
(Exercise!)

Remark: $\exists$ 8 or 9 different proofs of the Basel problem; see the ref. page.
Later in this course, I'll discuss another beautiful proof based on an eigenfcn expansion of Green's fcn for the Dirichlet-Laplacian on $[0, 1]$.