

Lecture 14: Fourier Series III

Note Title

1/27/2015

★ Derivatives, Integrals, & Uniform Convergence

$$\text{Let } f(\theta) \sim \sum c_n e^{in\theta} = \sum \hat{f}_n e^{in\theta}$$

And let's write

$$f'(\theta) \sim \sum c'_n e^{in\theta} = \sum \hat{f}'_n e^{in\theta}$$

$$F(\theta) = \int_0^\theta f(\phi) d\phi \sim \sum C_n e^{in\theta} = \sum \hat{F}_n e^{in\theta}$$

What are the relationship between c'_n , C_n and c_n ?

Answer: $c'_n = in c_n$, $C_n = \frac{c_n}{in}$ ($n \neq 0$)

You can see $\frac{d}{d\theta}$: roughening

$\int \cdot d\phi$: smoothing

More precisely,

Thm Let $f: 2\pi$ -periodic & $\in C(\mathbb{R}) \cap PS(\mathbb{R})$.

Then, $c'_n = in c_n$ (or $\hat{f}'_n = in \hat{f}_n$).

In terms of a_n, b_n , we have

$$a'_n = n b_n, \quad b'_n = -n a_n \quad (\text{Exercise!})$$

(Proof) $c'_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(\theta) e^{-in\theta} d\theta$

Int. by Parts

$$\equiv \frac{1}{2\pi} \left\{ f(\theta) e^{-in\theta} \Big|_{-\pi}^{\pi} + in \int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta \right\}$$

$$= \frac{1}{2\pi} \left\{ \underbrace{f(\pi) e^{-in\pi}}_{(-1)^n f(\pi)} - \underbrace{f(-\pi) e^{in\pi}}_{(-1)^n f(-\pi)} \right\} + in C_n$$

$$= in C_n \quad \equiv \equiv$$

$= (-1)^n f(\pi)$ since $f: 2\pi$ -per. & in $C(\mathbb{R})$.

As for the F.S. of an antiderivative of f , note first that

an antiderivative of a periodic fun \neq a periodic fun in general!

e.g., $f(\theta) \equiv 1$ is a periodic fun, but its antiderivative $F(\theta) = \theta$ is not periodic.

The key is $c_0 = \hat{f}_0 = 0$ or not.

Since $\int_0^\theta c_n e^{in\phi} d\phi$ is 2π -periodic if $n \neq 0$,

we can see $F(\theta)$ is 2π -periodic if $c_0 = \hat{f}_0 = 0$.

Thm Let $f \in PC(\mathbb{R})$, 2π -periodic, and $F(\theta) = \int_0^\theta f(\phi) d\phi$. If $c_0 = \hat{f}_0 = 0$, then

$$F(\theta) = C_0 + \sum_{n \neq 0} \frac{c_n}{in} e^{in\theta} = \frac{A_0}{2} + \sum_1^\infty \left(\frac{a_n}{n} \sin n\theta - \frac{b_n}{n} \cos n\theta \right)$$

where $C_0 = \hat{F}_0 = \frac{A_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) d\theta$.

$$\begin{aligned} \text{(Proof)} \quad F(\theta + 2\pi) - F(\theta) &= \int_0^{\theta+2\pi} f(\phi) d\phi = \int_{-\pi}^{\pi} f(\phi) d\phi \\ &= 2\pi C_0 = 0. \end{aligned}$$

↑
 $f: 2\pi$ -periodic

So, $F(\theta)$ is also 2π -periodic!

Moreover, $F \in C(\mathbb{R}) \cap PS(\mathbb{R})$ because $f \in PC(\mathbb{R})$.

Hence, the F.S. of F at $\theta = F(\theta) \quad \forall \theta \in \mathbb{R}$.

By applying the previous Thm to $F(\theta)$, we get

$$c_n = \hat{f}_n = in C_n = in \hat{F}_n \Rightarrow \hat{F}_n = \frac{\hat{f}_n}{in}, n \neq 0$$

These two thms suggest:

$\frac{d}{d\theta}$: high freq. coef \uparrow ; $\int \cdot d\phi$: high freq. coef. \downarrow

The previous conv. thm, i.e., $S_N[f](\theta) \rightarrow \frac{1}{2}[f(\theta^-) + f(\theta^+)]$
 $\forall \theta \in \mathbb{R}$ for $f \in PS(\mathbb{R})$, 2π -periodic, was about
the **pointwise** convergence of F.S.
Now we want to have a thm of **absolute & uniform** convergence!

Def. Suppose $\sum_1^\infty g_n(x)$ converges to $g(x)$ on $x \in S$,
 S : some set. Then if $\sum_1^\infty |g_n(x)|$ converges for
 $x \in S$ too, $\sum_1^\infty g_n(x)$ is said to **converge absolutely**
on S . If $\sup_{x \in S} |g(x) - \sum_1^N g_n(x)| \rightarrow 0 \quad \forall x \in S$,
then $\sum_1^\infty g_n(x)$ is said to **converge uniformly** to
 $g(x)$ on S .

Ex. $g_n(x) = \frac{2(-1)^{n+1}}{n} \sin nx$, $S = [-\pi, \pi]$.

$\Rightarrow \sum g_n(x)$, i.e., the F.S. of $g(x) = x$ (2π -per.),
does **not** converge uniformly to $g(x)$
because the uniform limit of a continuous fcn
must be continuous, but g is discontinuous at $x = \pm\pi$.

To check the abs. & unif. conv. of a series,
we can use the **Weierstrass M-test** (aka the
Comparison Test): If $\exists M_n \geq 0$ is a seq. s.t.
 $|g_n(x)| \leq M_n, \forall x \in S$ & $\sum_1^\infty M_n < \infty$,
then $\sum_1^\infty g_n(x)$ conv. abs. & unif. on S .

So, in our case of F.S., because $|C_n e^{in\theta}| = |C_n|$, if $\sum_{-\infty}^{\infty} |C_n| < \infty$, then the F.S. conv. abs. & unif., that is,
Thm (Sufficiency for abs. & unif. conv.)
If $f \in C(\mathbb{R}) \cap PS(\mathbb{R})$, 2π -periodic,
then the F.S. of f converges to f
 abs. & unif. on \mathbb{R} .

(Proof) To show: $\sum_{-\infty}^{\infty} |\hat{f}_n| < \infty$
 (Note that we already know $\sum_{-\infty}^{\infty} |\hat{f}_n|^2 < \infty$
 thanks to Bessel's inequality!)

Because of the cond. on f , we have

$$\hat{f}'_n = in \hat{f}_n \Rightarrow |\hat{f}_n| = \left| \frac{\hat{f}'_n}{n} \right| \text{ for } n \neq 0.$$

Bessel's inequality applied to f' gives us

$$\sum_{-\infty}^{\infty} |\hat{f}'_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(\theta)|^2 d\theta < \infty$$

$$\text{So, } \sum_{-\infty}^{\infty} |\hat{f}_n| = |\hat{f}_0| + \sum_{n \neq 0} \left| \frac{\hat{f}'_n}{n} \right| \leq |\hat{f}_0| + \left(\sum_{n \neq 0} \frac{1}{n^2} \right)^{\frac{1}{2}} \cdot \left(\sum_{n \neq 0} |\hat{f}'_n|^2 \right)^{\frac{1}{2}}$$

↑ Cauchy-Schwarz!

$$= |\hat{f}_0| + \frac{\pi}{\sqrt{3}} \left(\sum_{n \neq 0} |\hat{f}'_n|^2 \right)^{\frac{1}{2}} < \infty \quad \equiv \equiv \equiv$$

Remark: This sufficient cond. is not necessarily the sharpest one. We won't prove the following sharper thm's:

Thm (Bernstein, 1914) $f \in \text{Lip } \alpha(\mathbb{R}) = C^\alpha(\mathbb{R})$, $\alpha > \frac{1}{2}$
 \Rightarrow The F.S. of f conv. abs. & unif.

Thm (Zygmund, 1928) $f \in \text{Lip } \alpha(\mathbb{R}) \cap BV(\mathbb{R})$, $\alpha > 0$
 \Rightarrow The F.S. of f conv. abs. & unif.

Thm (Smoothness class & Fourier coef)

Suppose $f: 2\pi$ -periodic, $\in C^{k-1}(\mathbb{R})$, and $f^{(k-1)} \in PS(\mathbb{R})$, i.e., $f^{(k)}$ exists except perhaps at finitely many pts in each bdd. interval.

$$\text{Then, } \sum_{-\infty}^{\infty} |n^k \hat{f}_n|^2 < \infty.$$

In particular, $n^k \hat{f}_n \rightarrow 0$ as $n \rightarrow \pm\infty$

On the other hand, suppose the Fourier coef's satisfy $|\hat{f}_n| \leq C |n|^{-(k+\alpha)} \exists C > 0, \exists \alpha > 1$ for $n \in \mathbb{Z} \setminus \{0\}$. Then $f \in C^k(\mathbb{R})$.

(Proof) The first part: Apply the Deriv. Thm ($\hat{f}'_n = in \hat{f}_n$) k times to get $\hat{f}_n^{(k)} = (in)^k \hat{f}_n$.

Apply Bessel's ineq. to $f^{(k)}$ to get

$$\sum_{-\infty}^{\infty} |n^k \hat{f}_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(k)}(\theta)|^2 d\theta < \infty //$$

The second part:

$$\sum_{n \neq 0} |n^j \hat{f}_n| \leq C \sum_{n \neq 0} |n|^{-(k-j+\alpha)} \leq 2C \sum_{n > 0} n^{-\alpha} \text{ for } j \leq k$$

converges since $\alpha > 1$.

$\Rightarrow \sum (in)^j \hat{f}_n e^{in\theta}$ conv. abs. & unif. to $f^{(j)}$, $j \leq k$.

$\Rightarrow f^{(j)} \in C(\mathbb{R})$, $j \leq k$. $\Rightarrow f \in C^k(\mathbb{R})$. ///