

Lecture 15: Fourier Series IV

Note Title

* Fourier Series on a general interval
often, we need to consider the F.S. of a fun $f(x)$ on a more general interval, say $[\alpha, \beta]$ instead of $[-\pi, \pi]$.

Consider then a linear map from $[-\pi, \pi]$ to $[\alpha, \beta]$ via $x = p\theta + q$. Then

find p, q s.t. $\theta = -\pi \leftrightarrow x = \alpha$

$\theta = +\pi \leftrightarrow x = \beta$.

$$\Rightarrow p = \frac{\beta - \alpha}{2\pi}, \quad q = \frac{\alpha + \beta}{2}$$

$$\text{So, } \theta = \frac{2\pi}{\beta - \alpha} \left(x - \frac{\alpha + \beta}{2} \right)$$

$$\begin{aligned} \text{Hence, } \hat{g}_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x) e^{-2\pi i n \left(\frac{x - (\alpha + \beta)/2}{\beta - \alpha} \right)} dx \end{aligned}$$

$$\text{where } f(x) = g\left(\frac{2\pi}{\beta - \alpha} \left(x - \frac{\alpha + \beta}{2}\right)\right)$$

$$\text{i.e., } f\left(\frac{\beta - \alpha}{2\pi} \theta + \frac{\alpha + \beta}{2}\right) = g(\theta)$$

Let us call this $\hat{g}_n$ as the n th Fourier coef. of f defined on $[\alpha, \beta]$ with period $\beta - \alpha$, and denote by $\hat{f}_n$ (sorry for notational abuse!)

$$f(x) \sim \sum_{-\infty}^{\infty} \hat{f}_n e^{2\pi i n \left(\frac{x - (\alpha + \beta)/2}{\beta - \alpha} \right)}$$

$$\hat{f}_n = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x) e^{-2\pi i n \left(\frac{x - (\alpha + \beta)/2}{\beta - \alpha} \right)} dx$$

Remark: $[\alpha, \beta] = [0, 1], [-\frac{1}{2}, \frac{1}{2}], [-1, 1]$ are common.

★ Functions of Bounded Variations

Why are we interested in BV?

- Often chosen as a model for piecewise smooth signals & images
- Useful in data compression & statistical estimation
- Provide sharp info on the decay rate of the Fourier coeff's.

Let $g(x)$ be a fcn on a closed interval $I = [a, b]$. (I could be $\mathbb{R}$).

Let $D :=$ a subdivision of I , i.e.,
 $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.

Now, let's form the sum:

$$T_D(g) := \sum_{k=1}^n |g(x_k) - g(x_{k-1})|$$

Def. If $T_D(g) < \infty$ for all possible subdivision D , then g is said to be of bdd. var. in I , and the total variation of g in I is defined as

$$V_I[g] = V_a^b[g] := \sup_D T_D(g).$$

$BV(I) :=$ a set of all fns of bdd. var. in I .

Fact: • $|g(b) - g(a)| \leq V_a^b[g] < \infty$ ← Take $x_0 = a$
 $x_1 = b$.

• If $I \subset J$, then $V_I[g] \leq V_J[g]$

Thm 1. $g \in BV(I) \Rightarrow g$ is bdd. in I .

(Pf) $g(x) = g(a) + g(x) - g(a)$

$$\Rightarrow |g(x)| \leq |g(a)| + |g(x) - g(a)|$$

$$\leq |g(a)| + V_a^x[g]$$

$$\leq |g(a)| + V_a^b[g] < \infty. //$$

One can also show that $BV(I)$ is a
Banach space.

Thm 2. $g, h \in BV(I) \Rightarrow gh \in BV(I)$.

$$g, h \in BV(I), |h(x)| \geq \exists m > 0 \\ \Rightarrow g/h \in BV(I).$$

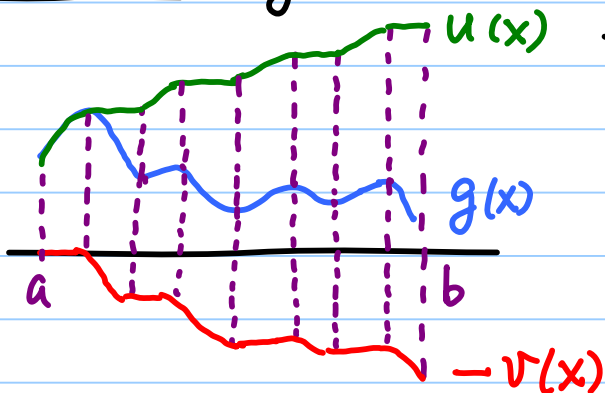
Thm 3. $\forall c \in (a, b), g \in BV[a, b]$
 $\Leftrightarrow g \in BV[a, c] \text{ \& } g \in BV[c, b]$.

Moreover, $V_a^b[g] = V_a^c[g] + V_c^b[g]$.

Remark : This can be generalized to
 $a < c_1 < c_2 < \dots < c_n < b$.

Thm 4 $g \in BV(I) \Leftrightarrow g$ can be written as
the difference of two
non decreasing fcn's.

say, $u(x) - v(x)$



Previously, we proved the thm:

$f: 2\pi$ -periodic & $\in PS(\mathbb{R})$

$$\Rightarrow S_N[f](0) \rightarrow \frac{1}{2} [f(0-) + f(0+)].$$

Hence if f is also in $C(\mathbb{R})$, then
 $S_N[f](0) \rightarrow f(0), \forall 0 \in \mathbb{R}.$

This thm is usually attributed to Dirichlet (1829). But it was made sharper thanks to the notion of BV by Jordan (1881):

Thm (Dirichlet - Jordan)

Suppose $f \in BV[\alpha, \beta]$. Then the following hold:

- (i) $S_N[f](x) \rightarrow \frac{1}{2} [f(x+) + f(x-)]$.
- (ii) Furthermore, if $f \in C(\alpha, \beta)$, then $S_N[f](x)$ converges to $f(x)$ uniformly $\forall x \in (\alpha + \gamma, \beta - \gamma), \forall \gamma > 0$.

In fact, thanks to the Riemann localisation principle (1892), the above becomes even sharper as follows:

Thm (Dirichlet - Jordan - Riemann)

- (i) If $f|_{N(x)} \in BV(N(x))$ where $N(x) = [x - \delta, x + \delta] \subset [\alpha, \beta]$
 $\exists \delta > 0$, then $S_N[f](y) \rightarrow \frac{1}{2} [f(y+) + f(y-)], \forall y \in N(x)$.
- (ii) If $f \in C(\alpha, \beta) \cap BV(\alpha, \beta)$, then $S_N[f](x) \rightarrow f(x)$
uniformly $\forall x \in (\alpha + \gamma, \beta - \gamma), \forall \gamma > 0$

Note: In the above two thm's, of course
 $\forall \gamma > 0$ s.t., $(\alpha + \gamma, \beta - \gamma)$ makes sense, i.e.,
in fact, $\gamma \in (0, (\beta - \alpha)/2)$.

M. Taibleson's Thm (1967) 1 page paper!

If $f \in BV[0,1]$, $f(x) \sim \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$,

then $c_k = O(1/k)$ as $k \rightarrow \infty$.

(Pf) Use the fact:

$$\int_{j/|k|}^{(j+1)/|k|} e^{-2\pi i k x} dx = 0, \quad k \neq 0, \quad j=0,1,\dots,|k|,$$

$$\Leftrightarrow = \frac{e^{-2\pi i (j+1) \frac{k}{|k|}} - e^{-2\pi i j \frac{k}{|k|}}}{-2\pi i k}$$

$$= \frac{1}{-2\pi i k} (e^{\mp 2\pi i (j+1)} - e^{\mp 2\pi i j}) = 0 //$$

Now, fix k , and let $a_j := \frac{j}{|k|}$, $j=0,1,\dots,|k|$.

Then define

$$g(x) := \sum_{j=0}^{|k|-1} f(a_j) \chi_{[a_j, a_{j+1}]}(x) \quad \text{a step fn approx } f.$$

Then, $c_k[g] = \int_0^1 g(x) e^{-2\pi i k x} dx$

The k th Fourier coeff. of g .

$$= \int_0^1 \sum_{j=0}^{|k|-1} f(a_j) \chi_{[a_j, a_{j+1}]}(x) e^{-2\pi i k x} dx$$

$$= \sum_{j=0}^{|k|-1} f(a_j) \int_{a_j}^{a_{j+1}} e^{-2\pi i k x} dx = 0.$$

$$c_k[f] = \int_0^1 f(x) e^{-2\pi i k x} dx$$

$$|c_k[f]| = |c_k[f] - c_k[g]| = |c_k[f-g]|$$

$$= \left| \int_0^1 (f(x) - g(x)) e^{-2\pi i k x} dx \right|$$

$$\leq \sum_{j=0}^{|k|-1} \int_{a_j}^{a_{j+1}} |f(x) - f(a_j)| dx$$

$$\leq \sum_{j=0}^{|k|-1} V_{a_j}^{a_{j+1}}[f] \underbrace{(a_{j+1} - a_j)}_{= 1/|k|}$$

Thm 3 $\rightarrow$

$$= \frac{1}{|k|} \underbrace{V_0^1[f]}_{\text{"const."}} = O(1/|k|). //$$

Thm (NS - J.F. Remy, 2006)

Let f be a 1-periodic fcn and $f \in C^m(\mathbb{R})$. Furthermore, let us assume that $f^{(m+1)}$ exists and in $BV[0,1]$. Then its Fourier coeff.

$c_k[f] = \hat{f}(k)$ decays as $O(|k|^{-m-2})$, where $m = 0, 1, \dots$.

(Pf) Use { the periodicity, i.e., $f^{(l)}(0) = f^{(l)}(1)$, $l = 0, \dots, m$.
integration by parts!

$$\begin{aligned} \hat{f}(k) &= \int_0^1 f(x) e^{-2\pi i k x} dx \\ &= \frac{e^{-2\pi i k x}}{-2\pi i k} f(x) \Big|_0^1 + \frac{1}{2\pi i k} \int_0^1 f'(x) e^{-2\pi i k x} dx \\ &= \frac{e^{-2\pi i k x}}{-(2\pi i k)^2} f'(x) \Big|_0^1 + \frac{1}{(2\pi i k)^2} \int_0^1 f''(x) e^{-2\pi i k x} dx \\ &= \dots = \frac{e^{-2\pi i k x}}{-(2\pi i k)^{m+1}} f^{(m)}(x) \Big|_0^1 + \frac{1}{(2\pi i k)^{m+1}} \int_0^1 f^{(m+1)}(x) e^{-2\pi i k x} dx \end{aligned}$$

By assumption, $f^{(m+1)} \in BV[0,1]$. So, can use the Taibleson Thm to get:

$$|\hat{f}(k)| \leq V_0^1[f^{(m+1)}] (2\pi)^{-m-1} |k|^{-m-2} //$$