

Lecture 17: Basics of L^2 Theory I

Note Title

2/5/2015

In order to study the Sturm-Liouville Theory, Green's fns, & integral eqn's, we need to know the very basics of the L^2 theory!

★ Convergence & Completeness

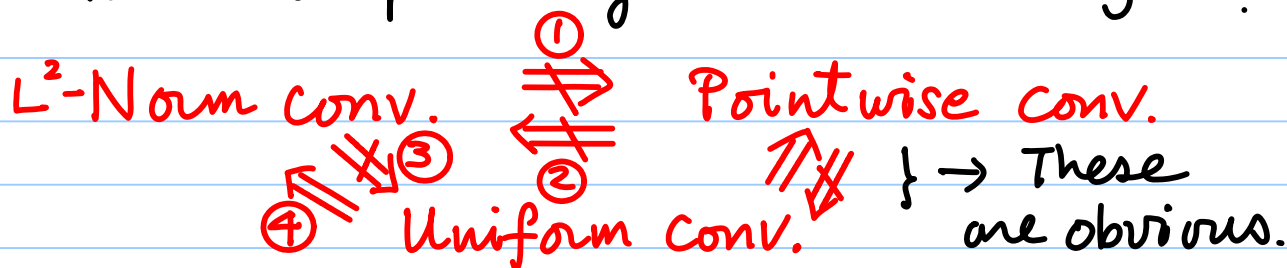
Def. Let f_n be a sequence in $PC[a, b]$.
 f_n **converges** to f in **norm** $\stackrel{\text{def}}{\iff} \|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$

Of course, the norm here is $\|f\| = \|f\|_2 := \sqrt{\int_a^b |f(x)|^2 dx}$.

Axioms of norm: Let $\mathcal{X}$ be a vector space over $\mathbb{C}$. For $\forall f, g \in \mathcal{X}, \forall \alpha \in \mathbb{C}$,
i) $\|f\| \geq 0$ with $\|0\| = 0$; ii) $\|\alpha f\| = |\alpha| \|f\|$; iii) $\|f + g\| \leq \|f\| + \|g\|$.

So, the norm conv. is equiv. to $\int_a^b |f_n(x) - f(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0$.

Relationship among various convergences:



①: Example WLOG, let $[a, b] = [0, 1]$.

$$f_n = \begin{cases} 1 & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

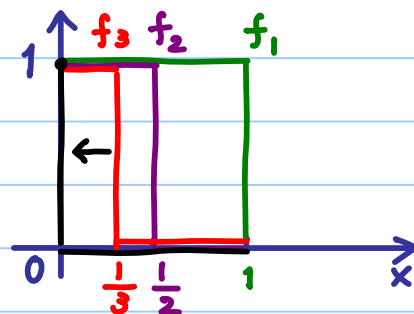
$$\|f_n\|^2 = \int_0^{\frac{1}{n}} 1^2 dx = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$$

So, $f_n \rightarrow 0$ in norm.

But, $f_n(0) \equiv 1, \forall n \in \mathbb{N}$.

So, f_n cannot conv. to 0 pointwise.

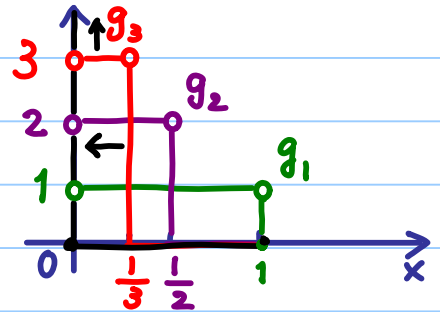
$f_n \rightarrow$ the unit impulse at $x = 0$. #



② : Example WLOG, let $[a, b] = [0, 1]$.

$$g_n = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} g_n(0) \equiv 0 & \forall n \in \mathbb{N}. \\ g_n(x) = 0 & \forall x > 0 \text{ with } n > \frac{1}{x} \end{cases}$$



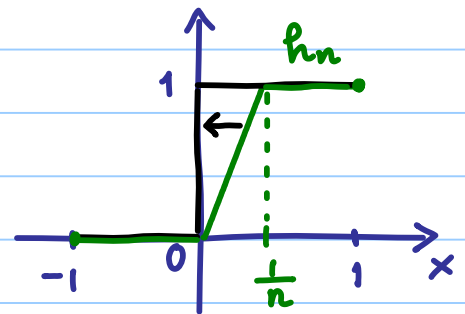
So, $g_n \rightarrow 0$ pointwise as $n \rightarrow \infty$

But, $\|g_n\|^2 = \int_0^{1/n} n^2 dx = n \rightarrow \infty$.

So it doesn't conv. to 0 in norm. #

③ : Example WLOG, let $[a, b] = [-1, 1]$.

$$h_n(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ nx & 0 \leq x \leq \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq 1 \end{cases}$$



$$\|h_n - \chi_{[0,1]}\|^2 = \int_0^{1/n} |nx - 1|^2 dx$$

the characteristic
fcn of $[0, 1]$.

$$= \int_0^{1/n} (n^2 x^2 - 2nx + 1) dx$$

$$= \left[\frac{n^2}{3} x^3 - nx^2 + x \right]_0^{1/n} = \frac{1}{3n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

So, $h_n \rightarrow \chi_{[0,1]}$ in norm.

But $\chi_{[0,1]}$ is a discontinuous fcn while each h_n is continuous. Hence, h_n cannot conv. uniformly to $\chi_{[0,1]}$. #

④ : Thm $f_n \rightarrow f$ uniformly on $[a, b] \Rightarrow f_n \rightarrow f$ in norm.

(Proof) $f_n \rightarrow f$ unif. $\Leftrightarrow \exists M_n > 0$, s.t. $|f_n(x) - f(x)| < M_n, \forall x \in [a, b]$
 $M_n \rightarrow 0$ as $n \rightarrow \infty$

But, $\|f_n - f\|^2 = \int_a^b |f_n(x) - f(x)|^2 dx \leq M_n^2 (b-a) \xrightarrow{n \rightarrow \infty} 0. ///$

Remark: If $f_n \rightarrow f$ in norm, then

$$\|f_n\| \rightarrow \|f\|, \quad \langle f_n, g \rangle \rightarrow \langle f, g \rangle, \quad \forall g \in \mathcal{X}$$

(Proof) $|\|f_n\| - \|f\|| \leq \|f_n - f\| \rightarrow 0.$

why this inequality holds?

Let $f_n - f =: g_n$. Then $f_n = g_n + f$

$$\Rightarrow \|f_n\| = \|g_n + f\| \leq \|g_n\| + \|f\|.$$

So, $\|f_n\| - \|f\| \leq \|g_n\| = \|f_n - f\|.$

Similarly, $\|f\| - \|f_n\| \leq \|f - f_n\|.$

Hence $|\|f_n\| - \|f\|| \leq \|f - f_n\| \rightarrow 0 //$

$\langle f_n, g \rangle \rightarrow \langle f, g \rangle, \quad \forall g \in \mathcal{X}$ can be shown similarly.

Now, let's discuss the completeness of a fun sp.

Def. $\mathcal{X}$ is complete w.r.t. $\|\cdot\|$

$$\iff \forall \{f_n\} \subset \mathcal{X} \text{ with } \|f_n - f\| \rightarrow 0 \Rightarrow \underline{f \in \mathcal{X}}$$

narrow def.

So, $C[a, b]$ is not complete with $\|\cdot\|_2 = \sqrt{\int_a^b |f|^2 dx}$

See Ex. ③! $PC[a, b]$ is not complete w.r.t. this norm either (will show this later).

A better def. of the completeness is the following:

Any Cauchy seq. $\{f_n\} \subset \mathcal{X}$ conv. to a limit $f \in \mathcal{X}$.
 $\{f_n\} \subset \mathcal{X}$ is a Cauchy seq. $\stackrel{\text{def}}{\iff} \|f_m - f_n\| \xrightarrow{m, n \rightarrow \infty} 0.$

For a general vector space $\mathcal{X}$,
 $\{f_n\}$: a convergent seq. in $\mathcal{X}$ $\stackrel{\text{⑤}}{\iff} \{f_n\}$: Cauchy in $\mathcal{X}$
 $\stackrel{\text{⑥}}{\iff}$

☺ Ex ③ implies ⑥. For ⑤, Let $f_n \rightarrow f \in \mathcal{X}$ in norm.
Then, $\|f_m - f_n\| = \|f_m - f + f - f_n\| \leq \|f_m - f\| + \|f - f_n\| \xrightarrow{n \rightarrow \infty} 0 //$

Hence, if every Cauchy seq. in $\mathcal{X}$ is a convergent seq. in $\mathcal{X}$, then such $\mathcal{X}$ is special, and is called **complete**.

* Already showed: $C[a, b]$: not complete w.r.t. $\|\cdot\|_2$
 yet " : complete w.r.t. $\|\cdot\|_\infty$
 $\|f\|_\infty := \sup_{a \leq x \leq b} |f(x)|$

* $PC[a, b]$ is not complete w.r.t. $\|\cdot\|_2$.

Why? Let $[a, b] = [0, 1]$.

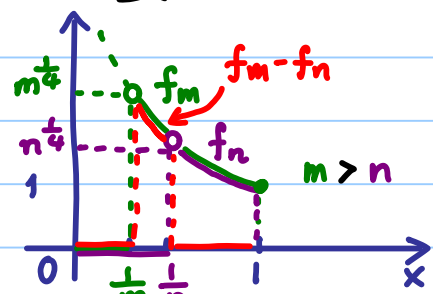
$$\text{Define } f_n(x) = \begin{cases} x^{-1/4} & \frac{1}{n} < x \leq 1 \\ 0 & 0 \leq x \leq \frac{1}{n} \end{cases}$$

$$\text{So, } f_m(x) - f_n(x) = x^{-1/4} \chi_{(\frac{1}{m}, \frac{1}{n})}(x)$$

$$\Rightarrow \|f_m - f_n\|_2^2 = \int_0^1 x^{-1/2} \chi_{(\frac{1}{m}, \frac{1}{n})}(x) dx = \int_{\frac{1}{m}}^{\frac{1}{n}} x^{-1/2} dx = 2\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right)$$

$$\text{But, } f_n(x) \rightarrow f(x) = \begin{cases} x^{-1/4} & 0 < x \leq 1 \\ 0 & x = 0 \end{cases} \xrightarrow{m, n \rightarrow \infty} 0$$

So, $f(0+) = +\infty \Rightarrow f \notin PC[0, 1]$, i.e.,
 PC is not complete w.r.t. $\|\cdot\|_2$! //



In order to have a fcn space $\mathcal{X}$ complete w.r.t. $\|\cdot\|_2$, we need **Lebesgue integral** instead of Riemann integral. However, we won't discuss Lebesgue integrals in detail in this course.

$$\text{Denote } L^2[a, b] := \left\{ f : \int_a^b |f(x)|^2 dx < \infty \right\}.$$

This is a space of **square-integrable** (L^2) fcn's. **Lebesgue**

In $L^2[a, b]$, the following **inner product** is well defined: $\langle f, g \rangle := \int_a^b f(x) \overline{g(x)} dx$

Why? Thanks to the Cauchy-Schwarz ineq.
we have $|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2 < \infty$
for $\forall f, g \in L^2[a, b]$. //

Remark: $\|f\| = 0$ in L^2 -norm

does not mean $f(x) \equiv 0$ everywhere in $[a, b]$.
but means $f(x) = 0$ **almost everywhere**
(a.e.) in $[a, b]$.

An extreme case:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ (rational numbers)} \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow \|f\| = 0$!! in terms of the Lebesgue int.
Riemann integral cannot be applied to
such a fcn.

Thm (a) $L^2[a, b]$ is complete w.r.t. $\|\cdot\|_2$.

(b) $\forall f \in L^2[a, b], \exists \{f_n\} \subset C[a, b]$ s.t. $\|f_n - f\| \rightarrow 0$.
In fact, one can also take $\{f_n\} \subset \underline{C^\infty[a, b]}$, $\|f_n - f\| \rightarrow 0$.

Bessel's Ineq.: If $\{\phi_n\}_1^\infty$ is an orthonormal set in
 $L^2[a, b]$, then $\sum_1^\infty |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2, \forall f \in L^2[a, b]$.

(Proof) Essentially the same as before.

The only difference is $\phi_n(x)$ instead of $\frac{1}{\sqrt{2\pi}} e^{in\theta}$. //

We know that

- $f \in PS(\mathbb{R}), 2\pi\text{-periodic} \Rightarrow S_N[f]$ conv. pointwise.

- $f \in PS(\mathbb{R}) \cap C(\mathbb{R}), 2\pi\text{-per} \Rightarrow S_N[f]$ conv. abs/unif.

Question: $f \in L^2[a, b] \Rightarrow \sum_1^\infty \langle f, \phi_n \rangle \phi_n \rightarrow f$ in norm?