

Lecture 18: Basics of L^2 Theory II

Note Title

2/7/2015

Recap: We already know that:

- $f \in \mathcal{PS}(\mathbb{R})$, 2π -periodic $\Rightarrow S_N[f]$ conv. pointwise.
- $f \in \mathcal{PS}(\mathbb{R}) \cap C(\mathbb{R})$, 2π -per $\Rightarrow S_N[f]$ conv. abs/unif.

Question: $f \in L^2[a, b] \Rightarrow \sum_1^\infty \langle f, \phi_n \rangle \phi_n \rightarrow f$ in norm?

Lemma If $f \in L^2[a, b]$, $\{\phi_n\}$: any ONset in $L^2[a, b]$,
then $\sum \langle f, \phi_n \rangle \phi_n$ conv. in norm and

$$\left\| \sum \langle f, \phi_n \rangle \phi_n \right\| \leq \|f\|.$$

(Proof) By Bessel's ineq., $\forall f \in L^2[a, b]$,
 $\sum_1^\infty |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2 < \infty$.

So, by the Pythagorean Thm, we have

$$\begin{aligned} \left\| \sum_M^N \langle f, \phi_n \rangle \phi_n \right\|^2 &= \sum_M^N \|\langle f, \phi_n \rangle \phi_n\|^2 \\ &= \sum_M^N |\langle f, \phi_n \rangle|^2 \underbrace{\|\phi_n\|^2}_{=1} \xrightarrow{M, N \rightarrow \infty} 0. \end{aligned}$$

Thus, the partial sums of $\sum \langle f, \phi_n \rangle \phi_n$ form a Cauchy seq. in $L^2[a, b]$, which is complete.

$\Rightarrow \sum_1^\infty \langle f, \phi_n \rangle \phi_n$ conv. in norm to $\exists f_{\text{en}} \in L^2[a, b]$

$$\begin{aligned} \text{Finally, } \left\| \sum_1^\infty \langle f, \phi_n \rangle \phi_n \right\|^2 &= \lim_{N \rightarrow \infty} \left\| \sum_1^N \langle f, \phi_n \rangle \phi_n \right\|^2 \\ &= \lim_{N \rightarrow \infty} \sum_1^N |\langle f, \phi_n \rangle|^2 \\ &= \sum_1^\infty |\langle f, \phi_n \rangle|^2 \leq \|f\|^2 \end{aligned}$$

Bessel's ineq.

Remark: An example of an ON **set** in $L^2[-1,1]$ but not an ON **B** of $L^2[-1,1]$:

$$\phi_n(x) = \begin{cases} \sqrt{2} \sin n\pi x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } -1 \leq x \leq 0. \end{cases}$$

Thm Let $\{\phi_n\}_1^\infty$ be an ON set in $L^2[a,b]$. Then the following cond's are equivalent:

- (a) $\langle f, \phi_n \rangle = 0, \forall n \in \mathbb{N} \Rightarrow f \equiv 0 \text{ (a.e.)}$.
- (b) $\forall f \in L^2[a,b], f = \sum \langle f, \phi_n \rangle \phi_n$ in norm.
- (c) $\forall f \in L^2[a,b], \|f\|^2 = \sum |\langle f, \phi_n \rangle|^2$ (**Parseval's equality**).

(Proof) (a) $\Rightarrow$ (b): By the lemma, $\sum \langle f, \phi_n \rangle \phi_n$ conv. in norm to a fcn in $L^2[a,b]$.

Need to show that fcn is in fact f .

To do so, consider $g = f - \sum \langle f, \phi_n \rangle \phi_n$.

Take the inner prod. with ϕ_m .

$$\begin{aligned} \Rightarrow \langle g, \phi_m \rangle &= \langle f, \phi_m \rangle - \sum \langle f, \phi_n \rangle \underbrace{\langle \phi_n, \phi_m \rangle}_{=\delta_{n,m}} \\ &= \langle f, \phi_m \rangle - \langle f, \phi_m \rangle = 0, \forall m \in \mathbb{N}. \end{aligned}$$

By (a), $g \equiv 0$, i.e., $f = \sum \langle f, \phi_n \rangle \phi_n$ (a.e.) //

(b) $\Rightarrow$ (c): $f = \sum \langle f, \phi_n \rangle \phi_n$.

$$\begin{aligned} \|f\|^2 &= \lim_{N \rightarrow \infty} \left\| \sum_1^N \langle f, \phi_n \rangle \phi_n \right\|^2 = \lim_{N \rightarrow \infty} \sum_1^N \|\langle f, \phi_n \rangle \phi_n\|^2 \\ &= \sum_1^\infty |\langle f, \phi_n \rangle|^2 // \end{aligned}$$

Pythagoras!

(c) $\Rightarrow$ (a): If $\|f\|^2 = \sum |\langle f, \phi_n \rangle|^2$ & $\langle f, \phi_n \rangle = 0, \forall n \in \mathbb{N}$,

then $\|f\| = 0$ is a must. So, $f \equiv 0$ (a.e.) //

Def. An ONset $\{\phi_n\}_1^\infty$ satisfying (a) - (c) of the thm is called a **complete ON set (CONS)** or **ON Basis (ONB)** of $L^2[a, b]$. $\{\langle f, \phi_n \rangle\}_1^\infty$ are called the **(generalized) Fourier coef's** of f w.r.t. $\{\phi_n\}_1^\infty$.
 $\sum_1^\infty \langle f, \phi_n \rangle \phi_n$ is called the **(generalized) Fourier series** of f .

Note that an Orthogonal Basis (OB) can always be transformed to ONB by simple normalization.

Thm $\{e^{inx}\}_{n \in \mathbb{Z}} : \text{OB for } L^2[-\pi, \pi]$

$\{\frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in \mathbb{Z}} : \text{ONB}$ "

$\{\cos nx\}_{n=0}^\infty \cup \{\sin nx\}_{n=1}^\infty : \text{OB}$ "

$\{\frac{1}{\sqrt{2\pi}}\} \cup \{\frac{1}{\sqrt{\pi}} \cos nx\}_{n=1}^\infty \cup \{\frac{1}{\sqrt{\pi}} \sin nx\}_{n=1}^\infty : \text{ONB}$ "

$\{\cos nx\}_{n=0}^\infty : \text{OB for } L^2[0, \pi]$

$\{\frac{1}{\sqrt{\pi}}\} \cup \{\sqrt{\frac{2}{\pi}} \cos nx\}_{n=1}^\infty : \text{ONB}$ "

$\{\sin nx\}_{n=1}^\infty : \text{OB}$ "

$\{\sqrt{\frac{2}{\pi}} \sin nx\}_{n=1}^\infty : \text{ONB}$ "

(Proof) We'll prove this for $\phi_n = \frac{1}{\sqrt{2\pi}} e^{inx}$.

The other cases are similar.

We already know these ϕ_n 's form an ON set.

The only thing we need to show: its **completeness!**

i.e., To show: $\forall f \in L^2[-\pi, \pi], \|f - \sum_1^N \langle f, \phi_n \rangle \phi_n\| \xrightarrow{N \rightarrow \infty} 0$

(If so, we know $\{\phi_n\}$: an ONB thanks to the Thm.)

Now, $\forall f \in L^2[-\pi, \pi], \forall \varepsilon > 0, \exists \tilde{f} \in C^\infty[-\pi, \pi]$ s.t.

$\|f - \tilde{f}\| \leq \varepsilon$ via the Thm in Lec. 17.

$$\begin{aligned}
\text{So, } & \| f - \sum_{-N}^N \langle f, \phi_n \rangle \phi_n \| \\
&= \| f - \tilde{f} + \tilde{f} - \sum_{-N}^N \langle \tilde{f}, \phi_n \rangle \phi_n + \sum_{-N}^N \langle \tilde{f}, \phi_n \rangle \phi_n - \sum_{-N}^N \langle f, \phi_n \rangle \phi_n \| \\
&\leq \| f - \tilde{f} \| + \| \tilde{f} - \sum_{-N}^N \langle \tilde{f}, \phi_n \rangle \phi_n \| + \| \sum_{-N}^N \langle \tilde{f} - f, \phi_n \rangle \phi_n \| \\
&\leq \varepsilon + \varepsilon + \left(\sum_{-N}^N |\langle \tilde{f} - f, \phi_n \rangle|^2 \right)^{\frac{1}{2}} \quad \text{Pythagoras} \\
&\quad \text{for } N \geq N_0(\varepsilon) \text{ because } \sum_{-N}^N \langle \tilde{f}, \phi_n \rangle \phi_n \rightarrow \tilde{f} \text{ unif.} \\
&\leq 2\varepsilon + \| \tilde{f} - f \| \leq 3\varepsilon. \\
&\quad \uparrow \text{Bessel's ineq.}
\end{aligned}$$

★ Another Example of ONB on $L^2[-1, 1]$:

Legendre Polynomials

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad \leftarrow \text{called the Rodrigues formula. } n=0, 1, 2, \dots$$

is called the n th Legendre polynomial.

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x), \dots$$

Thm The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ form an OB for $L^2[-1, 1]$, with $\|P_n\|^2 = \frac{2}{2n+1}$.

So, $\left\{ \sqrt{n + \frac{1}{2}} P_n(x) \right\}_{n=0}^{\infty}$ form an **ONB** for $L^2[-1, 1]$.

These can also be obtained by applying the **Gram-Schmidt** procedure to $\{1, x, x^2, x^3, \dots\}$.

$$\text{Let } \varphi_n(x) := \sqrt{n + \frac{1}{2}} P_n(x)$$

For a given $f \in L^2[-1, 1]$, suppose we have

$$f = \sum_{n=0}^{\infty} \alpha_n \varphi_n = \sum_{n=0}^{\infty} c_n P_n$$

Then, the relationship between α_n & c_n is $\sum \alpha_n \varphi_n = \sum \alpha_n \sqrt{n+\frac{1}{2}} P_n$

$$\text{and } \alpha_n = \langle f, \varphi_n \rangle = c_n$$

$$\Rightarrow c_n = \sqrt{n+\frac{1}{2}} \alpha_n = \sqrt{n+\frac{1}{2}} \langle f, \varphi_n \rangle$$

$$= \sqrt{n+\frac{1}{2}} \langle f, \sqrt{n+\frac{1}{2}} P_n \rangle$$

$$= (n+\frac{1}{2}) \langle f, P_n \rangle.$$

Of course, dealing with an ONB $\{\varphi_n\}$ is easier and more convenient.

Example: Expand $\chi_{[0,1]}(x)$ into a Legendre series!
 $\chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

$$c_n = (n+\frac{1}{2}) \int_{-1}^1 \chi_{[0,1]}(x) P_n(x) dx$$

$$= (n+\frac{1}{2}) \int_0^1 P_n(x) dx$$

$$= \frac{2n+1}{2^{n+1} n!} \int_0^1 \frac{d^n}{dx^n} (x^2-1)^n dx$$

$$= \frac{2n+1}{2^{n+1} n!} \left[\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right]_0^1$$

$$= - \frac{2n+1}{2^{n+1} n!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \Big|_{x=0}$$

$$= - \frac{2n+1}{2^{n+1} n!} \frac{d^{n-1}}{dx^{n-1}} \left(\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^{2k} \right) \Big|_{x=0}$$

Since $\frac{d^{n-1}}{dx^{n-1}} x^{2k} = 2k(2k-1)\cdots(2k-n+2)x^{2k-n+1}$ and we evaluate it at $x=0$, $C_n = 0$ unless $\underline{2k=n-1}$ or $n=0$.
 $\Leftrightarrow n=2k+1$

$$C_0 = \frac{1}{2} \int_0^1 \underline{P_0(x)} dx = \frac{1}{2}$$

$$C_{2k+1} = -\frac{4k+3}{2^{2k+2}(2k+1)!} (-1)^{2k+1-k} \binom{2k+1}{k} (2k)! \\ = (-1)^k \frac{4k+3}{4^{k+1}} \frac{(2k)!}{k!(k+1)!}$$

$$\Rightarrow \chi_{[0,1]}(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k (4k+3) (2k)!}{4^{k+1} k! (k+1)!} \underline{P_{2k+1}(x)} \quad //$$

odd only

* Interesting fact

Given a fcn $f \in L^2[-1,1]$, the best n -deg. poly. over $[-1,1]$ is given by $\sum_{k=0}^n c_k P_k(x)$, $c_k = (k + \frac{1}{2}) \langle f, P_k \rangle$.

$$\min_{p \in P_n} \|f - p\|^2 = \|f - \sum_{k=0}^n c_k P_k\|^2 \quad \leftarrow \text{in the least squares sense.}$$

Why? $\|f - p\|^2 = \|\sum_{k=0}^{\infty} c_k P_k - p\|^2 = \|\sum_{k=0}^{\infty} c_k P_k - \sum_{k=0}^n b_k P_k\|^2$

$$= \|\sum_{k=0}^n (c_k - b_k) P_k + \sum_{k=n+1}^{\infty} c_k P_k\|^2$$

Pythagoras $\rightarrow$

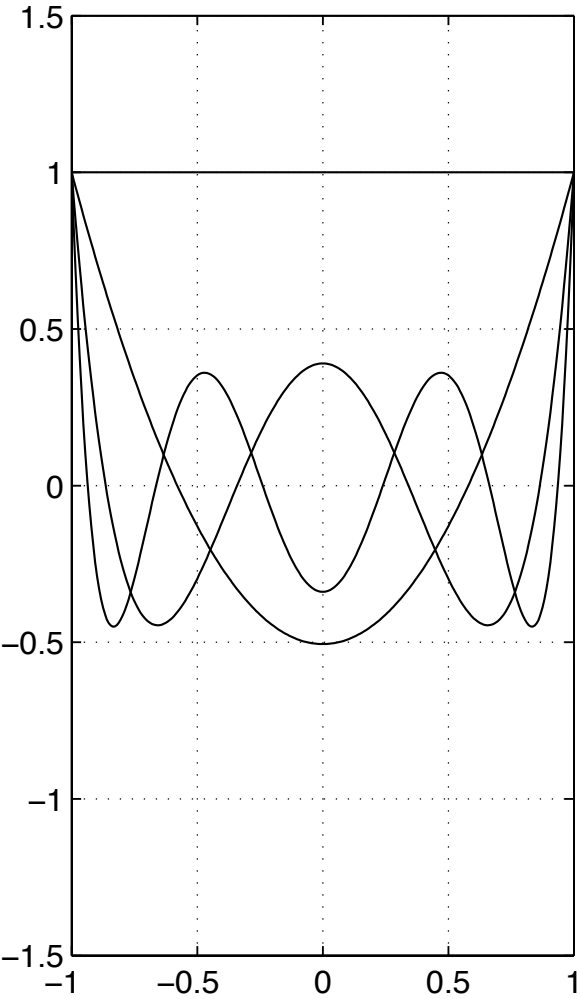
$$= \|\sum_{k=0}^n (c_k - b_k) P_k\|^2 + \|\sum_{k=n+1}^{\infty} c_k P_k\|^2$$

$$\geq \|\sum_{k=n+1}^{\infty} c_k P_k\|^2$$

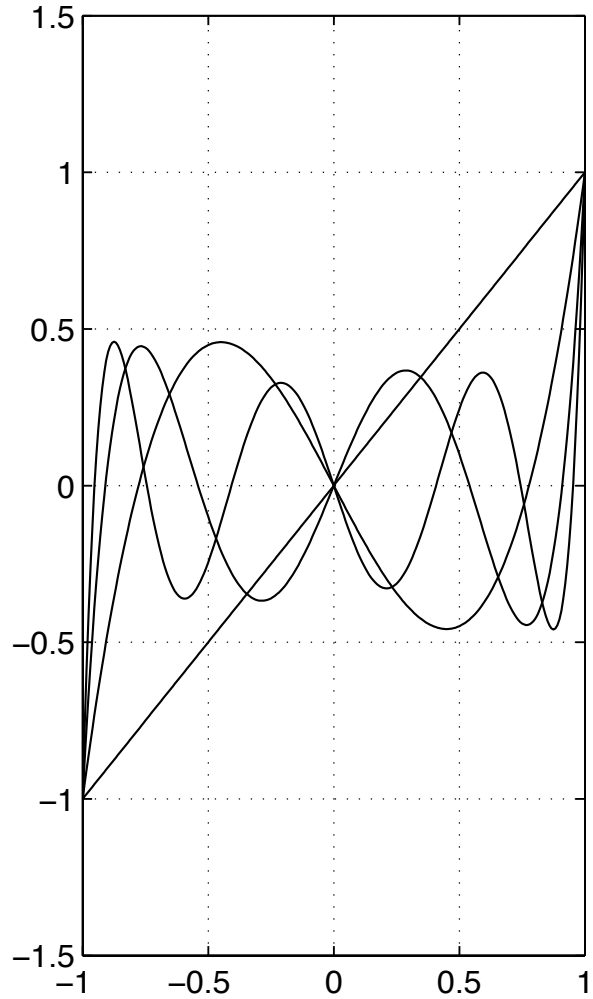
$\hookrightarrow$

$$= \text{iff } b_k = c_k, k=0,1,\dots,n. \quad //$$

Even Legendre Polynomials



Odd Legendre Polynomials



Approximation of a discontinuous function

