

# Lecture 19: Basics of $L^2$ Theory III

Note Title

2/10/2015

## ★ Other Types of $L^2$ spaces

### (1) Weighted $L^2$ space

Let  $w(x) \in (0, \infty)$  for a.e.  $x \in [a, b]$ .

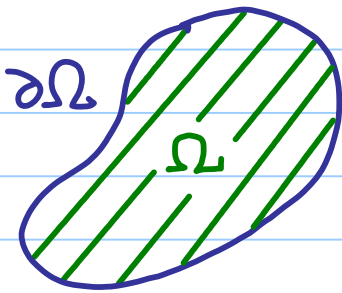
$$L^2_w[a, b] := \left\{ f: [a, b] \rightarrow \mathbb{C} \mid \int_a^b |f(x)|^2 w(x) dx < \infty \right\}$$

$$\langle f, g \rangle_w := \int_a^b f(x) \overline{g(x)} w(x) dx, \quad \|f\|_w := \sqrt{\langle f, f \rangle_w}$$

$\Rightarrow$  Used frequently in Sturm-Liouville Theory  
Orthogonal Polynomials

### (2) Higher-dimensions

Instead of the interval  $[a, b]$ , consider a **domain**  $\Omega \subset \mathbb{R}^d$



$$L^2(\Omega) := \left\{ f: \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f(x)|^2 dx < \infty \right\}$$

Thm  $L^2(\Omega)$  is **complete**.

If  $f \in L^2(\Omega)$ , then  $\exists \{f_n\} \subset L^2(\Omega)$  s.t.  
 $\|f_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover,  
one can take  $\{f_n\} \subset C_0^\infty(\Omega) \subset C_0(\Omega)$ .  
 $\hookrightarrow$  have compact support in  $\Omega$ .

## ★ Hilbert Space

Def. A vector space  $\mathcal{H}$  is called a **Hilbert space** if

- (1) inner product is defined on elements in  $\mathcal{H}$   
(thus the norm is defined as  $\|f\| = \sqrt{\langle f, f \rangle}$ ); and
- (2)  $\mathcal{H}$  is **complete** w.r.t. this norm.

Def. A vector space  $B$  is called a **Banach space** if

- (1) a norm is defined on elements in  $B$ ; and
- (2)  $B$  is **complete** w.r.t. this norm.

Def. A set  $M$  is called a **metric space** if  
a **distance** (i.e., **metric**) is defined among elem's of  $M$ .

$$\left\{ \begin{array}{c} \text{Inner product} \\ \text{spaces} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Normed} \\ \text{spaces} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Metric} \\ \text{spaces} \end{array} \right\}$$

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$$\left\{ \begin{array}{c} \text{Hilbert} \\ \text{spaces} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Banach} \\ \text{spaces} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Complete metric} \\ \text{spaces} \end{array} \right\}$$

e.g.  $\hookrightarrow L^2(\Omega)$

$\hookrightarrow L^p(\Omega), 1 \leq p \leq \infty$

$\hookrightarrow \mathbb{R}$  with  $d(x,y) = |x-y|/(1+|x-y|)$

See highly informative math.stackexchange.com posting!

Another important example of Hilbert space:

$$\ell^2(\mathbb{N}) := \left\{ \mathbb{C} = (c_j)_1^\infty, c_j \in \mathbb{C} \mid \sum_1^\infty |c_j|^2 < \infty \right\}$$

$$\text{For } \mathbb{C}, \mathbb{d} \in \ell^2(\mathbb{N}), \langle \mathbb{C}, \mathbb{d} \rangle := \sum_1^\infty c_j \bar{d}_j, \|\mathbb{C}\|_2 = \sqrt{\sum_1^\infty |c_j|^2}$$

Can show  $\{\mathbb{C}_n\} \subset \ell^2(\mathbb{N}) : \text{Cauchy} \Rightarrow \mathbb{C}_n \rightarrow \exists \mathbb{C} \in \ell^2(\mathbb{N})$

Thm Any Hilbert space is **isomorphic** to  $\ell^2(\mathbb{N})$ .

(Hence,  $L^2(\Omega)$  is isomorphic to  $\ell^2(\mathbb{N})$ , which is referred to as **the Riesz-Fischer Thm.**)

Def. Two Hilbert spaces  $\mathcal{H}$  &  $\mathcal{H}'$  are said to be **isomorphic** to each other if  $\exists$  a bijection  $\Phi: \mathcal{H} \rightarrow \mathcal{H}'$  s.t.

$$(1) \Phi(\alpha x + \beta y) = \alpha \Phi(x) + \beta \Phi(y), \quad \forall x, y \in \mathcal{H}, \forall \alpha, \beta \in \mathbb{C}.$$

$$(2) \langle x, y \rangle_{\mathcal{H}} = \langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}'}$$

(Proof) Suppose  $\{\phi_n\}_1^\infty$  be an ONB for  $\mathcal{H}$ .

Then, we can consider a map  $\Phi: \mathcal{H} \rightarrow \ell^2(\mathbb{N})$  by  $\Phi f := \{\langle f, \phi_n \rangle\}_1^\infty, \quad \forall f \in \mathcal{H}.$

Thanks to Parseval's equality  $\|f\|_{\mathcal{H}}^2 = \sum_1^\infty |\langle f, \phi_n \rangle|^2$ , so clearly  $\Phi f \in \ell^2(\mathbb{N})$  and  $\|f\|_{\mathcal{H}} = \|\Phi f\|_2$  (**isometry**)

(2) is also satisfied via Parseval,  $\langle f, g \rangle_{\mathcal{H}} = \sum \langle f, \phi_n \rangle \langle g, \phi_n \rangle = \langle \Phi f, \Phi g \rangle_2$

So,  $\Phi$  is **one-to-one** from  $\mathcal{H}$  to  $\ell^2(\mathbb{N})$ . **injection**

Remains to show:  $\Phi$  is also "onto" (surjection).

Take any  $\mathbb{C} = (c_1, c_2, \dots) \in \ell^2(\mathbb{N})$ , i.e.,  $\sum |c_j|^2 < \infty$ .

Then consider the following sequence in  $\mathcal{H}$ :

$$f_1 = c_1 \phi_1, f_2 = c_1 \phi_1 + c_2 \phi_2, \dots, f_n = \sum_{j=1}^n c_j \phi_j, \dots$$

$$\Rightarrow \text{For } m > n, \|f_m - f_n\|_{\mathcal{H}}^2 = \sum_{j=n+1}^m |c_j|^2 \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

So,  $\{f_n\}$ : Cauchy in  $\mathcal{H}$ .  $\{\phi_j\}$ : ONB

That implies that  $f_n \rightarrow \exists f \in \mathcal{H}$  thanks to its completeness.

For this  $f$ , we clearly have  $\Phi f = \mathbb{C}$ , and  $\mathbb{C} \in \ell^2(\mathbb{N})$  was arbitrary. Hence we are done! //

## ★ The Dominated Convergence Thm

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , be a domain, and let  $\{g_n\}, g, \phi$  be fcn's on  $\Omega$  s.t.

$$\begin{cases} (a) \phi(x) \geq 0 \text{ and } \int_{\Omega} \phi(x) dx < \infty; \\ (b) |g_n(x)| \leq \phi(x), \forall n \in \mathbb{N}, \forall x \in \Omega; \text{ and} \\ (c) g_n(x) \xrightarrow{n \rightarrow \infty} g(x), \text{ a.e. } x \in \Omega. \end{cases}$$

$$\text{Then, } \int_{\Omega} g_n(x) dx \xrightarrow{n \rightarrow \infty} \int_{\Omega} g(x) dx.$$

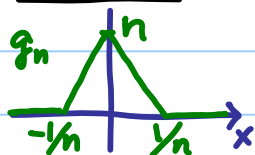
Ex. 1:  $d=1, \Omega = \mathbb{R}$ .  $g_n(x) = e^{ix/n} \phi(x)$ ,  $\phi \geq 0, \int_{\Omega} \phi < \infty$ .

Then,  $|g_n(x)| \leq |e^{ix/n} \phi(x)| = |\phi(x)| = \phi(x)$ .

also, we see  $g_n(x) \rightarrow \phi(x)$ , so by the D.C. Thm,

$$\lim \int_{\Omega} g_n(x) dx = \int_{\Omega} \lim g_n(x) dx = \int_{\Omega} \phi(x) dx < \infty. //$$

Ex. 2: Consider the fcn in the figure.

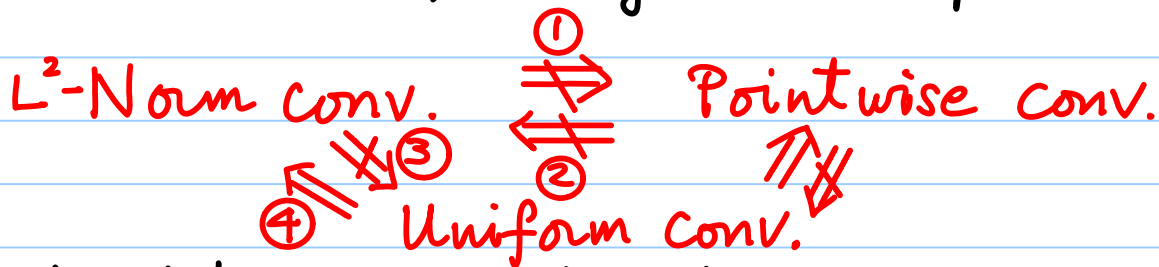


$g_n(x) \rightarrow 0$  a.e., but  $\int_{\Omega} g_n(x) dx = \frac{1}{2} \cdot n \cdot \frac{2}{n} = 1$ .

So,  $\lim \int g_n(x) dx = 1$ . On the other hand,

there is no  $\phi$  satisfying (a), (b). Moreover,  $\int \lim g_n(x) dx = 0$ . The D.C. Thm doesn't hold. The idea of  $\delta$  fcn &  $L^2$  the theory of distribution is necessary to deal with such an example!

Now, recall the following relationship :



With a bit more assumption, however, we can show Pointwise conv.  $\Rightarrow L^2$ -norm conv. as follows:

Thm Let  $\{f_n\} \subset L^2(\Omega)$ ,  $f_n \rightarrow f$  pointwise.

If  $\exists \psi \in L^2(\Omega)$  s.t.  $|f_n(x)| \leq |\psi(x)|$  a.e.  $x \in \Omega$ , then  $f_n \rightarrow f$  in norm.

(Proof)  $|f(x)| = \lim_{n \rightarrow \infty} |f_n(x)| \leq |\psi(x)|$  a.e.  $x \in \Omega$ .

$$\text{So, } |f_n(x) - f(x)|^2 \leq (|f_n(x)| + |f(x)|)^2 \leq |2\psi(x)|^2$$

Now apply the D.C. Thm with  $g_n(x) = |f_n(x) - f(x)|^2$ ,  $g(x) = 0$ , and  $\phi(x) = |2\psi(x)|^2$  to get:

$$\lim \int_{\Omega} g_n(x) dx = \int_{\Omega} \lim g_n(x) dx = \int_{\Omega} g(x) dx = 0$$

$$= \|f_n - f\|_{L^2(\Omega)}^2$$

$$\text{So, } \|f_n - f\|^2 \rightarrow 0. //$$

## ★ Best Approximation in $L^2$

• If  $\{\phi_n\}_1^\infty$  is an ONB of  $L^2(\Omega)$ , then  $f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n$   $\forall f \in L^2(\Omega)$ .

• If  $\{\phi_n\}$  is an ON set, but not complete in  $L^2(\Omega)$ , then  $\forall f \in L^2(\Omega)$ ,  $\exists$  residual error  $f - \sum \langle f, \phi_n \rangle \phi_n$  and  $\sum \langle f, \phi_n \rangle \phi_n = \exists \tilde{f} \in L^2(\Omega)$ .

could be a finite subset of an ONB.

The last lemma in Lecture 18.

It turns out that  $\tilde{f}$  is the best linear approx. of  $f$  in  $L^2(\Omega)$  in the following sense:

Thm  $\|f - \sum \langle f, \phi_n \rangle \phi_n\| \leq \|f - \sum \alpha_n \phi_n\|$   
for arbitrary choice of  $\{\alpha_n\}$  with  $\sum |\alpha_n|^2 < \infty$ .  
= holds iff  $\alpha_n = \langle f, \phi_n \rangle$ ,  $\forall n$ .

(Proof) This is really the least squares approx.!

$$f - \sum \alpha_n \phi_n = \underbrace{f - \sum \langle f, \phi_n \rangle \phi_n}_{\perp \text{ to } \phi_j \in \{\phi_n\}} + \underbrace{\sum (\langle f, \phi_n \rangle - \alpha_n) \phi_n}_{\text{a linear comb. of } \{\phi_n\}}$$

$$\odot \langle f, \phi_j \rangle - \sum \langle f, \phi_n \rangle \langle \phi_n, \phi_j \rangle = \delta_{nj}$$

So, the Pythagorean Thm applies:

$$\|f - \sum \alpha_n \phi_n\|^2 = \|f - \sum \langle f, \phi_n \rangle \phi_n\|^2 + \underbrace{\sum |\langle f, \phi_n \rangle - \alpha_n|^2}_{\geq 0}$$

and clearly = holds iff  $\alpha_n = \langle f, \phi_n \rangle$ . ///

Cor.  $\{\phi_n\}_1^\infty$ : an ONB for  $L^2(\Omega)$ . Then for any  $f \in L^2(\Omega)$ , the  $N$ th partial sum  $\sum \langle f, \phi_n \rangle \phi_n$  is the best linear approx. in  $L^2$ -norm to  $f$  among all linear combinations of  $\{\phi_1, \dots, \phi_N\}$ . the least squares approx.

Note that  $\{\phi_1, \dots, \phi_N\}$  are selected independently from  $f$ .

If we choose  $N$  basis fcn's **dependent on**  $f$ , i.e.,  
 $\{\phi_{\lambda_1}, \dots, \phi_{\lambda_N}\} \subset \{\phi_n\}^\infty$ , where  $\lambda_1, \dots, \lambda_N$  depend on  $f$ ,  
then  $\sum_{j=1}^N \langle f, \phi_{\lambda_j} \rangle \phi_{\lambda_j}$  is better than  $\sum_{j=1}^N \langle f, \phi_j \rangle \phi_j$  in general.

Example (Obvious) Say  $N=100$ ,  $f(x) = \phi_{101}(x)$ .

Then  $f - \sum_{j=1}^{100} \langle f, \phi_j \rangle \phi_j = \phi_{101} \neq 0$ .

But clearly  $\lambda_1 = 101$ , and  $f = \phi_{101}$  is just a one term!  
This way of approximation is called **nonlinear approx.**  
In practice,  $\{\lambda_1, \dots, \lambda_N\}$  are chosen so that  $|\langle f, \phi_{\lambda_j} \rangle|$   
are the largest  $N$  expansion coeff's of  $f$  w.r.t.  $\{\phi_n\}$ .