

Lecture 22: Green's Functions I

Note Title

2/21/2015

- Want to **diagonalize** the self-adj. op. \mathcal{L} arising in the Sturm-Liouville theory. \mathcal{L} \rightarrow **Differential op.**
- If \mathcal{L} were a **compact** self-adj. op., then it's pretty close to a matrix, i.e., one can apply tools similar to those in linear algebra (e.g., eigenvalue decomposition, etc.)
- Unfortunately, \mathcal{L} is far from being compact! In fact, \mathcal{L} is **unbounded**!

Ex. Let $\mathcal{L} = \frac{d^2}{dx^2}$ on $[0,1]$ with $D(\mathcal{L}) = \{f \in H^2(0,1) \mid f(0) = f(1) = 0\}$.
Then, $\|\mathcal{L}\| := \sup_{f \in D(\mathcal{L})} \frac{\|\mathcal{L}f\|_2}{\|f\|_2}$

Take $f_n = \sqrt{2} \sin n\pi x \in D(\mathcal{L})$. $\|f_n\|_2 = 1$.

$\Rightarrow \mathcal{L}f_n = -(n\pi)^2 \sqrt{2} \sin n\pi x$. $\|\mathcal{L}f_n\|_2 = (n\pi)^2$.

$\Rightarrow \|\mathcal{L}f_n\| / \|f_n\| = (n\pi)^2 \rightarrow \infty$!

Hence $\|\mathcal{L}\| = \infty$, i.e., unbdd. //

- Therefore, we investigate \mathcal{L}^{-1} instead. \mathcal{L}^{-1} is an **integral op.**, and turns out to be **compact** in our case!
- We'll review some basic properties of compact op's later. But first, let's proceed with deriving \mathcal{L}^{-1} !

Consider an RSL: $\mathcal{L}f = g$, i.e., $(pf')' + qf = g$, where $D(\mathcal{L}) = \{f \in H^2(a,b) \mid \alpha f(a) + \alpha' f'(a) = 0, \beta f(b) + \beta' f'(b) = 0\}$, $g \in L^2(a,b)$. Want to do: $f = \mathcal{L}^{-1}g$.

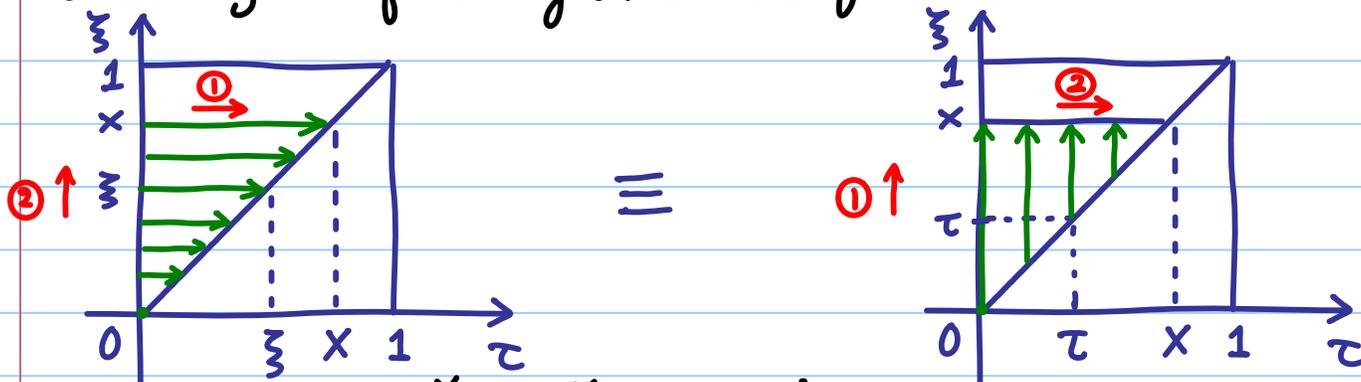
Let's look at the simplest case first, i.e., $\mathcal{L} = \frac{d^2}{dx^2}$ on $[0,1]$ with the Dirichlet B.C.: $f(0) = f(1) = 0$.

Integrate $\mathcal{L}f = f'' = g$ twice to get

$$f(x) = \int_0^x \left(\int_0^{\xi} g(\tau) d\tau \right) d\xi + Ax + B.$$

integration const's

Let's swap the order of integration by considering the region of integration as follows:



$$\begin{aligned} \text{So, } f(x) &= \int_0^x \left(\int_{\tau}^x g(\tau) d\xi \right) d\tau + Ax + B \\ &= \int_0^x (x - \tau) g(\tau) d\tau + Ax + B \end{aligned}$$

↪ a fun of τ !

Now, $f(0) = B = 0$

$$f(1) = \int_0^1 (1 - \tau) g(\tau) d\tau + A = 0.$$

$$\begin{aligned} \Rightarrow f(x) &= \int_0^x (x - \tau) g(\tau) d\tau - x \int_0^1 (1 - \tau) g(\tau) d\tau \\ &= \int_0^x (x - \tau) g(\tau) d\tau - \left\{ \int_0^x x(1 - \tau) g(\tau) d\tau + \int_x^1 x(1 - \tau) g(\tau) d\tau \right\} \\ &= \int_0^x \tau(x - 1) g(\tau) d\tau + \int_x^1 x(1 - \tau) g(\tau) d\tau \\ &=: \int_0^1 k(x, \tau) g(\tau) d\tau =: \mathcal{K} g(x) \end{aligned}$$

where $k(x, \tau) := \begin{cases} \tau(x - 1) & \text{if } 0 \leq \tau \leq x; \\ x(\tau - 1) & \text{if } x \leq \tau \leq 1. \end{cases}$

\mathcal{K} : an integral op.

$k(\cdot, \cdot)$: the kernel fun of \mathcal{K} .

⇒ Mathematica fig. show here.

In this case, $k(\cdot, \cdot)$ has a special name:

Green's fcn for \mathcal{L} (or the BVP associated w. \mathcal{L})

Now, for a general case of RSL: $\mathcal{L}f = g$, $f \in D(\mathcal{L})$, we use **the method of variation of parameters** due to Lagrange. First, let's define an **extended op.**:
 $M := \frac{d}{dx} \left(p \frac{d}{dx} \cdot \right) + q \cdot$, $D(M) = H^2(a, b) \supset D(\mathcal{L})$
 no explicit B.C.

Now suppose we can find two linearly indep. sol's u, v of the corresponding homogeneous eqn. $Mf = 0$. Then we look for a sol. of $\mathcal{L}f = g$ of the form:

$$f = \varphi u + \psi v, \quad \exists \varphi, \psi \in C^1[a, b].$$

$$\Rightarrow f' = \varphi u' + \psi v' + \varphi' u + \psi' v$$

$$\text{Let's choose } \varphi, \psi \text{ s.t. } \varphi' u + \psi' v = 0 \quad \text{--- ①}$$

$$\Rightarrow f' = \varphi u' + \psi v'$$

$$\Rightarrow \mathcal{L}f = (pf')' + qf$$

$$= p' \varphi u' + p' \psi v' + p \varphi' u' + p \varphi u'' + p \psi' v' + p \psi v''$$

$$= \varphi' p u' + \varphi (p' u' + p u'' + q u) \leftarrow = M u = 0 + q \varphi u + q \psi v$$

$$+ \psi' p v' + \psi (p' v' + p v'' + q v) \leftarrow = M v = 0$$

$$= p(\varphi' u' + \psi' v') \quad \text{--- ②}$$

$$\text{①} \times p u' - \text{②} \times u \quad p(\cancel{\varphi' u' u} + \psi' u v) - p(\cancel{\varphi' u' u} + \psi' u v') = -u q$$

$$\text{i.e., } \psi' p(uv' - vu') = u q \quad \text{--- ③}$$

By Lagrange's identity, we have

$$u \underbrace{M v}_{=0} - v \underbrace{M u}_{=0} = (p(uv' - vu'))' = 0.$$

So, $p(uv' - vu') = \text{const.}$, say, c .

\Rightarrow ③ becomes $c \psi' = u q$.

Similarly, we can also get $c \varphi' = -v q$.

So, **assuming $c \neq 0$** , we can set

$$\varphi(x) = \frac{1}{c} \left(\int_x^b v(\tau) g(\tau) d\tau + \underline{A} \right) \rightarrow \text{Integration Const's}$$

$$\psi(x) = \frac{1}{c} \left(\int_a^x u(\tau) g(\tau) d\tau + \underline{B} \right)$$

as long as $g \in C[a, b]$, $\varphi, \psi \in C'[a, b]$ and satisfy $c\varphi' = -vg$, $c\psi' = ug$.

And in turn, $\varphi'u + \psi'v = 0$ and $f = \varphi u + \psi v$ is a sol. of $Lf = g$. Using the B.C., we expect A, B to be determined.

Yet, \exists two potential problems:

Ⓐ Can we really assume $c \neq 0$?

Ⓑ Can we extend the result to $g \in L^2[a, b]$ from $g \in C[a, b]$?

We first deal with Ⓐ and briefly discuss Ⓑ later. (full discussion on Ⓑ requires the knowledge of measure theory.)

Let's proceed! $Mf = 0$ has nontrivial (i.e., nonzero)

$$\text{sol's } f = u, \text{ s.t. } \alpha u(a) + \alpha' u'(a) = 0$$

$$f = v, \text{ s.t. } \beta v(b) + \beta' v'(b) = 0$$

thanks to the Existence Thm for 2nd order ODEs.

Now set $f(x) = \varphi(x)u(x) + \psi(x)v(x)$, $f' = \varphi u' + \psi v'$

$$= \frac{1}{c} u(x) \left\{ \int_x^b v(\tau) g(\tau) d\tau + A \right\} + \frac{1}{c} v(x) \left\{ \int_a^x u(\tau) g(\tau) d\tau + B \right\}$$

By our derivation, we know $Lf = g$, but f must satisfy the B.C. at both ends.

$$f(a) = \frac{1}{c} u(a) \left\{ \int_a^b v(\tau) g(\tau) d\tau + A \right\} + \frac{1}{c} v(a) B$$

$$f'(a) = \varphi(a) u'(a) + \psi(a) v'(a)$$

$$= \frac{1}{c} u'(a) \left\{ \int_a^b v(\tau) g(\tau) d\tau + A \right\} + \frac{1}{c} v'(a) B$$

$$\text{So, } 0 = \alpha f(a) + \alpha' f'(a) = \frac{1}{c} \left\{ \int_a^b v(\tau) g(\tau) d\tau + A \right\} (\alpha u(a) + \alpha' u'(a)) + \frac{B}{c} (\alpha v(a) + \alpha' v'(a)) \stackrel{=0}{=} 0$$

$\Rightarrow B=0$. Similarly, we get $A=0$. $\neq 0$

So, we now have: $\varphi(x) = \frac{1}{c} \int_a^x v(\tau) g(\tau) d\tau$
If $f = \varphi u + \psi v$ with $\psi(x) = \frac{1}{c} \int_a^x u(\tau) g(\tau) d\tau$ $x \in [a, b]$

then f is a sol. of $Lf = g$ satisfying the B.C.'s!

Now, we'll prove the following thm, which takes care of (a):

Thm Suppose that 0 is **not** an eigenval. of the RSL.

Then $\forall g \in C[a, b]$, $Lf = g$ (incl. B.C.'s) has the unique sol. $f(x) = \int_a^b k(x, \tau) g(\tau) d\tau$

$$\text{where } k(x, \tau) := \begin{cases} \frac{1}{c} v(x) u(\tau) & \text{if } a \leq \tau \leq x \leq b \\ \frac{1}{c} u(x) v(\tau) & \text{if } a \leq x \leq \tau \leq b \end{cases}$$

u, v are non-zero real sol's of $M_1 u = 0, M_2 v = 0$, and $c = p(uv' - vu') = \text{const.}$ $= M + \text{B.C. at } a$ $M + \text{B.C. at } b$

Furthermore, f'' exists and $f'' \in C[a, b]$. //

First, we prove:

Lemma Under the assumption of the above thm,

$$u v' - v u' \neq 0 \quad \forall x \in [a, b] \quad =: W[u, v](x)$$

Wronskian!

(Proof) Suppose $u v' - v u' = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = 0$ at $\exists x_0 \in [a, b]$

Then $\exists (\gamma, \delta) \neq (0, 0)$ s.t.

$$\begin{cases} \gamma u(x_0) + \delta v(x_0) = 0 \\ \gamma u'(x_0) + \delta v'(x_0) = 0 \end{cases} \leftarrow \begin{bmatrix} u(x_0) & v(x_0) \\ u'(x_0) & v'(x_0) \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow f = \gamma u + \delta v$ is a sol. of $Mf = 0$ with $\begin{cases} f(x_0) = 0 \\ f'(x_0) = 0 \end{cases}$.

\Rightarrow By the Fundamental Existence Thm for 2nd order ODEs, $f(x) \equiv 0 \quad \forall x \in [a, b]$.

$\Rightarrow \gamma u + \delta v \equiv 0$, but $(\gamma, \delta) \neq (0, 0)$

$\Leftrightarrow u, v$: linearly dep. and $u = -\frac{\delta}{\gamma} v$.

$\Rightarrow \beta u(b) + \beta' u'(b) = 0$ since $\beta v(b) + \beta' v'(b) = 0$.

$\Rightarrow u$ satisfies $\mathcal{L}u = 0$, $u \neq 0$.

$\Leftrightarrow u$: an eigenfcn of \mathcal{L} corresp. to 0 eigenvalue, which contradicts with the assumption! #

(Proof of Thm) $p > 0$ & $uv' - vu' \neq 0$, $\forall x \in [a, b]$ via Lemma.

$\Rightarrow c = p(uv' - vu') \neq 0$.

Since 0 is not an eigval of \mathcal{L} , $\mathcal{L}f = g$ certainly has at most one sol. in $D(\mathcal{L})$. The foregoing calculations have already shown that $f = \varphi u + \psi v$ is such a sol. It remains to check that $f \in C^2[a, b]$.

u'' exists since u is a sol. to $M, f = 0$ (obvious!)

and $u'' = -\frac{p'u'}{p} - \frac{qu}{p} \in C[a, b] \Rightarrow u \in C^2[a, b]$.

Similarly, $v \in C^2[a, b]$.

Now, $\varphi', \psi' \in C[a, b]$ because $c\varphi' = uq$, $c\psi' = -vq$.

Since $f' = \varphi u' + \psi v' \in C^1[a, b]$, i.e., $f'' \in C[a, b]$. ///