

Lecture 24: Green's Functions III

Note Title

2/24/2015

★ Toward the Spectral Thm

Thm Let T be a self-adj. op. on $\mathcal{H}$.

Then all eigval's of T are **real**, and eigfns of T corresp. to distinct eigval's are **orthogonal**.
(We already did this for an SL system assuming that eig pairs exist. The proof is the same.)

Thm Let K be a compact self-adj. op. on $\mathcal{H}$.

Then, $\sigma(K) := \{\text{the set of eigval's of } K\} \subset \mathbb{R}$, which is either **finite** or a **countable seq. tending to 0**.

(Proof) Suppose K has infinitely many eigval's that are not tending to 0. Then $\exists \varepsilon > 0$ s.t. $\#\{n \in \mathbb{N} \mid |\lambda_n| > \varepsilon\} = +\infty$. Pick a (sub) seq. $\{\mu_n\}_{n=1}^{\infty} \subset \{\lambda_n\}_{n=1}^{\infty}$ and $|\mu_n| > \varepsilon \forall n \in \mathbb{N}$ s.t. $\mu_n \neq \mu_m$ if $n \neq m$.

Let $K\varphi_n = \mu_n\varphi_n$, $\|\varphi_n\| = 1$. Then such $\{\varphi_n\}_{n=1}^{\infty}$ form an ON seq. (set) and
 $\|K\varphi_n - K\varphi_m\|^2 = \|\mu_n\varphi_n - \mu_m\varphi_m\|^2 = \mu_n^2 + \mu_m^2 > 2\varepsilon^2$
 $\Rightarrow \{K\varphi_n\}_{n=1}^{\infty}$ has no Cauchy subseq., i.e., no convergent subseq. This contradicts K : compact. #

Thm (Spectral Thm) Let K be a compact self-adj. op. on $\mathcal{H}$. Then $\exists$ a finite or infinite ON seq. $\{\varphi_n\}$ of eigfns of K with corresp. real eigval's $\{\lambda_n\}$ s.t. $Kx = \sum \lambda_n \langle x, \varphi_n \rangle \varphi_n$.
If $\{\lambda_n\}$ is an infinite seq., then $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

Remark: At this point, $\{\varphi_n\}$ may or may not be an ON **B**.

In fact, if $\mathcal{H}$ is not **separable**, then no ON seq. is complete in $\mathcal{H}$. But most of $\mathcal{H}$ of our interest are **separable**. So, $\{\varphi_n\}$ can be extended to an ONB in a **separable** $\mathcal{H}$.

Def. A Hilbert space is **separable** if it admits a **countable** complete ON set, i.e., ONB.

- An example of **non-separable** Hilbert space:
Let $L := \{f \in C(\mathbb{R}) \mid p(f) := (\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx)^{\frac{1}{2}} < \infty\}$.
 $p(\cdot)$ is just a **seminorm**, i.e., $p(f) = 0 \not\Rightarrow f \equiv 0$ a.e.
e.g., f with a finite support $\Rightarrow p(f) = 0$.

Consider a quotient space $L/N(p)$ (i.e., equiv. modulo $N(p)$).
 $= \{f \in L \mid p(f) = 0\}$.

That is, for $x, y \in L$, we write $[x], [y] \in L/N(p)$, and $[x] = \{x + n \in L \mid n \in N(p)\}$, $[x] = [y]$ if $x - y \in N(p)$. Define the norm $\|[x]\|_{L/N(p)} := \inf_{y \in [x]} p(x - y)$. Then $L/N(p)$ becomes a normed space. Let $\mathcal{H}$ be the completion of $L/N(p)$ w.r.t. $\|\cdot\|_{L/N(p)}$ and the inner prod.

$$\langle f, g \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) \overline{g(x)} dx.$$

Now the fcn's $\{e^{i\lambda t}\}_{\lambda \in \mathbb{R}}$ are pairwise ON elements of $\mathcal{H}$.
 $\Rightarrow \exists$ uncountable ON seq. in $\mathcal{H}$. $\Rightarrow \mathcal{H}$: **not separable**. //

This space is called the space of almost periodic fcn's and denoted by B^2 to honor A.S. Besicovitch.

★ Compactness of the inverse of an RSL operator

Our foregoing discussion already proved:

Thm Suppose 0 is not an eigval of $\mathcal{L}$ (an RSL op.). Let $\mathcal{K}$ be the integral op. on $L^2(a, b)$ with $k(x, \tau)$ as the Green's fn. Then $\mathcal{K}$ is a compact self-adj. op. and for $\forall g \in C[a, b]$, $\mathcal{L}f = g \Rightarrow f = \mathcal{K}g$. ///

Remark: This thm **almost** states that $\mathcal{K} = \mathcal{L}^{-1}$.

However, two things prevent us from stating so:

- ① $g \in C[a, b]$, not yet shown for $g \in L^2(a, b)$.
- ② Range($\mathcal{L}$) $\subsetneq L^2(a, b)$ at this point due to the B.C. imposed.
 $= D(\mathcal{L}^{-1})$

Can we **extend** $D(\mathcal{L})$ so that $\text{Range}(\mathcal{L}) = L^2(a, b)$?

$\Rightarrow$ Yes! But, properly resolving ② requires measure theory. So, we'll avoid dealing with ②. Here, we can say a bit more about ①:

Lemma Suppose 0 is not an eigval of $\mathcal{L}$ (an RSL op.)

Then $\forall g \in L^2(a, b)$, $\mathcal{K}g$ is differentiable and

$$(*) (\mathcal{K}g)' = \varphi u' + \psi v' \quad \text{where} \quad \begin{cases} \varphi(x) = \frac{1}{c} \int_x^b v(\tau) g(\tau) d\tau, \\ \psi(x) = \frac{1}{c} \int_a^x u(\tau) g(\tau) d\tau. \end{cases}$$

(Proof) By the def. of $\mathcal{K}$, we know $\mathcal{K}g = \varphi u + \psi v$, $u, v \in C^1[a, b]$ by the Fund. Existence Thm for 2nd order ODEs. If $g \in C[a, b]$, then by the Fund. Thm. Calc. $\mathcal{K}g \in C^1[a, b]$, and $(\mathcal{K}g)' = \varphi' u + \varphi u' + \psi' v + \psi v' = \varphi u' + \psi v' \leftarrow \varphi u + \psi v = 0$.

If $g \in L^2(a, b)$ (i.e., could be discontinuous), then $\varphi, \psi \notin C^1[a, b]$ in general.

But (*) does not involve φ', ψ' . So $\exists$ a chance!

Certainly, $\varphi u' + \psi v' \in C[a, b]$.

So, (*) is equivalent to

$$(**) \quad K g(x) = K g(a) + \int_a^x (\varphi u' + \psi v') d\tau, \quad \forall x \in [a, b]$$

$$\begin{aligned} \varphi(a) = 0 &\rightarrow \varphi(a) u(a) + \int_a^x (\varphi u' + \psi v') d\tau \\ &= \frac{1}{c} u(a) \int_a^b v(\tau) g(\tau) d\tau + \int_a^x (\varphi u' + \psi v') d\tau \end{aligned}$$

Let's introduce 3 mappings: $\tilde{K}, M, N: L^2(a, b) \rightarrow C[a, b]$.

$$\left. \begin{aligned} \tilde{K} g &:= K g \\ M g(x) &:= \frac{u(a)}{c} \int_a^b v(\tau) g(\tau) d\tau \\ N g(x) &:= \int_a^x (\varphi u' + \psi v') d\tau \end{aligned} \right\} \quad \forall g \in L^2(a, b).$$

These are continuous (i.e., bdd.) maps w.r.t.

$\|\cdot\|_2$ in $L^2(a, b)$ and $\|\cdot\|_\infty$ in $C[a, b]$.

Let's prove this for N .

An antiderivative of an $L^2(a, b)$ fcn $\in C[a, b]$.

So, $\varphi, \psi \in C[a, b] \Rightarrow \varphi u' + \psi v' \in C[a, b]$.

Now, $\forall x \in [a, b]$.

$$|N g(x)| \leq \int_a^x |\varphi u' + \psi v'| d\tau$$

$$\leq (b-a) \{ \|\varphi\|_\infty \|u'\|_\infty + \|\psi\|_\infty \|v'\|_\infty \}$$

$$\begin{aligned} |\varphi(x)| &= \frac{1}{c} \left| \int_a^b v(\tau) g(\tau) d\tau \right| \quad \rightarrow \text{Cauchy-Schwarz} \\ &\leq \frac{1}{c} \int_a^b |v(\tau) g(\tau)| d\tau \leq \frac{1}{c} \|v\|_2 \|g\|_2 \end{aligned}$$

Take $\sup_{x \in [a, b]}$ on the LHS. $\Rightarrow \|\varphi\|_\infty \leq \frac{1}{c} \|v\|_2 \|g\|_2$

Similarly, $\|\psi\|_\infty \leq \frac{1}{c} \|u\|_2 \|g\|_2$

So, $|N g(x)| \leq \frac{b-a}{c} \{ \|v\|_2 \|u'\|_\infty + \|u\|_2 \|v'\|_\infty \} \|g\|_2.$

So, $N: L^2(a, b) \rightarrow C[a, b]$ is a bdd. linear op.
Hence $\tilde{K} - M - N$ is a continuous op. from $L^2(a, b)$ to $C[a, b]$, and $\tilde{K} - M - N \equiv 0$ on $\underline{C[a, b]} \subset L^2(a, b)$. So, $\tilde{K} - M - N \equiv 0$ on $L^2(a, b)$ dense in $L^2(a, b)$. $\Rightarrow (**)$, hence $(*)$, are valid for $\forall g \in L^2(a, b)$. $///$

Thm Suppose 0 is not an eigval of L .

Then (i) 0 is not an eigval of K either.

(ii) λ is an eigval of $L \iff \frac{1}{\lambda}$ is an eigval of K
" $\lambda \in \sigma(L)$ " " $\frac{1}{\lambda} \in \sigma(K)$ "

Furthermore, the eigfcn of L corresp. to λ coincides with
" K " " $\frac{1}{\lambda}$ ".

(Proof) (i) Suppose $Kg = 0$. Then $\varphi u + \psi v = 0$.
By the lemma on page 3, $\varphi u' + \psi v' = (Kg)' = 0$.
 $\Rightarrow \begin{bmatrix} u & v \\ u' & v' \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

By the other lemma ($uv' - v u' \neq 0 \forall x \in [a, b]$)
we must have $\varphi \equiv 0 \equiv \psi$.

$\iff \int_x^b v(\tau) g(\tau) d\tau \equiv 0 \equiv \int_a^x u(\tau) g(\tau) d\tau, \forall x \in [a, b]$.

$\iff \int_x^b v g \equiv 0 \equiv \int_a^x u g$ a.e., i.e., in $L^2(a, b)$ sense.

From the above lemma again, u, v cannot vanish simultaneously. $\Rightarrow g \equiv 0$ a.e., i.e., $0 \notin \sigma(K)$. $///$

(ii) Suppose that $\lambda \in \sigma(L)$ with corresp. eigfcn $f \in D(L)$, i.e., $Lf = \lambda f$, $f \neq 0$, $\lambda \neq 0$. Since $f \in D(L)$, $\lambda f \in C[a, b]$.
So, by the Green' fcn thm, $Lf = \lambda f \iff f = K(\lambda f) = \lambda Kf$

Since $\lambda \neq 0$, this is equivalent to $Kf = \frac{1}{\lambda} f$,
i.e., $\frac{1}{\lambda} \in \sigma(K)$ and f is the corresp. eigfcn.

Conversely, suppose that $\mu \in \sigma(K)$ with corresp. eigfcn $g \in L^2(a,b)$
i.e., $Kg = \mu g$, $g \neq 0$, $\mu \neq 0 \iff g = \frac{1}{\mu} Kg \xRightarrow{\text{Lemma}} g \in C[a,b]$

Hence, by the Green's fcn thm, $Lf = g$ has the sol.
 $f = Kg$. So, $LKg = g \iff L\mu g = g \iff Lg = \frac{1}{\mu} g$. ///