Lecture 26: Layer Potentials & Integral Eqn's I

Note Title 3/7/2015

- In the next 4 lectures, we will study the solution of the Dirichlet & Neumann problems for the Laplacian in $\mathbb{R}^d$ by the method of layer potentials $\Rightarrow$ reduces to solving the so-called boundary integral eqn's.
- Why is this approach important?
  + The associated integral op's: Compact!
  + The unknown fcn (often called "charge") to solve is only defined on the bdry. $\Rightarrow$ Dimension Reduction
  + Yet, once we obtain the "charge", we can evalute the solution of the Dirichlet / Neumann problems at any point $x \in \mathbb{R}^d$ in a simple manner.
  + Both interior & exterior problems can be treated consistently.
  + It leads to well-conditioned & superb numerical algorithms $\Rightarrow$ ask Prof. Jim Bremer!

☆ Review and some facts of a Fundamental Sol.

Lecture 10 discussed the fundamental sol. for $-\Delta$. We'll review it here and discuss more facts of it.

Def. Let $\mathcal{L}$ = a const. coefficient linear partial differential operator. Then the distribution Φ s.t. $\mathcal{L}\Phi = \underset{\text{the Dirac delta}}{\delta}$ is called a fundamental solution for $\mathcal{L}$ (a.k.a. a free space Green's fcn for $\mathcal{L}$).

So, if we have a problem $\mathcal{L}u = f$ in $\mathbb{R}^d$ then take $u = \Phi * f$. convolution

$$\Rightarrow \mathcal{L}u = \mathcal{L}(\Phi * f) = (\mathcal{L}\Phi) * f$$
$$= \delta * f = f \ \checkmark$$

In Lecture 10, we defined the fund. sol. for $-\Delta$ as follows:

$$\Phi(x,y) := \begin{cases} -\frac{1}{2}|x-y| & \text{if } d=1 \\ -\frac{1}{2\pi}\ln|x-y| & \text{if } d=2 \\ \frac{|x-y|^{2-d}}{(d-2)\omega_d} & \text{if } d\geq 3 \end{cases}$$

where $\omega_d := 2\pi^{d/2}/\Gamma(d/2)$, i.e., the surface area of the unit ball in $\mathbb{R}^d$.

Note that $d=3$ leads to the famous $\frac{1}{4\pi|x-y|}$.

In this lecture and the rest, following the same sign convention with the SL systems, we will use the fund. sol. of Δ instead of $-\Delta$, and write its fund. sol. as $N(x,y) := -\Phi(x,y)$.

bdry shape is C^1-smooth

<u>Thm</u> Let Ω be a bdd. domain with C^1 bdry $\Gamma = \partial\Omega$. <u>If</u> $u \in C^1(\bar{\Omega})$ is harmonic in Ω, <u>then</u>

$$(*)\ u(x) = \int_\Gamma [u(y)\partial_{\nu_y} N(x,y) - \partial_{\nu_y} u(y) N(x,y)]\, d\sigma(y), \ \forall x \in \Omega,$$

where ν_y is the outward unit normal vector at $y \in \Gamma$, and $\partial_{\nu_y}\cdot$ is the normal derivative of $\cdot$ at $y \in \Gamma$.

(Proof) We only consider $d \geq 2$ ($d=1$ is trivial).

Let $N^\varepsilon(x,y)$ be the regularized version of $N(x,y)$:

$$N^\varepsilon(x,y) := \begin{cases} \frac{1}{\pi}\ln(|x-y|^2+\varepsilon^2) & \text{if } d=2, \\ \frac{(|x-y|^2+\varepsilon^2)^{(2-d)/2}}{(2-d)\omega_d} & \text{if } d\geq 3. \end{cases}$$

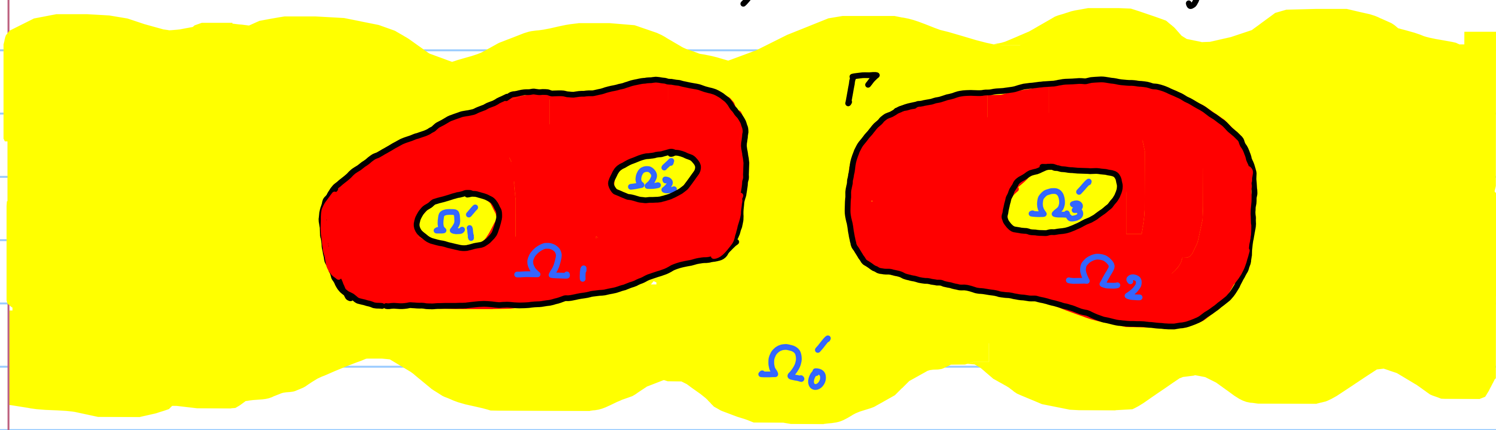
Since $\Delta u = 0$ in Ω, by Green's 2nd identity,

$$\int_\Omega u(y)\Delta_y N^\varepsilon(x,y)\,dy = \int_\Gamma [u(y)\partial_{\nu_y} N^\varepsilon(x,y) - \partial_{\nu_y} u(y) N^\varepsilon(x,y)]\,d\sigma(y)$$

As $\varepsilon \to 0$, the RHS of this eq. $\to$ the RHS of (*), $\forall x \in \Omega$.
Singularities of N do not matter here since $x \in \Omega$, $y \in \partial\Omega = \Gamma$ i.e., $x \neq y$. Now, the LHS of the above eq. is simply $u * (\Delta N^\varepsilon)(x)$ if we set $u(x) = 0$, $\forall x \in \mathbb{R}^d \setminus \overline{\Omega}$
Viewing ΔN^ε as an approximate identity, we can show that $u * (\Delta N^\varepsilon)(x) \longrightarrow u(x)$ as $\varepsilon \to 0$, $\forall x \in \Omega$. ///

☆ The Problem Setup

$\Omega \subset \mathbb{R}^d$, bdd., $\Gamma = \partial\Omega$ has C^2 smoothness.
$\Omega' := \mathbb{R}^d \setminus \overline{\Omega}$. Both Ω & Ω' are allowed to be disconnected, consist of finite number of connected components, i.e., $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m$, $|\Omega_j| < \infty$
$\Omega' = \Omega_0' \cup \Omega_1' \cup \cdots \cup \Omega_m'$, Ω_0': unbdd. $|\Omega_j'| < \infty$, $j = 1:m$.



$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega' = \Omega_0' \cup \Omega_1' \cup \Omega_2' \cup \Omega_3'$$

- To deal with the Neumann problem, we need the following normal derivatives & spaces of fcns:

$C_\nu(\Omega) := \{ u \in C^1(\Omega) \cap C(\bar{\Omega}) \mid \partial_{\nu_-} u(x) \text{ exists } \forall x \in \Gamma \}$,
where $\partial_{\nu_-} u(x) := \lim_{t \uparrow 0} \nu(x) \cdot \nabla u(\underbrace{x + t\nu(x)}_{\in \Omega})$
interior norm. deriv.

$C_\nu(\Omega') := \{ u \in C^1(\Omega') \cap C(\bar{\Omega}') \mid \partial_{\nu_+} u(x) \text{ exists } \forall x \in \Gamma \}$,
where $\partial_{\nu_+} u(x) := \lim_{t \downarrow 0} \nu(x) \cdot \nabla u(\underbrace{x + t\nu(x)}_{\in \Omega'})$
exterior norm. deriv.

In both cases, the convergence is uniform on Γ.

We can now state precisely the problems we want to solve:

- **The Interior Dirichlet Problem (IDP):** Given $f \in C(\Gamma)$, find $u \in C(\bar{\Omega})$ s.t. $\Delta u = 0$ in Ω and $u = f$ on Γ.
- **The Exterior Dirichlet Problem (EDP):** Given $f \in C(\Gamma)$, find $u \in C(\bar{\Omega}')$ s.t. $\Delta u = 0$ in $\Omega' \cup \{\infty\}$ and $u = f$ on Γ.
- **The Interior Neumann Problem (INP):** Given $f \in C(\Gamma)$, find $u \in C_\nu(\Omega)$ s.t. $\Delta u = 0$ in Ω and $\partial_{\nu_-} u = f$ on Γ.
- **The Exterior Neumann Problem (ENP):** Given $f \in C(\Gamma)$, find $u \in C_\nu(\Omega')$ s.t. $\Delta u = 0$ in $\Omega' \cup \{\infty\}$ and $\partial_{\nu_+} u = f$ on Γ.

Remarks: 1) For the exterior problems, we require u to be harmonic at ∞, i.e., $|u(x)| = O(|x|^{2-d})$ as $|x| \to \infty$, for the uniqueness of the solution.

2) Since ν is always taken to be outward normal vector of Ω, ∂_{ν_+} in ENP is taken along the inward-pointing normal to Ω'; if we want the outward normal to Ω', replace f by $-f$.

Let's discuss the uniqueness thm's first.

Prop. If u solves IDP with $f = 0$, then $u \equiv 0$.

(Proof) Use the uniqueness thm proved in Lecture 10. ///

Prop. If u solves EDP with $f=0$, then $u\equiv 0$.
(Proof) See Folland. (It can transform to IDP, then use the previous prop.) ///

Prop. If u solves INP with $f=0$, then $u=$ const. on each component Ω_j of Ω.
(Proof) By Green's 1st identity,

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \underbrace{\Delta u}_{=0 \text{ in } \Omega} dx + \int_\Gamma u \underbrace{\partial_{\nu_-} u}_{=0 \text{ on } \Gamma} d\sigma = 0.$$

$\Rightarrow \nabla u = 0$ on Ω, so u is locally const. on Ω. ///

Prop. If u solves ENP with $f=0$, then $u=$ const. on each component Ω_j' of Ω', and $u=0$ on $\underbrace{\Omega_0'}_{\text{the unbdd. component}}$ when $d>2$.
(Proof) Let $r>0$ be large enough so that $\overline{\Omega}\subset B_r=$ the ball of radius r in $\mathbb{R}^d$. By Green's 1st id,

$$\int_{B_r\setminus\Omega} |\nabla u|^2 dx = -\int_{B_r\setminus\Omega} u \underbrace{\Delta u}_{=0 \text{ in } B_r\setminus\Omega} dx - \int_\Gamma u \underbrace{\partial_{\nu_+} u}_{=0 \text{ on } \Gamma} d\sigma + \int_{\partial B_r} u \partial_r u \, d\sigma$$

$$= \int_{\partial B_r} u \partial_r u \, ds(x) \quad \text{where } \partial_r : \text{the radial deriv.}$$

Since u is harmonic at ∞, we have $|u(x)| = O(|x|^{2-d})$ and $|\partial_r u(x)| = O(|x|^{1-d})$ as $|x| = r\to\infty$. Hence, we have

$$\left|\int_{\partial B_r} u \partial_r u \, d\sigma\right| \leq C r^{2-d} r^{1-d} \underbrace{\int_{\partial B_r} 1}_{=\omega_d r^{d-1}} \leq C\omega_d r^{2-d} \xrightarrow[r\to\infty]{} 0 \text{ if } d>2.$$

So, $\nabla u = 0$ on Ω', and $u=0$ on Ω_0' since $|u(x)| = O(|x|^{2-d})$.
If $d=2$, we can show $|\partial_r u(x)| = O(r^{-2})$, so the same argument shows that $\left|\int_{\partial B_r} u\partial_r u \, d\sigma\right| = O(r^{-1})$. $\Rightarrow$ Similarly, $\nabla u = 0$, i.e., u is locally const. on Ω'. ///

Remark: Green's identities hold for $C^2(\overline{\Omega})$ fcns. Yet, in the above Prop's, u is in C^1. This is not a problem since we can replace Ω by Ω_t whose bdy is $\Gamma_t := \{ x + t\nu(x) \mid x \in \Gamma \}$, and pass to the limit $t \uparrow 0$ or $t \downarrow 0$. The def.'s of ∂_{ν_-} & ∂_{ν_+} are designed precisely to make this argument work!

We'll see that both IDP & EDP are always solvable. For INP & ENP, we need some necessary cond's.

<u>Prop.</u> <u>If</u> INP has a sol., <u>then</u> $\int_{\partial\Omega_j} f \, d\sigma = 0$, $j = 1:m$.

(Proof) This follows immediately from <u>Cor.</u> in Lecture 10:
<u>Cor.</u> u: harmonic on $\Omega \Rightarrow \int_\Gamma \partial_\nu u \, d\sigma = 0$. ///

<u>Prop.</u> <u>If</u> ENP has a sol., <u>then</u> $\int_{\partial\Omega_j'} f \, d\sigma = 0$, $j = \begin{cases} 1:m', & d > 2 \\ 0:m', & d = 2 \end{cases}$

(Proof) The same Cor. leads to $\int_{\partial\Omega_j'} f \, d\sigma = 0$ for $j = 1:m'$.
If $d = 2$, let r be large enough so that $\bar{\Omega} \subset B_r$.
Then the same Cor. gives $\int_{\partial B_r} \partial_r u - \int_{\partial\Omega_0'} \partial_{\nu_+} u = 0$.
But $|\partial_r u(x)| = O(|x|^{-2})$.
So, as $r \to \infty$, $\int_{\partial\Omega_0'} \partial_{\nu_+} u = 0 = \int_{\partial\Omega_0'} f \, d\sigma$. ///