# Lecture 28: Layer Potentials & Integral Eqn's III

Note Title 3/12/2015

## ☆ Single Layer Potentials

Recall the single layer potential

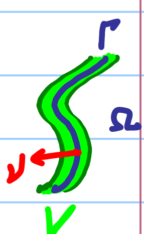
(SLP) $u(x) = \int_{\Gamma} N(x,y)\,\phi(y)\,d\sigma(y)$

with moment $\phi \in C(\Gamma)$. This will be used to solve INP & ENP! Because of the properties of $N(\cdot,\cdot)$, we can easily see: 1) $u$ is harmonic on $\mathbb{R}^d \setminus \Gamma$; 2) $u$ is harmonic at $\infty$ if $d>2$ since $|u(x)| = O(|x|^{2-d})$; If $d=2$, this still holds iff $\int_\Gamma \phi(y)\,d\sigma(y) = 0$, as we'll see; 3) $N|_{\Gamma\times\Gamma}$ is a continuous kernel of order $d-2$.

$\Rightarrow$ $u$ is well-defined on $\Gamma$! In fact, we have

**Prop.** If $\phi \in C(\Gamma)$ and $u$ is defined in (SLP), then $u \in C(\mathbb{R}^d)$, i.e., (SLP) does not have a jump discontinuity, which is a contrast to (DLP).

(Proof) See the end page of this lecture. ///




Now let's consider $\partial_\nu u$ in (SLP) in the tubular neighborhood $V = \{x + t\nu(x) \in \mathbb{R}^d \mid \forall x \in \Gamma, \forall t \in (-\varepsilon, \varepsilon)\}$, $\exists \varepsilon > 0$, of $\Gamma$. At a point in $V$, $\partial_\nu u(x + t\nu(x))$ is defined as

$$\partial_\nu u(x + t\nu(x)) = \nu(x) \cdot \nabla u(x + t\nu(x)).$$

Then, for $x \in V \setminus \Gamma$, we have

$$(*) \quad \partial_\nu u(x) = \int_\Gamma \underbrace{\partial_{\nu_x}}_{\neq \partial_{\nu_y} \text{ used in (DLP)!}} N(x,y)\,\phi(y)\,d\sigma(y)$$

Now, note $N(x,y) = N(y,x)$! So, $\partial_{\nu_x} N(x,y) = \partial_{\nu_y} N(y,x)$. Recall our notation for convenience: $K(x,y) := \partial_{\nu_y} N(x,y)$ for $x, y \in \Gamma$. Hence, it's natural to set $K^*(x,y) = K(y,x)$.

$\Rightarrow$ RHS of $(*)$ makes sense for $x \in \Gamma$ if we interpret it as

$(\mathrm{SLP}\partial_\nu\Gamma)$ $\quad \int_\Gamma K^*(x,y)\,\phi(y)\,d\sigma(y) =: T_{K^*}\phi$.

Since $K$ is a continuous kernel of order $d-2$, so is $K^*$.

$\Rightarrow (\mathrm{SLP}\partial_\nu\Gamma)$ defines a continuous fcn on $\Gamma$. Moreover, since $K$ is real-valued, $T_{K^*}$ is truly an adjoint of $T_K$ as an operator in $L^2(\Gamma)$, i.e., $\langle T_{K^*}f, g\rangle = \langle f, T_K g\rangle$ $\forall f, g \in L^2(\Gamma)$.

$\Rightarrow T_{K^*}\phi = T_K^*\phi$ !!

So far, so good! But as you might expect, similar to the relationship between (DLP) & (DLP$\Gamma$), $\exists$ a jump discontinuity between (*) defined on $V\setminus\Gamma$ & $(\mathrm{SLP}\partial_\nu\Gamma)$ defined on $\Gamma$:

<u>Thm</u> Suppose $\phi \in C(\Gamma)$ and $u$ is defined on $\mathbb{R}^d$ by (SLP). Then, $u|_{\bar\Omega} \in C_\nu(\Omega)$ & $u|_{\bar\Omega'} \in C_\nu(\Omega')$, and on $\Gamma$, we have

$$\partial_{\nu_\pm} u = \pm\frac{1}{2}\phi + T_K^*\phi,$$

where $\partial_{\nu_+}$: exterior norm. deriv. & $\partial_{\nu_-}$: interior norm. deriv.

Compare this with (DLP$\Gamma$): $u_\pm = \mp\frac{1}{2}\phi + T_K\phi$ !

<u>Cor.</u> $\quad \phi = \partial_{\nu_+}u - \partial_{\nu_-}u$.

(Proof of Thm) First of all, $u$ in (SLP) $\in C(\mathbb{R}^d)$.

Consider the DLP: $v(x) = \int_\Gamma \partial_{\nu_y} N(x,y)\phi(y)\,d\sigma(y)$, $x \in \mathbb{R}^d\setminus\Gamma$.

Now define the following fcn $f$ in the tubular neighborhood $V$ of $\Gamma$ as

$$f(x) := \begin{cases} v(x) + \partial_\nu u(x) & \text{if } x \in V\setminus\Gamma; \\ T_K\phi(x) + T_K^*\phi(x) & \text{if } x \in \Gamma. \end{cases}$$

<u>Claim</u>: $f \in C(V)$.

(Proof) See the end pages of this lecture.

Let's proceed. So, $f = v + \partial_\nu u$ extends continuously across $\Gamma$ $\Rightarrow$ Can use the DLP Thm in Lecture 27:

$$T_K\phi(x) + T_K^*\phi(x) = \underbrace{v_-(x)}_{= \frac{1}{2}\phi(x) + T_K\phi(x)} + \partial_{\nu_-} u(x).$$

$$\Rightarrow \quad \partial_{\nu_-} u(x) = -\frac{1}{2}\phi(x) + T_K^*\phi(x), \quad x \in \Gamma. \ \checkmark$$

Similarly, $T_K\phi(x) + T_K^*\phi(x) = \underbrace{v_+(x)}_{= -\frac{1}{2}\phi(x) + T_K\phi(x)} + \partial_{\nu_+} u(x)$.

$$\Rightarrow \quad \partial_{\nu_+} u(x) = +\frac{1}{2}\phi(x) + T_K^*\phi(x), \quad x \in \Gamma. \ ///$$

To proceed toward the sol. of I/ENPs, we need the following 3 lemmas.

**Lemma 1:** If $\phi \in C(\Gamma)$ & $\frac{1}{2}\phi + T_K^*\phi = f$, then $\int_\Gamma \phi = \int_\Gamma f$.

(Proof) By the Prop. of DLP with $\phi \equiv 1$, together with Fubini's Thm, we have

$$\int_\Gamma f(x)\,d\sigma(x) = \frac{1}{2}\int_\Gamma \phi(x)\,d\sigma(x) + \int_\Gamma\int_\Gamma K(y,x)\phi(y)\,d\sigma(y)\,d\sigma(x)$$
$$= \frac{1}{2}\int_\Gamma \phi(x)\,d\sigma(x) + \int_\Gamma \frac{1}{2}\phi(y)\,d\sigma(y) = \int_\Gamma \phi(y)\,d\sigma(y). \ ///$$

**Lemma 2:** Suppose $d=2$. If $\phi \in C(\Gamma)$, then $u$ in (SLP) is harmonic at $\infty$ $\Longleftrightarrow$ $\int_\Gamma \phi = 0$.
Hence $u \to 0$ as $x \to \infty$ in this case.

(Proof) We have

$$u(x) = \frac{1}{2\pi}\int_\Gamma (\ln|x-y| - \ln|x|)\phi(y)\,d\sigma(y) + \frac{\ln|x|}{2\pi}\int_\Gamma \phi(y)\,d\sigma(y).$$

Since $\ln|x-y| - \ln|x| \to 0$ uniformly as $x \to \infty$, $\forall y \in \Gamma$.
Hence, the result follows. ///

**Lemma 3**: Suppose $d=2$. If $\phi \in C(\Gamma)$, $\int_\Gamma \phi = 0$, and $u$ in (SLP) $\equiv$ const. on $\bar{\Omega}$, then $\phi \equiv 0$ (hence $u \equiv 0$).

(Proof) By Lemma 2, $u$ is harmonic at $\infty$, so if $u = c$ on $\bar{\Omega}$, $u$ solves EDP with $f = c$ on $\Gamma$. But the sol. to this problem is, via the uniqueness prop. of EDP, $u \equiv c$. Then via Cor.: $\phi = \partial_{\nu_+} u - \partial_{\nu_-} u = 0 - 0$ ///

**Summary of Lectures 26 — 28:**

Given a bdry data $f \in C(\Gamma)$, solving I/E D/N Ps reduces to finding the moment $\phi \in C(\Gamma)$ satisfying the bdry. integral eqn's:

IDP: $\frac{1}{2}\phi + T_K \phi = f$, EDP: $-\frac{1}{2}\phi + T_K \phi = f$;

INP: $-\frac{1}{2}\phi + T_K^* \phi = f$, ENP: $\frac{1}{2}\phi + T_K^* \phi = f$.

Once, $\phi$ is found, then the sol. $u$ is obtained via (DLP) for I/EDP, and (SLP) for I/ENP!!

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(Proof of Prop: $u$ via (SLP) with $\phi \in C(\Gamma) \Rightarrow u \in C(\mathbb{R}^d)$)

We only need to prove the continuity of $u$ on $\Gamma$ since we already know $u \in C(\mathbb{R}^d \setminus \Gamma)$.

Given $x_0 \in \Gamma$, $\delta > 0$, let $B_\delta := \{ y \in \Gamma \mid |x_0 - y| < \delta \}$.

Then,

$$|u(x) - u(x_0)| \le \int_{B_\delta} |N(x,y)\phi(y)|\, d\sigma(y) + \int_{B_\delta} |N(x_0,y)\phi(y)|\, d\sigma(y)$$
$$+ \int_{\Gamma \setminus B_\delta} |N(x,y) - N(x_0,y)|\, |\phi(y)|\, d\sigma(y).$$

Since $\phi$: bdd. and $N(x,y) = \begin{cases} O(|x-y|^{2-d}) & \text{if } d>2 \\ O(\ln|x-y|^{-1}) & \text{if } d=2 \end{cases}$ for $x \approx y$,

the first two terms of the RHS above are both $O(\delta)$ if $d>2$, $O(\delta \ln \delta^{-1})$ if $d=2$ since $\int_{B_\delta} d\sigma = \omega_d \delta^{d-1}$.
Hence, given $\varepsilon > 0$, we can choose $\delta$ small so that these two terms $\leq \varepsilon/3$. How about the third term for $\Gamma \setminus B_\delta$? If we require $|x - x_0| < \frac{1}{2}\delta$, then $x \neq y$, so the integrand is bdd. and $\to 0$ uniformly as $x \to x_0$. So again we can choose $\delta$ small to make the third term $\leq \varepsilon/3$. ///

(Proof of Claim: $f := \begin{cases} v + \partial_\nu u & \text{on } V\setminus\Gamma \\ T_K\phi + T_K^*\phi & \text{on } \Gamma \end{cases} \in C(V)$.)

Since $f|_{V\setminus\Gamma} \in C(V\setminus\Gamma)$, $f|_\Gamma \in C(\Gamma)$, it suffices to show that $x_0 \in \Gamma$, $x = x_0 + t\nu(x_0) \in V\setminus\Gamma$, $\lim\limits_{t\to 0} f(x) = f(x_0)$ unif.

$f(x) - f(x_0) = \int_\Gamma [\partial_{\nu_y} N(x,y) + \partial_{\nu_x} N(x,y) - \partial_{\nu_y} N(x_0,y) - \partial_{\nu_x} N(x_0,y)]\, \phi(y)\, d\sigma(y)$

As before, we split $\int_\Gamma = \int_{B_\delta} + \int_{\Gamma\setminus B_\delta}$. By the same argument, $\int_{\Gamma\setminus B_\delta} [\cdot] \to 0$ as $x \to x_0$ while

$$\left|\int_{B_\delta} [\cdot]\right| \leq \|\phi\|_\infty \left\{\int_{B_\delta} |(\partial_{\nu_x} + \partial_{\nu_y}) N(x,y)|\, d\sigma(y) + \int_{B_\delta} |(\partial_{\nu_x} + \partial_{\nu_y}) N(x_0,y)|\, d\sigma(y)\right\}$$
$$\leq 2\delta \|\phi\|_\infty \to 0 \text{ as } \delta \to 0$$ if each integral is $\leq \delta$. But that is the case because

Def. of norm. deriv. in V.

$$\partial_{\nu_x} N(x,y) = \frac{(x-y)\cdot\nu(x_0)}{\omega_d |x-y|^d}, \quad \partial_{\nu_y} N(x,y) = \frac{-(x-y)\cdot\nu(y)}{\omega_d |x-y|^d}$$

So, $(\partial_{\nu_x} + \partial_{\nu_y}) N(x,y) = (x-y)\cdot[\nu(x_0) - \nu(y)]/(\omega_d |x-y|^d)$.
But $|\nu(x_0) - \nu(y)| = O(|x_0 - y|)$ since $\nu$ is a $C^1$ fcn thanks to the assumption of $\Gamma$: $C^2$ smooth. Moreover, $|x-y| \geq C|x_0 - y|$ since $x = x_0 + t\nu(x_0)$. $\Rightarrow |(\partial_{\nu_x} + \partial_{\nu_y}) N(x,y)| \leq C' |x_0 - y|^{2-d}$.
Integrating both sides over $B_\delta$, we see it $\leq C' \int_0^\delta r^{2-d} \cdot \omega_{d-1} r^{d-2}\, dr = C''\delta$ (note that $B_\delta \subset \Gamma \subset \mathbb{R}^{d-1}$). ///