Lecture 29: Layer Potentials & Integral Eqn's IV

Note Title 3/15/2015

☆ Solutions of the Problems

Given a bdry data $f \in C(\Gamma)$, solving I/E D/N Ps reduces to finding the moment $\phi \in C(\Gamma)$ satisfying the bdry. integral eqn's:

IDP: $\frac{1}{2}\phi + T_K\phi = f$, EDP: $-\frac{1}{2}\phi + T_K\phi = f$;

INP: $-\frac{1}{2}\phi + T_K^*\phi = f$, ENP: $\frac{1}{2}\phi + T_K^*\phi = f$.

Once, ϕ is found, then the sol. u is obtained via (DLP) for I/EDP, and (SLP) for I/ENP!!

Can we solve these bdry integral eqn's?

$\Rightarrow$ Always yes! for I/EDPs.

yes! for INP iff $\int_{\partial\Omega_j} f = 0$, $j = 1:m$.

yes! for ENP iff $\int_{\partial\Omega_j'} f = 0$, $j = \begin{cases} 1:m' & (d>2) \\ 0:m' & (d=2) \end{cases}$.

Hence, we have reached the following culminating

Thm. With the notation & terminology of Lect. 26, we have:

- IDP has a unique solution for every $f \in C(\Gamma)$.
- EDP has a unique solution for every $f \in C(\Gamma)$.
- INP for $f \in C(\Gamma)$ has a sol. iff $\int_{\partial\Omega_j} f = 0$, $j = 1:m$. The sol. is unique modulo fcns that are const. on each Ω_j.
- ENP for $f \in C(\Gamma)$ has a sol. iff $\int_{\partial\Omega_j'} f = 0$, $j = 1:m'$, and also for $j = 0$ in case $d = 2$. The sol. is unique modulo fcns that are const. on Ω_j', $j = 1:m'$, and also on Ω_0' in case $d = 2$.

(Proof) See, e.g., Folland; Kress; Colton, ...

In the proof, the eigenspaces of T_K & T_K^* for the eigenvalues $\pm\frac{1}{2}$ play important roles as you can easily imagine. $\Rightarrow$ Theory of Fredholm Integral Eqn's of 2nd Kind, the Fredholm alternative, ··· ///

Rather than the proof of the above thm, let's work out an example!

Example: The IDP for the unit disk in $\mathbb{R}^2$.

Note that one does not really need to follow our bdry. integral eqn. approach because $\exists$ other approaches. The power of the bdry. integral approach will be seen more clearly for more complicated domain. This 2D disk is just a simple illustration of our approach.

First of all, let's recall

$$\left.\begin{aligned} N(x,y) &= \frac{1}{2\pi}\ln|x-y| \\ \partial_{\nu_y}N(x,y) &= -\frac{(x-y)\cdot\nu(y)}{2\pi|x-y|^2} \end{aligned}\right\} x, y \in \mathbb{R}^2$$

Since $\Omega = \{(r\cos\theta, r\sin\theta)^T \in \mathbb{R}^2 \mid r \in [0,1), \theta \in [-\pi,\pi]\}$, $\Gamma = \partial\Omega = \{(\cos\theta, \sin\theta)^T \in \mathbb{R}^2 \mid \theta \in [-\pi,\pi]\}$, we want to write $\partial_{\nu_y}N(x,y)$ in the polar coordinates.

Let $x = (r\cos\theta, r\sin\theta)^T \in \Omega$, $y = (\cos\eta, \sin\eta)^T \in \Gamma$.

Then, $\nu(y) = (\cos\eta, \sin\eta)^T = y$!!

So, $\partial_{\nu_y}N(x,y) = -\dfrac{x\cdot y - 1}{2\pi|x-y|^2} = \dfrac{1}{2\pi}\,\dfrac{1-r\cos(\theta-\eta)}{1-2r\cos(\theta-\eta)+r^2}$ (*)

Now, let's consider what happens as $x \in \Omega$ approaches Γ from inside Ω, i.e., as $r \uparrow 1$. Clearly (*) $\xrightarrow[r\uparrow 1]{} \frac{1}{4\pi}$.

Hence, $\frac{1}{2}\phi + T_K\phi = f$, $f, \phi \in C(\Gamma)$ reduces to

$$\frac{1}{2}\phi(\theta) + \underbrace{\int_{-\pi}^{\pi}\frac{1}{4\pi}\phi(\eta)\,d\eta}_{=:\,c,\ \text{a const.}} = f(\theta), \quad \theta \in [-\pi,\pi].$$

$\Rightarrow$ $\phi = 2(f-c)$. Inserting this into the def. of c to get,

$$c = \frac{1}{4\pi}\int_{-\pi}^{\pi} 2(f(\eta)-c)\,d\eta \Rightarrow c = \frac{1}{4\pi}\int_{-\pi}^{\pi} f(\eta)\,d\eta$$

So, $\phi(\theta) = 2f(\theta) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)\,d\eta$!

Now, insert this ϕ into (DLP) to get

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r\cos(\theta-\eta)}{1-2r\cos(\theta-\eta)+r^2}\,\phi(\eta)\,d\eta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-r\cos(\theta-\eta)}{1-2r\cos(\theta-\eta)+r^2} f(\eta)\,d\eta - \frac{1}{4\pi^2}\int_{-\pi}^{\pi} f(\eta)\,d\eta \cdot \int_{-\pi}^{\pi} \underbrace{\frac{1-r\cos(\theta-\eta)}{1-2r\cos(\theta-\eta)+r^2}}_{=1+\sum_{1}^{\infty} r^n\cos n(\theta-\eta)}\,d\eta$$

Exercise !

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-r\cos(\theta-\eta)}{1-2r\cos(\theta-\eta)+r^2} f(\eta)\,d\eta - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)\,d\eta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\theta-\eta)+r^2} f(\eta)\,d\eta$$

$$= \underbrace{P_r * f}_{\text{convolution}}(\theta) \qquad r\in[0,1],\ \theta\in[-\pi,\pi],$$

where $P_r(\theta) := \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}$ is called the **Poisson kernel** for the 2D unit disk. ///

☆ Summary

- $d=1$:

Wave/Heat/Pot. Eqn. with const. coeff's $\xrightarrow{\text{sep. var's}}$ $\left\{ \begin{array}{l} f''+\lambda f = 0 \text{ with} \\ \text{Dirichlet/Neumann B.C.} \end{array}\right.$ $\longrightarrow$ Fourier Series Expansion

Wave/Heat/Pot. Eqn. with variable coeff's $\xrightarrow{\text{sep. var's}}$ $\left\{ \begin{array}{l} \text{Sturm-Liouville system} \\ \text{with self-adjoint B.C.} \end{array}\right.$ $\longrightarrow$ Green's Fcn's + Eigenfcn Expansion

- $d \geq 2$:

I/E D/N Ps for Δ with C^2 bdry $\longrightarrow$ DLP, SLP $\longrightarrow$ Bdry. Integral Eqn's

They are all **linear 2nd order differential eqn's** !

☆ Where can you go from here?

• $d = 1$:

+ Forward Problems (Existence, uniqueness, computation of sol's)
  - SSL systems $\Rightarrow$ Orthogonal Polynomials, Special Fcn's, …
  - Higher order systems
  - Nonlinear systems

) $\Rightarrow$ Dynamical systems, bifurcation, chaos, …

+ Inverse Problems (Recovery of parameters/coeff's from data)
  - Inverse spectral prob.: p, q in an SL sys. from $\{\lambda_n\}$
  - Inverse nodal prob.: p, q in an SL sys. from zeros of eigfcn's.
  - …

• $d \geq 2$:

+ Forward Problems
  - Domain shape: Lipschitz domains; fractal bdries, …
  - Bdry cond.: mixed B.C. (part D, part N); oblique deriv. …
  - Variable coeff.'s: e.g., $\nabla \cdot (a \nabla u) = f$, $a \in C^1(\Omega)$, …
  - Higher order: e.g., polyharmonic $\Delta^p u = f$, $p \geq 1$, …
  - Nonlinear: e.g., p-Laplacian $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f$, $p \geq 1$

+ Inverse Problems
  - Inverse Scattering Prob., Imaging, Tomography, …
  - Inverse Spectral Prob.
  - Inverse Nodal Prob.
  - Spectral Geometry: "Can we hear the shape of a drum?" — Mark Kac ('66)
  - …

• All these have been generalized for:

$\left\{ \begin{array}{l} \Omega: \text{ a Riemannian manifold (continuum)} \\ \Omega: \text{ a graph (discrete)} \\ \Omega: \text{ a metric graph (edges are continuum)} \end{array} \right.$

$\exists$ Still so many things to do, to which you can contribute a lot!!