

MAT 271: Applied & Computational Harmonic Analysis

Supplementary Notes V by Naoki Saito

The Balian-Low Theorem

Suppose $\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ constitutes a windowed Fourier frame of $L^2(\mathbb{R})$ with $\Delta x \Delta \xi = 1$ (which includes the case of an orthonormal basis). Then, either $\sigma_x(g) = \infty$ or $\sigma_\xi(g) = \infty$.

Proof. We only prove here the orthonormal basis case due to Battle [1]. For the general non-orthogonal case, which includes the Gabor frame, see [2].

Our strategy here is the following: Assume $\sigma_x(g) < \infty$ and $\sigma_\xi(g) < \infty$, then lead to contradiction. Let us consider the inner product, $\langle xg, g' \rangle$, which also appeared in the proof of the inequality of the Heisenberg uncertainty principle. Note that xg is in $L^2(\mathbb{R})$ so as g' , because

$$\|xg\|^2 = \int x^2 |g(x)|^2 dx = \sigma_x^2(g) < \infty,$$

since the mean of g is 0 and $\|g\|^2 = 1$. Recognizing that $\mathcal{F}g' = \xi \widehat{g}(\xi)$ and $\sigma_\xi^2(g) < \infty$, we can show $g' \in L^2(\mathbb{R})$.

Now, we have the following:

$$\begin{aligned} \langle xg, g' \rangle &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle xg, g_{m,n} \rangle \langle g_{m,n}, g' \rangle \\ &\stackrel{(a)}{=} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle g_{-m, -n}, xg \rangle \langle -(g')_{m,n}, g \rangle \\ &\stackrel{(b)}{=} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle g_{-m, -n}, xg \rangle \langle -g', g_{-m, -n} \rangle \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle -g', g_{m,n} \rangle \langle g_{m,n}, xg \rangle \\ &= -\langle g', xg \rangle. \end{aligned} \tag{1}$$

Here, (a) was derived by the following computations.

$$\begin{aligned} \langle xg, g_{m,n} \rangle &= \int xg(x) \overline{g(x - m\Delta x)} e^{-2\pi i n \Delta \xi x} dx \\ \text{[by change of variable } y = x - m\Delta x] &= \int (y + m\Delta x) g(y + m\Delta x) \overline{g(y)} e^{-2\pi i n \Delta \xi y} dy \cdot e^{-2\pi i n \Delta \xi m \Delta x} \\ \text{[since } e^{-2\pi i n m \Delta \xi \Delta x} = e^{-2\pi i n m} = 1] &= \int (x + m\Delta x) g(x + m\Delta x) \overline{g(x)} e^{-2\pi i n \Delta \xi x} dx \\ &= \langle g_{-m, -n}, xg \rangle + m\Delta x \langle g_{-m, -n}, g \rangle \\ &= \langle g_{-m, -n}, xg \rangle. \end{aligned} \tag{2}$$

The last equality holds since $g_{-m, -n}$ is orthogonal to $g = g_{0,0}$ for $(m, n) \neq (0, 0)$, and if $(m, n) = (0, 0)$, then $m\Delta x = 0$ at any rate.

Similarly, we can proceed:

$$\begin{aligned}
\langle g_{m,n}, g' \rangle &= \int g_{m,n}(x) \overline{g'(x)} dx \\
\text{[by integration by parts]} &= (g_{m,n}(x))' \overline{g'(x)} \Big|_{-\infty}^{\infty} - \int (g_{m,n}(x))' \overline{g'(x)} dx \\
\text{[since } g, g' \in L^2(\mathbb{R})] &= - \int (g_{m,n}(x))' \overline{g'(x)} dx \\
&= \langle -(g')_{m,n}, g \rangle - 2\pi i n \Delta \xi \langle g_{m,n}, g \rangle \\
&= \langle -(g')_{m,n}, g \rangle,
\end{aligned}$$

where we used the same logic as in the last equality of (2).

As for (b) in (1), we use the same change of variable as in (2) to get $\langle -(g')_{m,n}, g \rangle = \langle -g', g_{-m,-n} \rangle$. Now, let us consider a function $f \in C_c^\infty(\mathbb{R})$, i.e., a C^∞ -function vanishing as $|x| \rightarrow \infty$. Then,

$$\begin{aligned}
\langle xf, f' \rangle &= \int xf(x) \overline{f'(x)} dx \\
\text{[by integration by parts]} &= xf(x) \overline{f'(x)} \Big|_{-\infty}^{\infty} - \int f'(x) (xf'(x) + f(x)) dx \\
\text{[since } f \in C_c^\infty(\mathbb{R})] &= - \int x \overline{f'(x)} f'(x) dx - \int |f(x)|^2 dx \\
&= -\langle f', xf \rangle - \|f\|^2.
\end{aligned}$$

Now, $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{H} = \{f \in L^2 \mid xf \in L^2, f' \in L^2\}$. Hence, the function g under consideration must satisfy

$$\langle xg, g' \rangle = -\langle g', xg \rangle - \|g\|^2.$$

Combining this with (1), we conclude $\|g\| = 0$. This contradicts the condition $\|g\| = 1$. □

References

- [1] G. BATTLE, *Heisenberg proof of the Balian-Low theorem*, Lett. Math. Phys., 15 (1988), pp. 175–177.
- [2] I. DAUBECHIES AND A. J. E. M. JANSSEN, *Two theorems on lattice expansions*, IEEE Trans. Inform. Theory, 39 (1993), pp. 3–6.