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THE INTERACTION OF THE 3D NAVIER–STOKES EQUATIONS WITH A MOVING NONLINEAR KOITER ELASTIC SHELL*

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Abstract. We study a moving boundary value problem consisting of a viscous incompressible fluid moving and interacting with a nonlinear elastic solid shell. The fluid motion is governed by the Navier–Stokes equations, while the shell is modeled by the nonlinear Koiter shell model, consisting of bending and membrane tractions as well as inertia. The fluid is coupled to the solid shell through continuity of displacements and tractions (stresses) along the moving material interface. We prove existence and uniqueness of solutions in Sobolev spaces.

Key words. Navier–Stokes, free boundary, Koiter shell, geometric motion

AMS subject classifications. 74F10, 35Q30, 74K25, 35Q74, 74B20, 74H20, 74H25, 76D05

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1. Introduction.

1.1. The problem statement and background.

Fluid-solid interaction problems involving moving material interfaces have been the focus of active research since the nineties. The first problem solved in this area was for the case of a rigid body moving in a viscous fluid (see [11], [15] and also the early works of [19] and [17] for a rigid body moving in a Stokes flow in the full space). The case of an elastic body moving in a viscous fluid was considerably more challenging because of some apparent regularity incompatibilities between the parabolic fluid phase and the hyperbolic solid phase. The first existence results in this area were for regularized elasticity laws, such as in [12] for a *finite* number of elastic modes, in [2], [4], and [3] for hyperviscous elasticity laws, and in [16], in which a phase-field regularization "fattens" the sharp interface via a diffuse-interface model.

The treatment of classical elasticity laws for the solid phase, without any regularizing term, was considered only recently in [7] for the three-dimensional (3D) linear St. Venant–Kirchhoff constitutive law and in [8] for quasi-linear elastodynamics coupled to the Navier–Stokes equations. Some of the basic new ideas introduced in those works concerned a functional framework that scales in a hyperbolic fashion (and is therefore driven by the solid phase), the introduction of approximate problems either penalized with respect to the divergence-free constraint in the moving fluid domain or smoothed by an appropriate parabolic artificial viscosity in the solid phase (chosen in an asymptotically convergent and consistent fashion), and the tracking of the motion of the interface by difference quotients techniques.

The complimentary fluid-solid interaction problem, studied herein, consists of the motion of a viscous incompressible fluid enclosed by a moving thin nonlinear elastic

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solid shell. The main mathematical differences with respect to the previous problem of a deformable solid body moving inside of the fluid is that the shell encloses the fluid and is mathematically the *boundary* of the fluid. Our companion paper [5] treats the case of a viscous incompressible fluid enclosed by a moving thin nonlinear elastic *fluid* shell. In that paper, a moving boundary problem that models the motion of a viscous incompressible Newtonian fluid inside of a deformable elastic structure of Willmore type was studied. The shell model comprises degenerate elliptic operators which provide the regularity in only a *variable* direction, the normal direction of the moving shell. With the exception of [5], the only cases considered until now consisted of regularized problems, wherein the elliptic degeneracy occurs along a *fixed direction*, such as in [4] or [14].

We are concerned here with establishing the existence and uniqueness of solutions to the time-dependent incompressible Navier–Stokes equations interacting with an elastic solid shell of Koiter type (see [1], [6] for a detailed account of Koiter shells). The solid shell energy is a nonlinear function of the first and second fundamental forms of the moving boundary.

The Koiter shell produces a boundary condition, consisting of degenerate elliptic (hyperbolic with inertia) operators that do not provide optimal regularity. In particular, our estimates require short time, and additionally for the 3D case we also require a small shell thickness ε. The main results of this paper can be summarized as follows.

Main result. For the two-dimensional (2D) fluid, we assume the Koiter shell has inertia and an arbitrary shell thickness. For the 3D fluid, we assume the Koiter shell does not have inertia and require the shell thickness to be much smaller than the kinematic viscosity of the fluid. Under these assumptions, given sufficiently smooth initial data that satisfy compatibility conditions, there exists a unique solution on a short time interval $[0,T]$, where T depends on the initial data.

The precise statements of the theorems for existence and uniqueness for the 2D and 3D fluid cases are given in Theorems 4.1 and 4.2, respectively.

We now introduce the precise mathematical statement of the problem we are interested in. Let $\mathfrak{n} = 2$ or 3 and $\Omega \subseteq \mathbb{R}^{\mathfrak{n}}$ denote an open bounded domain with boundary $\Gamma := \partial\Omega$. For each $t \in (0,T]$, we wish to find the domain $\Omega(t)$, a divergence-free velocity field $u(t,\cdot)$, a pressure function $p(t,\cdot)$ on $\Omega(t)$, and a volume-preserving transformation $\eta(t,\cdot) : \Omega \to \mathbb{R}^{\mathfrak{n}}$ such that

$$\Omega(t) = \eta(t,\Omega)\,, \tag{1.1a}$$

$$\eta_t(t,x) = u(t,\eta(t,x))\,, \tag{1.1b}$$

$$u_t + \nabla_u u - \nu\Delta u = -\nabla p + f \qquad \text{in} \;\; \Omega(t)\,, \tag{1.1c}$$

$$\operatorname{div} u = 0 \qquad \text{in} \;\; \Omega(t)\,, \tag{1.1d}$$

$$-(\nu \operatorname{Def} u - p\mathrm{Id})n = \mathfrak{t}_{shell} \qquad \text{on} \;\; \Gamma(t)\,, \tag{1.1e}$$

$$u(0,x) = u_0(x) \qquad \forall x \in \Omega\,, \tag{1.1f}$$

$$\eta(0,x) = x \qquad \forall x \in \Omega\,, \tag{1.1g}$$

where ν is the kinematic viscosity; $n(t,\cdot)$ is the outward pointing unit normal to $\Gamma(t)$; $\Gamma(t) := \partial\Omega(t)$ denotes the boundary of $\Omega(t)$; $\operatorname{Def} u$ is twice the rate of deformation tensor of u, given in coordinates by $u^i_{,j} + u^j_{,i}$, where $u^i_{,j}$ denotes $\frac{\partial u^i}{\partial x^j}$; Id is the identity matrix; and $\mathfrak{t}_{shell}$ is the traction imparted onto the fluid by the elastic solid shell, which we describe next.

Let $g_{\alpha\beta}$ denote the induced metric and $b_{\alpha\beta}$ denote the covariant component of the second fundamental form on $\Gamma(t)$, and let $\mathfrak{g}$ and $\mathfrak{b}$ denote the induced metric and the covariant component of the second fundamental form of the *equilibrium configuration* Γ, respectively. With

$$\varepsilon \text{ denoting the thickness of the Koiter shell,}$$

the energy required to deform a Koiter shell Γ from the equilibrium state to the location $\Gamma(t)$ is $\varepsilon E_{mem} + \frac{\varepsilon^3}{3} E_{ben}$, where E_{mem} and E_{ben} are the so-called *membrane* and *bending* energies, respectively. The membrane energy E_{mem} is

$$E_{mem} = \frac{1}{4} \int_\Gamma a^{\alpha\beta\gamma\delta} (g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})(g_{\gamma\delta} - \mathfrak{g}_{\gamma\delta}) dS \tag{1.2}$$

and the bending energy E_{ben} is

$$E_{ben} = \int_\Gamma a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})(b_{\gamma\delta} - \mathfrak{b}_{\gamma\delta}) dS \,, \tag{1.3}$$

where, with $\lambda/2$ and $\mu/2$ denoting the Lamé constants, the elasticity tensor $a^{\alpha\beta\gamma\delta}$ is defined as

$$a^{\alpha\beta\gamma\delta} = \frac{4\lambda\mu}{\lambda + 2\mu} \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\gamma\delta} + 2\mu(\mathfrak{g}^{\alpha\gamma}\mathfrak{g}^{\beta\delta} + \mathfrak{g}^{\alpha\delta}\mathfrak{g}^{\beta\gamma}) \,. \tag{1.4}$$

We will use $\bar\sigma$ as a parameter denoting either the absence or the inclusion of inertia in our shell model. In particular, we define

$$\bar\sigma = \begin{cases} 0\,, & \text{no inertia present,} \\ 1\,, & \text{inertia present.} \end{cases}$$

The traction vector $\mathfrak{t}_{shell}$ can then be expressed as

$$\mathfrak{t}_{shell} = \bar\sigma\varepsilon\mathfrak{t}_{ine} + \varepsilon\mathfrak{t}_{mem} + \frac{\varepsilon^3}{3}\mathfrak{t}_{ben} \,,$$

where $\mathfrak{t}_{ine}$ is the traction due to the inertia, given in Lagrangian coordinate by

$$\mathfrak{t}_{ine} = \eta_{tt} \,.$$

$\mathfrak{t}_{mem}$ and $\mathfrak{t}_{ben}$ are the traction due to the membrane and bending energies, respectively, and can be computed from the first variation of the energy function E_{shell}. We will provide an explicit form for both of these traction vectors in section 2.1

1.2. Notation.

1.2.1. Einstein summation convention. Repeated Latin indices are summed from 1 to $\mathfrak{n}$, while repeated Greek indices are summed from 1 to $\mathfrak{n} - 1$. For example,

$$f^\alpha g_\alpha := \sum_{\alpha=1}^{\mathfrak{n}-1} f^\alpha g_\alpha \qquad \text{and} \qquad f^i g_i := \sum_{i=1}^{\mathfrak{n}} f^i g_i \,.$$

1.2.2. The tangential derivative. Let $\{U_\ell\}_{\ell=1}^K$ denote an open covering of Γ, such that for each $\ell \in \{1, 2, \ldots, K\}$, with

$$V_\ell = B(0, r_\ell), \text{ denoting the open ball of radius } r_\ell \text{ centered at the origin,}$$
$$V_\ell^+ = V_\ell \cap \{x_{\mathfrak{n}} > 0\},$$
$$V_\ell^- = V_\ell \cap \{x_{\mathfrak{n}} < 0\},$$

there exist for $s \geq 3$ H^s-class charts θ_ℓ which satisfy

$$\theta_\ell : V_\ell \to U_\ell \text{ is an } H^s \text{ diffeomorphism,}$$
$$\theta_\ell(V_\ell^+) = U_\ell \cap \Omega,$$
$$\theta_\ell(V_\ell \cap \{x_{\mathfrak{n}} = 0\}) = U_\ell \cap \Gamma.$$

Next, for $L > K$, let $\{U_\ell\}_{\ell=K+1}^L$ denote a family of open balls of radius r_ℓ contained in Ω such that $\{U_\ell\}_{\ell=1}^L$ is an open cover of Ω, and let

$$\{\xi_\ell\}_{\ell=1}^L \text{ denote a } \mathcal{C}^\infty \text{ partition of unity subordinate to this covering of } \Omega.$$

For a differentiable function f on Ω, we use $\bar\partial f$ if $\mathfrak{n} = 3$ or f' if $\mathfrak{n} = 2$ to denote the tangential derivative of f in $U_\ell \cap \Omega$. The αth component of the tangential derivative of f is given by

$$f_{,\alpha} = \bar\partial_\alpha f = \frac{\partial}{\partial x_\alpha}[f \circ \theta_\ell] \circ \theta_\ell^{-1} = \left[(Df \circ \theta_\ell)\frac{\partial \theta_\ell}{\partial x_\alpha}\right] \circ \theta_\ell^{-1}.$$

When no chart is specified, the notation $\bar\partial f$ is used to denote the tangential derivative of f in some coordinate chart.

We use $f_{,i}$ to denote the ith component of Df, where Df is the gradient of f, or

$$f_{,i} = \frac{\partial f}{\partial x_i}.$$

1.2.3. The identity map e. The identity map on $\mathbb{R}^n$ is denoted by $\mathbf{e}$ so that $\mathbf{e}(x) = x$. For $\alpha = 1, 2$, we use the notation $\mathbf{e}_{,\alpha}$ to denote the two tangent vectors to the reference material interface Γ; more specifically, in any local coordinate chart V_ℓ, $\mathbf{e}_{,\alpha}$ denotes the tangent vectors $\frac{\partial \theta_\ell}{\partial x_\alpha}$. Note that

$$[(Df) \circ \theta_\ell] \cdot \mathbf{e}_{,\alpha} = f_{,\alpha} \circ \theta_\ell \quad \text{or} \quad (f_{,j} \circ \theta_\ell)\mathbf{e}^j_{,\alpha} = f_{,\alpha} \circ \theta_\ell.$$

1.2.4. Sobolev norms on Ω and Γ. We will use the notation $H^s(\Omega)$ ($H^s(\Gamma)$) to denote either $H^s(\Omega; \mathbb{R})$ ($H^s(\Gamma; \mathbb{R})$) for the Sobolev space of scalar functions or $H^s(\Omega; \mathbb{R}^{\mathfrak{n}})$ ($H^s(\Gamma; \mathbb{R}^{\mathfrak{n}})$) for the Sobolev space of vector valued functions.

For $s > 0$ we denote the $H^s(\Omega)$-norm and $H^s(\Gamma)$-norm by

$$\|w\|_s = \|w\|_{H^s(\Omega)} \qquad \text{and} \qquad |w|_s = \|w\|_{H^s(\Gamma)},$$

with $s = 0$ denoting the L^2-norm. For $s = -1$, $\|w\|_{-1}$ is defined as the norm of the dual space of $H^1(\Omega)$, that is, $\|w\|_{-1} = \|w\|_{H^1(\Omega)'}$.

1.2.5. The functional framework for solutions. For $T > 0$, we set

$$\mathcal{V}^1(T) = \Big\{ v \in L^2(0,T;H^1(\Omega)) \,\Big|\, v_t \in L^2(0,T;H^1(\Omega)') \Big\}\,;$$
$$\mathcal{V}^2(T) = \Big\{ v \in L^2(0,T;H^2(\Omega)) \,\Big|\, v_t \in L^2(0,T;L^2(\Omega)) \Big\}\,;$$
$$\mathcal{V}^3(T) = \Big\{ v \in L^2(0,T;H^3(\Omega)) \,\Big|\, v_t \in L^2(0,T;H^1(\Omega)) \Big\}\,;$$
$$\mathcal{V}^k(T) = \Big\{ v \in L^2(0,T;H^k(\Omega)) \,\Big|\, v_t \in \mathcal{V}^{k-2}(T) \Big\} \qquad \text{for} \quad k \geq 4$$

with norms

$$\|v\|^2_{\mathcal{V}^1(T)} = \|v\|^2_{L^2(0,T;H^1(\Omega))} + \|v_t\|^2_{L^2(0,T;H^1(\Omega)')}\,;$$
$$\|v\|^2_{\mathcal{V}^2(T)} = \|v\|^2_{L^2(0,T;H^2(\Omega))} + \|v_t\|^2_{L^2(0,T;L^2(\Omega))}\,;$$
$$\|v\|^2_{\mathcal{V}^3(T)} = \|v\|^2_{L^2(0,T;H^3(\Omega))} + \|v_t\|^2_{L^2(0,T;H^1(\Omega))}\,;$$
$$\|v\|^2_{\mathcal{V}^k(T)} = \|v\|^2_{L^2(0,T;H^k(\Omega))} + \|v_t\|^2_{\mathcal{V}^{k-2}(T)} \quad \text{for } k \geq 4.$$

We then introduce the space (of vectors with vanishing Lagrangian divergence)

$$\mathcal{V}_v = \Big\{ w \in H^1(\Omega) \,\Big|\, w \in H^2(\Gamma)\,, A^j_i(t) w^i_{,j} = 0 \;\forall\, t \in [0,T] \Big\}$$

and

$$\mathcal{V}_v(T) = \Big\{ w \in L^2(0,T;H^1(\Omega)) \,\Big|\, w \in L^2(0,T;H^2(\Gamma))\,, A^j_i(t) w^i_{,j} = 0 \;\forall\, t \in [0,T] \Big\}\,,$$

where the matrix A is defined by (2.4) with $\eta(t) = \mathbf{e} + \int_0^t v(s)ds$. We remind the reader again that $\mathbf{e}$ is the identity map satisfying $\mathbf{e}(x) = x$.

1.2.6. The functional framework for the forcing function f. As explained below, we have to assume that $f(t)$ is defined on an open set which properly contains $\Omega(t)$, and without loss of generality, we may assume that $f(t)$ is defined on $\mathbb{R}^{\mathfrak{n}}$.

For $T > 0$, we set

$$\mathcal{F}^1(T) = \Big\{ f \in L^2(0,T;H^1(\mathbb{R}^{\mathfrak{n}})) \,\Big|\, f_t \in L^2(0,T;H^1(\mathbb{R}^{\mathfrak{n}})') \Big\}\,;$$
$$\mathcal{F}^2(T) = \Big\{ f \in L^2(0,T;H^2(\mathbb{R}^{\mathfrak{n}})) \,\Big|\, f_t \in L^2(0,T;L^2(\mathbb{R}^{\mathfrak{n}})) \Big\}\,;$$
$$\mathcal{F}^3(T) = \Big\{ f \in L^2(0,T;H^3(\mathbb{R}^{\mathfrak{n}})) \,\Big|\, f_t \in L^2(0,T;H^1(\mathbb{R}^{\mathfrak{n}})) \Big\}\,;$$
$$\mathcal{F}^k(T) = \Big\{ f \in L^2(0,T;H^k(\mathbb{R}^{\mathfrak{n}})) \,\Big|\, f_t \in \mathcal{F}^{k-2}(T) \Big\} \qquad \text{for} \quad k \geq 4$$

with norms

$$\|f\|^2_{\mathcal{F}^1(T)} = \|f\|^2_{L^2(0,T;H^1(\mathbb{R}^{\mathfrak{n}}))} + \|f_t\|^2_{L^2(0,T;H^1(\mathbb{R}^{\mathfrak{n}})')}\,;$$
$$\|f\|^2_{\mathcal{F}^2(T)} = \|f\|^2_{L^2(0,T;H^2(\mathbb{R}^{\mathfrak{n}}))} + \|f_t\|^2_{L^2(0,T;L^2(\mathbb{R}^{\mathfrak{n}}))}\,;$$
$$\|f\|^2_{\mathcal{F}^3(T)} = \|f\|^2_{L^2(0,T;H^3(\mathbb{R}^{\mathfrak{n}}))} + \|f_t\|^2_{L^2(0,T;H^1(\mathbb{R}^{\mathfrak{n}}))}\,;$$
$$\|f\|^2_{\mathcal{F}^k(T)} = \|f\|^2_{L^2(0,T;H^k(\mathbb{R}^{\mathfrak{n}}))} + \|f_t\|^2_{\mathcal{F}^{k-2}(T)} \quad \text{for } k \geq 4.$$

1.2.7. Inner products and duality pairings. Given a Hilbert space X, we let $(\cdot,\cdot)_X$ denote the inner product in X, and let $\langle \cdot,\cdot \rangle_X$ denote the X-X' duality. In particular, without specifying X, $\langle \cdot,\cdot \rangle$ is used to denote $\langle \cdot,\cdot \rangle_{H^1(\Omega)}$, and $\langle \cdot,\cdot \rangle_\Gamma$ is used to denote the duality pairing between $H^2(\Gamma)$ and $H^{-2}(\Gamma)$.

1.2.8. H^s-norm of Γ. We defined the H^k-norm of Γ to be

$$|\Gamma|_k^2 := \sum_{\ell=1}^{K} \int_{\mathbb{R}}^{\mathfrak{n}-1} \xi_\ell |\partial^k_{\alpha_1\cdots\alpha_k}\theta_\ell|^2 dx_1 \cdots dx_{\mathfrak{n}-1}\,.$$

The H^s-norm for any real $s \geq 0$ is defined by interpolation. We say that Γ is of class H^s (or $\Gamma \in H^s$) whenever $|\Gamma|_s < \infty$.

1.2.9. Inner products and contractions. Given two vectors v and w in $\mathbb{R}^{\mathfrak{n}}$, the inner product of v and w is denoted by $v \cdot w$, which in component is defined as

$$v \cdot w = v^i w_i = \sum_{i=1}^{\mathfrak{n}} v^i w_i\,.$$

For two matrices A and B, the contraction between A and B, denoted by $A : B$, is the trace of the product of A and B, which in component is defined as

$$A : B = \mathrm{Tr}(AB) = A^j_i B^i_j = \sum_{i,j=1}^{\mathfrak{n}} A^i_j B^j_i\,.$$

1.2.10. The transpose of matrices. Given any matrix $\mathcal{A}$, we use $\mathcal{A}^t$ to denote its transpose.

1.2.11. The temporal trace of functions at $t = 0$. Throughout the paper, almost all of the functions that we consider are functions of both x and t. Given such a function $h(x,t)$, we will often drop the explicit dependence on the variable x when expressing the restriction to $t = 0$. Specifically, we will write

$$h(0) \text{ to denote } h(x,0)\,.$$

1.3. Outline of the paper. In section 2, $\mathfrak{t}_{mem}$ and $\mathfrak{t}_{ben}$ are computed in Lagrangian coordinates (which is widely used in the theory of elasticity) for $\mathfrak{n} = 2$ and $\mathfrak{n} = 3$, and (1.1) is reformulated in Lagrangian coordinates.

Section 3 is devoted to a number of technical lemmas that will be used repeatedly throughout the paper. Of significant (and perhaps independent) interest is a new set of estimates for Stokes-type elliptic equations with Sobolev-class coefficients. Such estimates are vital to our subsequent analysis.

The main theorems concerning existence and uniqueness of solutions are presented in section 4, wherein we first provide a detailed discussion of compatibility conditions.

In section 5, we introduce the linearized and regularized problems. The regularization requires smoothing certain variables as well as the introduction of a certain artificial viscosity term on the boundary of the fluid domain. Weak solutions of this regularized problem are established via a penalization scheme which approximates the incompressibility of the fluid.

In section 6, we establish the regularity theory for our weak solution using energy estimates for the regularized problem with constants depending on the artificial viscosity.

Section 7 is devoted to obtaining improved elliptic-type estimates for the first and second fundamental forms, g and b; more precisely, we analyze the boundary condition (1.1e) and use energy estimates to find the optimal regularity of g and b. In the case that $\mathfrak{n} = 2$ we require that $T > 0$ be taken sufficiently small, while in the case that $\mathfrak{n} = 3$, we additionally require the shell thickness ε to be taken much smaller than ν.

Having these estimates, we turn to section 8, wherein we establish that our a priori energy estimates are indeed independent of the artificial viscosity parameter κ, and are thus able to pass to the limit as $\kappa \to 0$.

Section 9 is devoted to the proof of the uniqueness of solutions to the fluid-shell interaction, and section 10 provides a list of notation.

1.4. User's guide for the reader. Our analysis makes use of rather technical and nonstandard compatibility conditions, solutions of nonlinear parabolic equations with nonlinear time-dependent constraints, and elliptic-type estimates, wherein the inequalities rely crucially on the small thickness of the shell ε, as well as time $T > 0$ taken sufficiently small.

In order to introduce these ideas in the simplest possible settings, we have included the sections 4.1, 5.3, and 7.3 as a *user's guide* for the reader.

We encourage the reader to read these sections in detail before proceeding with the more complicated subsequent (analogous) treatment of the solid-shell model.

2. The Lagrangian formulation.

2.1. The computation of the traction vector.

2.1.1. The case $\mathfrak{n} = 3$. In this subsection, we compute the variation of (1.2) and (1.3). We first note that if $\underline{\delta}$ is a differential operator, then

$$\underline{\delta} n = -g^{\sigma\tau} n \cdot (\underline{\delta}\eta,_\tau)\eta,_\sigma \,.$$

Therefore,

$$\begin{aligned}
&\underline{\delta}\Big[a^{\alpha\beta\gamma\delta}(\eta,_{\alpha\beta} \cdot n)(\eta,_{\gamma\delta} \cdot n)\Big] \\
&= 2a^{\alpha\beta\gamma\delta}(\eta,_{\alpha\beta} \cdot n)\Big[(\underline{\delta}\eta),_{\gamma\delta} \cdot n + \eta,_{\gamma\delta} \cdot \underline{\delta} n\Big] \\
&= 2a^{\alpha\beta\gamma\delta}(\eta,_{\alpha\beta} \cdot n)\Big[n \cdot (\underline{\delta}\eta),_{\gamma\delta} - g^{\sigma\tau}(\eta_{\gamma\delta} \cdot \eta,_\sigma)(n \cdot (\underline{\delta}\eta,_\tau))\Big] \,.
\end{aligned}$$

Since $b_{\alpha\beta} = \eta,_{\alpha\beta} \cdot n$, we find that

$$\begin{aligned}
&\underline{\delta}\int_\Gamma a^{\alpha\beta\gamma\delta} b_{\alpha\beta} b_{\gamma\delta} dS = \underline{\delta}\int_\Gamma a^{\alpha\beta\gamma\delta}(\eta,_{\alpha\beta} \cdot n)(\eta,_{\gamma\delta} \cdot n) dS \\
&= \int_\Gamma \frac{2}{\sqrt{a}}\Big\{\Big[\sqrt{a} a^{\alpha\beta\gamma\delta} b_{\alpha\beta} n\Big],_{\gamma\delta} + \Big[\sqrt{a} a^{\alpha\beta\gamma\delta} g^{\sigma\tau} b_{\alpha\beta}(\eta,_{\gamma\delta} \cdot \eta,_\sigma) n\Big],_\tau\Big\}\underline{\delta}\eta dS \,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\underline{\delta}\int_\Gamma \Big[a^{\alpha\beta\gamma\delta}\mathfrak{b}_{\alpha\beta}(\eta,_{\gamma\delta} \cdot n)\Big] dS \\
&= \int_\Gamma \frac{1}{\sqrt{a}}\Big\{\Big[\sqrt{a} a^{\alpha\beta\gamma\delta}\mathfrak{b}_{\alpha\beta} n\Big],_{\gamma\delta} + \Big[\sqrt{a} a^{\alpha\beta\gamma\delta} g^{\sigma\tau}\mathfrak{b}_{\alpha\beta}(\eta,_{\gamma\delta} \cdot \eta,_\sigma) n\Big],_\tau\Big\}\underline{\delta}\eta dS \,.
\end{aligned}$$

As a consequence, the bending traction is

$$\begin{aligned}
\mathcal{L}_b(\eta) = {} & \frac{2}{\sqrt{a}}\Big[\sqrt{a} a^{\alpha\beta\gamma\delta}(b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) n\Big],_{\gamma\delta} \\
& + \frac{2}{\sqrt{a}}\Big[\sqrt{a} a^{\alpha\beta\gamma\delta} g^{\sigma\tau}(b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})(\eta,_{\gamma\delta} \cdot \eta,_\sigma) n\Big],_\tau \,.
\end{aligned} \tag{2.1}$$

As for the membrane traction, by $g_{\alpha\beta} = \eta_{,\alpha} \cdot \eta_{,\beta}$,

$$\underline{\delta}\Big[a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})(g_{\gamma\delta} - \mathfrak{g}_{\gamma\delta})\Big] = 4a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})(\eta_{,\gamma} \cdot \underline{\delta}\eta_{,\delta}),$$

and hence the membrane traction is

$$\mathcal{L}_m(\eta) = -\frac{4}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta_{,\gamma}\Big]_{,\delta}. \tag{2.2}$$

Remark 1. In complying with the standard notation of the shell theory, we use $\sqrt{a}$ instead of $\sqrt{\mathfrak{g}}$ in (2.1) and (2.2) to denote the square root of the determinant of the metric tensor on Γ.

2.1.2. The case $\mathfrak{n} = 2$. For the case $\mathfrak{n} = 2$, the bending energy (1.3) and membrane energy (1.2) are expressed as

$$E_{ben} = \int_\Gamma |\mathbf{e}'|^{-3}(b - \mathfrak{b})^2 dS_0\,, \quad E_{mem} = \int_\Gamma |\mathbf{e}'|^{-3}(g - \mathfrak{g})^2 dS_0\,,$$

where $\mathfrak{g} = \mathbf{e}' \cdot \mathbf{e}'$ is the first fundamental form and $\mathfrak{b} = \mathbf{e}'' \cdot n_0$ is the covariant component second fundamental form of the unstressed initial boundary. Similar computations show that the bending traction $\mathcal{L}_b$ is given by

$$\mathcal{L}_b(\eta) = 2\Big[|\mathbf{e}'|^{-3}(b - \mathfrak{b})n\Big]'' + \Big[|\mathbf{e}'|^{-3}g^{-1}g'(b - \mathfrak{b})n\Big]'$$

and the membrane traction $\mathcal{L}_m$ is

$$\mathcal{L}_m(\eta) = -4\Big[|\mathbf{e}'|^{-3}(|\eta'|^2 - |\mathbf{e}'|^2)\eta'\Big]'.$$

2.2. Lagrangian formulation of the fluid-shell interaction problem. Let $\eta(t,x)$ denote the Lagrangian particle placement field, a volume-preserving embedding of Ω onto $\Omega(t) \subseteq \mathbb{R}^{\mathfrak{n}}$ satisfying

$$\eta_t(x,t) = u(\eta(x,t),t) = (u \circ \eta)(x,t)\,, \tag{2.3}$$

and denote the *inverse deformation* matrix, the inverse of $\nabla\eta(x,t)$, by

$$A(x,t) = [\nabla\eta(x,t)]^{-1}\,. \tag{2.4}$$

Let $v = u \circ \eta$ denote the Lagrangian or material velocity field, $q = p \circ \eta$ the Lagrangian pressure function, and $F = f \circ \eta$ the forcing function in the material frame. The coupled fluid-structure problem has the following Lagrangian description:

$$\eta_t = v \quad \text{in } (0,T) \times \Omega\,, \tag{2.5a}$$

$$v^i_t - \nu A^j_\ell (A^k_\ell v^i_{,k})_{,j} + A^k_i q_{,k} = F^i \quad \text{in } (0,T) \times \Omega\,, \tag{2.5b}$$

$$A^j_i v^i_{,j} = 0 \quad \text{in } (0,T) \times \Omega\,, \tag{2.5c}$$

$$-\Big[\nu(D_A v)^j_i - q\mathrm{Id}^i_j\Big]A^\ell_j N_\ell = \Big[\varepsilon\mathcal{L}_m(\eta) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\eta)\Big]^i + \bar{\sigma}\varepsilon\eta^i_{tt} \quad \text{on } (0,T) \times \Gamma\,, \tag{2.5d}$$

$$v(0) = u_0 \quad \text{on } \{t=0\} \times \Omega\,, \tag{2.5e}$$

$$\eta(0) = \mathbf{e} \quad \text{on } \{t=0\} \times \Omega\,, \tag{2.5f}$$

where $(D_A v)^j_i = A^k_i v^j_{,k} + A^k_j v^i_{,k}$. We also note that in order to have F well-defined, f has to be defined on a domain Ω^+ containing $\bar{\Omega}$ (or in other words, $\Omega \subset\subset \Omega^+$). Without loss of generality, we may assume that f is defined on $\mathbb{R}^{\mathfrak{n}}$.

3. Preliminary results.

3.1. Differentiating the inverse deformation matrix $\boldsymbol{A}$. In this subsection we list a very useful identity concerning the differentiation of the cofactor matrix A for reference. Let $\underline{\delta}$ be a differential operator such as ∂_t or D_x; then

$$\underline{\delta} A^j_i = -A^j_r \underline{\delta} \eta^r_{,s} A^s_i \,.$$

For example, when $\underline{\delta} = \partial_t$,

$$(A^j_i)_t = -A^j_r v^r_{,s} A^s_i \,.$$

3.2. Velocity and pressure estimates at time $\boldsymbol{t = 0}$. Before stating the main theorem, we provide estimates for the time derivatives of the velocity and pressure at $t = 0$; these quantities are important in the statement of the main theorems, as well as in the proof. We denote $\partial_t^k q(0)$ and $\partial_t^k v(0)$ by q_k and w_k, respectively. Note that in particular $u_0 = v(0)$ and $q_0 = p(0)$.

The case without inertia is much simpler than the case with inertia, so we first describe how q_k and w_k are computed in this case. Evaluating (2.5b) at $t = 0$, it is easy to see that w_1 satisfies

$$\mathrm{w}_1 = \nu \Delta u_0 - \nabla q_0 + f(0). \tag{3.1}$$

Since $\operatorname{div} \mathrm{w}_1 = (A^j_i v^i_{t,j})(0)$, by the incompressibility condition (2.5c),

$$(A^j_i v^i_{t,j})(0) = [A^j_i v^i_{,j}]_t(0) - [(A^j_i)_t v^i_{,j}](0) = [A^j_r v^r_{,s} A^s_i v^i_{,j}](0) = u^j_{0,i} u^i_{0,j} \,. \tag{3.2}$$

Taking the divergence of both sides of (3.1), we find that q_0 satisfies

$$-\Delta q_0 = u^j_{0,i} u^i_{0,j} - \operatorname{div} f(0) \qquad \text{in} \quad \Omega \,.$$

In order to invert $-\Delta$, a boundary condition is needed. By letting $t = 0$ in (2.5d) and projecting the resulting equation onto the normal direction, we find that q_0 must satisfy

$$q_0 = \nu (\operatorname{Def} u_0)^j_i N_i N_j \qquad \text{on} \quad \Gamma \,,$$

where the fact that $\mathcal{L}_m(\eta)$ and $\mathcal{L}_b(\eta)$ both vanish at time $t = 0$ is used. Therefore, q_0 solves the elliptic equation

$$-\Delta q_0 = u^j_{0,i} u^i_{0,j} - \operatorname{div} f(0) \qquad \text{in} \quad \Omega \,, \tag{3.3a}$$

$$q_0 = \nu (\operatorname{Def} u_0)^j_i N_i N_j \qquad \text{on} \quad \Gamma \,. \tag{3.3b}$$

By elliptic regularity, for $k \geq 2$,

$$\begin{aligned} \|q_0\|^2_k &\leq C\Big[\|u^j_{0,i} u^i_{0,j}\|^2_{k-2} + \|\operatorname{div} f(0)\|^2_{k-2} + |(\operatorname{Def} u_0)^j_i N_i N_j|^2_{k-0.5}\Big] \\ &\leq \mathcal{P}(\|u_0\|_{k+1}, \|f(0)\|_{k-1}, |\Gamma|_{k+0.5}) \,, \end{aligned} \tag{3.4}$$

where $\mathcal{P}$ denotes a polynomial of its arguments. Therefore,

$$\begin{aligned} \|\mathrm{w}_1\|^2_{k-1} &\leq C\Big[\|u_0\|^2_{k+1} + \|q_0\|^2_k + \|f(0)\|^2_{k-1}\Big] \\ &\leq \mathcal{P}(\|u_0\|_{k+1}, \|f(0)\|_{k-1}, |\Gamma|_{k+0.5}) \,. \end{aligned} \tag{3.5}$$

For the case with inertia, q_0 satisfies (3.3a) and the boundary condition

$$q_0 = \nu(\operatorname{Def} u_0)^j_i N_i N_j + \varepsilon \mathrm{w}_1 \cdot N \quad \text{on} \quad \Gamma, \tag{3.6}$$

while w_1 is unknown; therefore, the Dirichlet boundary condition cannot be used to solve for q_0. Instead, using (3.1) in (3.6), we find q_0 by solving the following equations:

$$-\Delta q_0 = u^j_{0,i} u^i_{0,j} - \operatorname{div} f(0) \quad \text{in} \quad \Omega\,, \tag{3.7a}$$

$$\varepsilon \frac{\partial q_0}{\partial N} + q_0 = \nu(\operatorname{Def} u_0)^j_i N_i N_j + \varepsilon\nu \Delta u_0 \cdot N + \varepsilon f(0)\cdot N \quad \text{on} \quad \Gamma\,. \tag{3.7b}$$

Equation (3.7) is a Robin-type problem for q_0, and there exists a unique solution q_0 satisfying

$$\begin{aligned}\|q_0\|^2_k &\le C\Big[\|u^j_{0,i}u^i_{0,j}\|^2_{k-2} + \|\operatorname{div} f(0)\|^2_{k-2} + \frac{1}{\varepsilon^2}|(\operatorname{Def} u_0)^j_i N_i N_j|^2_{k-1.5} \\ &\qquad + |\Delta u_0 \cdot N|_{k-1.5} + |f(0)\cdot N|_{k-1.5}\Big] \\ &\le \mathcal{P}(\varepsilon^{-1}, \|u_0\|_{k+1}, \|f(0)\|_{k-1}, |\Gamma|_{k+0.5})\end{aligned} \tag{3.8}$$

and consequently

$$\|\mathrm{w}_1\|^2_{k-1} \le \mathcal{P}(\varepsilon^{-1}, \|u_0\|_{k+1}, \|f(0)\|_{k-1}, |\Gamma|_{k+0.5}). \tag{3.9}$$

Remark 2. Estimates (3.4) and (3.5) are ε-independent, while (3.8) and (3.9) depend on ε. This is due to the different boundary conditions (3.3b) and (3.7b) used to solve for q_0.

Time differentiating (2.5b) and then evaluating the resulting equation at $t = 0$, it is easy to see that w_2 satisfies

$$\mathrm{w}^i_2 = -q_{1,i} + u^k_{0,i} q_{0,k} - \nu\Big[u^j_{0,k}u^i_{0,kj} + u^k_{0,jj}u^i_{0,k} + u^k_{0,j}u^i_{0,kj} - \Delta \mathrm{w}^i_1\Big] + F^i_t(0)\,. \tag{3.10}$$

Similarly to (3.2), we find that

$$\begin{aligned}\operatorname{div} \mathrm{w}_2 &= (A^j_i v^i_{tt,j})(0) = [A^j_i v^i_{,j}](0) - [(A^j_i)_{tt} v^i_{,j}](0) - 2[(A^j_i)_t v^i_{t,j}](0) \\ &= -2u^i_{0,j}u^j_{0,k}u^k_{0,i} + 3u^j_{0,i}\mathrm{w}^i_{1,j}\,.\end{aligned} \tag{3.11}$$

Taking the divergence of both sides of (3.10), as well as time differentiating the boundary condition (2.5d) and evaluating at $t = 0$, we find that q_1 satisfies

$$\begin{aligned}-\Delta q_1 &= (3\mathrm{w}^j_{1,i} - 2u^j_{0,k}u^k_{0,i})u^i_{0,j} + u^k_{0,ii}q_{0,k} + u^k_{0,i}q_{0,ki} + \nu\Delta \operatorname{div} \mathrm{w}_1 \\ &\quad - \nu\Big[2u^j_{0,ik}u^i_{0,kj} + u^k_{0,ijj}u^i_{0,k}\Big] + \operatorname{div} F_t(0) \quad \text{in} \quad \Omega\,,\end{aligned} \tag{3.12a}$$

$$\begin{aligned}\bar\sigma\varepsilon\frac{\partial q_1}{\partial N} + q_1 &= \Big[\nu(\operatorname{Def} \mathrm{w}_1)^j_i - 2\nu u^k_{0,j}u^i_{0,k} + q_0 u^j_{0,i}\Big]N_i N_j + \bar\sigma\varepsilon u^k_{0,i}q_{0,k}N_i \\ &\quad - \bar\sigma\nu\varepsilon\Big[2u^j_{0,k}u^i_{0,kj} + u^k_{0,jj}u^i_{0,k}\Big]N_i + \bar\sigma\nu\varepsilon\Delta\mathrm{w}_1\cdot N + \bar\sigma\varepsilon F_t(0)\cdot N \\ &\quad + \Big[\varepsilon\mathcal{L}_m(\eta) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\eta)\Big]^i_t(0)N^i \quad \text{on} \quad \Gamma\,.\end{aligned} \tag{3.12b}$$

Therefore, solving for q_1 by a Dirichlet problem if $\bar\sigma = 0$ or by a Robin problem if $\bar\sigma = 1$, we find that for $k \ge 4$,

$$\|q_1\|^2_{k-3+\bar\sigma} + \|\mathrm{w}_2\|^2_{k-4+\bar\sigma} \le \mathcal{P}(\varepsilon^{-1}, \|u_0\|_{k+1}, \|f\|_{\mathcal{F}^k(T)})\,,$$

where $\|f\|_{\mathcal{F}^k(T)}$ is defined in section 1.2 with $\mathbb{R}^n$ replacing Ω. Later on we will use $\mathcal{M}(\varepsilon, u_0, f)$ to denote the quantity $\mathcal{P}(\varepsilon^{-1}, \|u_0\|_{2\mathfrak{n}-1}, \|f\|_{\mathcal{F}^{2\mathfrak{n}-2}(T)})$, and by the argument above,

$$\|q_0\|^2_{2\mathfrak{n}-2} + \|\mathrm{w}_1\|^2_{2\mathfrak{n}-3} + (\mathfrak{n}-2)\Big[\|q_1\|^2_{2\mathfrak{n}-5+\bar{\sigma}} + \|\mathrm{w}_2\|^2_{2\mathfrak{n}-6+\bar{\sigma}}\Big] \leq C\mathcal{M}(\varepsilon, u_0, f)\,.$$

Remark 3. For the case without inertia, the estimates for q_1 and w_2 are independent of ε.

3.3. Elliptic regularity results for Stokes-type systems. In establishing the regularity theory of the Navier–Stokes equations, the regularity result of Stokes-type problems in Lagrangian coordinates is crucial for our analysis. The usual Stokes problem, set in Eulerian variables, is a constant-coefficient elliptic PDE set on the moving domain $\Omega(t)$. When we fix the domain, using the Lagrangian flow map $\eta(t)$, we map the moving-domain elliptic equation to a Stokes-type system set on the fixed domain Ω, but the change of variables maps the Laplace operator into a second order elliptic operator with Sobolev-class coefficients which depend on both space and time.

While estimates for solutions to the Eulerian Stokes equations on a fixed domain are classical, the estimates for the remapped Stokes-type systems are not very well studied, and we are not aware of a reference for such equations in the literature. As such, we develop the necessary elliptic regularity estimates for Stokes-type systems with incompressibility constraints and set in Lagrangian variables.

To provide a general presentation, we begin our estimates with some basic assumptions. Later, we will verify that these assumptions hold for our remapped and Lagrangian Stokes equations.

Basic assumptions. Suppose that $\eta : \Omega \to \mathbb{R}^{\mathfrak{n}}$ is given with the following properties: for some fixed $s \in \mathbb{N}$ with $r = \max\{3, s\}$,

1. (*smoothness*) $\eta \in H^r(\Omega)$,
2. (*invertibility*) η is a diffeomorphism onto its range with $\det\nabla\eta = 1$,
3. (*inverse matrix*) $A(x) := (\nabla\eta)^{-1}(x)$,
4. (*near identity map*) $\|A - \mathrm{Id}\|_{r-1} < \epsilon_s \ll 1$,

3.3.1. The Stokes problem set on $\eta(\Omega)$. Our first objective is to obtain the elliptic estimates of the solution of the Stokes problem on the domain $\eta(\Omega)$ (here we do not assume that η is the flow of u), with bounds that depend on assumption 3. In particular, we are concerned with the following problem:

(3.13a) $$-\Delta u + \nabla p = f \qquad \text{in } \eta(\Omega)\,,$$

(3.13b) $$\operatorname{div} u = 0 \qquad \text{in } \eta(\Omega)\,,$$

(3.13c) $$u = g \qquad \text{on } \eta(\Gamma)\,.$$

Letting $v = u \circ \eta$ and $q = p \circ \eta$, we use η to change variables and set the problem on the domain Ω as follows:

(3.14a) $$-[A^j_\ell A^k_\ell v^i_{,j}]_{,k} + A^k_i q_{,k} = F \qquad \text{in } \Omega\,,$$

(3.14b) $$A^j_i v^i_{,j} = 0 \qquad \text{in } \Omega\,,$$

(3.14c) $$v = G \qquad \text{on } \Gamma\,,$$

where $F = f \circ \eta$ and $G = g \circ \eta$, and g, G satisfy the solvability condition

$$\int_{\eta(\Gamma)} g \cdot n dS_\eta = \int_\Gamma G^i A^j_i N_j dS = 0\,.$$

If $\eta = \mathbf{e}$ and Ω is a smooth domain, then the standard elliptic regularity results for (3.13) are that for $s \geq 1$,

$$\|u\|_1^2 + \|p\|_0^2 \leq C\Big[\|f\|_{-1}^2 + |g|_{0.5}^2\Big], \tag{3.15a}$$

$$\|u\|_s^2 + \|p\|_{s-1}^2 \leq C_s\Big[\|f\|_{s-2}^2 + |g|_{s-0.5}^2\Big]; \tag{3.15b}$$

see [18] for the existence, uniqueness, and regularity. In the case that the domain Ω has specified Sobolev regularity, C_1 depends on $|\Gamma|_2^2$, while C_s for $s \geq 2$ depends on $|\Gamma|_{r-0.5}$, where $r = \max\{3, s\}$ is defined above.

Remark 4. If $g = 0$ in (3.15a), then (3.15a) can be refined as

$$\|u\|_1^2 + \|p\|_0^2 \leq C\|f\|_{H^{-1}(\Omega)}^2. \tag{3.16}$$

Lemma 3.1 (existence, uniqueness, and regularity for (3.14)). *Suppose that η satisfies the basic assumptions stated above, $F \in H^1(\Omega)'$, and $G \in H^{0.5}(\Gamma)$. Then for $\epsilon_s > 0$ taken sufficiently small, there exists a unique weak solution $(v, q) \in H^1(\Omega) \times L^2(\Omega)$ of* (3.14)*, which satisfies, for a generic constant $C > 0$,*

$$\|v\|_1^2 + \|q\|_0^2 \leq C\Big[\|F\|_{-1}^2 + |G|_{0.5}^2\Big]. \tag{3.17}$$

Furthermore, if $F \in H^{s-2}(\Omega)$ and $G \in H^{s-0.5}(\partial\Omega)$, then

$$\|v\|_s^2 + \|q\|_{s-1}^2 \leq C\Big[\|F\|_{s-2}^2 + |G|_{s-0.5}^2\Big]. \tag{3.18}$$

Proof. We employ a topological fixed-point argument. We rewrite (3.14) as

$$\begin{aligned}
-\Delta v + \nabla q &= [(A_\ell^j A_\ell^k - \mathrm{Id}_\ell^j \mathrm{Id}_\ell^k) v^i_{,j}]_{,k} + (A_i^k - \mathrm{Id}_i^k) q_{,k} + F && \text{in } \Omega, \\
\operatorname{div} v &= (\mathrm{Id}_i^j - A_i^j) v^i_{,j} && \text{in } \Omega, \\
v &= G && \text{on } \Gamma.
\end{aligned}$$

Let $(\tilde{v}, \tilde{q}) \in H^s(\Omega) \times H^{s-1}(\Omega)$ be given so that

$$\|\tilde{v}\|_s^2 + \|\tilde{q}\|_{s-1}^2 \leq M,$$

where M will be determined later (in (3.22)). Let (v, q) be the unique solution to

$$-\Delta v + \nabla q = [(A_\ell^j A_\ell^k - \mathrm{Id}_\ell^j \mathrm{Id}_\ell^k) \tilde{v}^i_{,j}]_{,k} + (A_i^k - \mathrm{Id}_i^k) \tilde{q}_{,k} + F \quad \text{in } \Omega, \tag{3.19a}$$

$$\operatorname{div} v = (\mathrm{Id}_i^j - A_i^j) \tilde{v}^i_{,j} \quad \text{in } \Omega, \tag{3.19b}$$

$$v = G \quad \text{on } \Gamma. \tag{3.19c}$$

Let φ be a scalar function so that

$$\begin{aligned}
\Delta \varphi &= (\mathrm{Id}_i^j - A_i^j) \tilde{v}^i_{,j} && \text{in } \Omega, \\
\frac{\partial \varphi}{\partial N} &= \frac{1}{|\Gamma|} \int_\Omega (\mathrm{Id}_i^j - A_i^j) \tilde{v}^i_{,j} dx && \text{on } \Gamma,
\end{aligned}$$

and set $w = v - \nabla\varphi$. Then w satisfies

$$\begin{aligned}
-\Delta w + \nabla q = [(A_\ell^j A_\ell^k - \mathrm{Id}_\ell^j \mathrm{Id}_\ell^k) \tilde{v}^i_{,j}]_{,k} + (A_i^k - \mathrm{Id}_i^k) \tilde{q}_{,k}& \quad \text{in } \Omega, \\
- \nabla \Delta \varphi + F&
\end{aligned} \tag{3.20a}$$

$$\operatorname{div} w = 0 \quad \text{in } \Omega, \tag{3.20b}$$

$$w = G - \nabla\varphi \quad \text{on } \Gamma. \tag{3.20c}$$

For $s \geq 2$, it follows from (3.15b) that

$$\begin{aligned}\|w\|_s^2 + \|q\|_{s-1}^2 &\leq C\Big[\|[(A_\ell^j A_\ell^k - \text{Id}_\ell^j \text{Id}_\ell^k)\tilde{v}^i_{,j}]_{,k}\|_{s-2}^2 + \|(A_i^k - \text{Id}_i^k)\tilde{q}_{,k}\|_{s-2}^2 \\ &\qquad + \|\nabla\Delta\varphi\|_{s-2}^2 + \|F\|_{s-1}^2 + |G - \nabla\varphi|_{s-0.5}^2\Big] \\ &\leq C\Big[\|A^t A - \text{Id}^t \text{Id}\|_{r-1}^2 \|\tilde{v}\|_s^2 + \|A - \text{Id}\|_{r-1}^2 (\|\tilde{v}\|_s^2 + \|\tilde{q}\|_{s-1}^2) \\ &\qquad + \|\varphi\|_{s+1}^2 + \|F\|_{s-2}^2 + |G|_{s-0.5}^2\Big].\end{aligned}$$

Hence, for $s \geq 2$, by $\|\varphi\|_{H^{s+1}(\Omega)} \leq C\epsilon_s \|\tilde{v}\|_s$,

$$\|w\|_s^2 + \|q\|_{s-1}^2 \leq C\Big[\|F\|_{s-2}^2 + |G|_{s-0.5}^2 + \epsilon_s M\Big].$$

For $s = 1$, we decompose w as $w = w_1 + w_2$, where $w_1 \in H_0^1(\Omega)$ is the unique weak solution to

$$-\Delta w_1 + \nabla q_1 = \Big[(A_\ell^j A_\ell^k - \text{Id}_\ell^j \text{Id}_\ell^k)\tilde{v}^i_{,j}\Big]_{,k} + (A_i^k - \text{Id}_i^k)\tilde{q}_{,k} - \nabla\Delta\varphi \quad \text{in } \Omega, \tag{3.21a}$$

$$\operatorname{div} w_1 = 0 \quad \text{in } \Omega, \tag{3.21b}$$

$$w_1 = 0 \quad \text{on } \Gamma, \tag{3.21c}$$

and $w_2 \in H^1(\Omega)$ is the unique weak solution of

$$\begin{aligned} -\Delta w_2 + \nabla q_2 &= F && \text{in } \Omega, \\ \operatorname{div} w_2 &= 0 && \text{in } \Omega, \\ w_2 &= G - \nabla\varphi && \text{on } \Gamma. \end{aligned}$$

The estimate for w_2 is an immediate consequence of (3.15a); we find that

$$\|w_2\|_1^2 + \|q_2\|_0^2 \leq C\Big[\|F\|_{-1}^2 + |G - \nabla\varphi|_{0.5}^2\Big] \leq C\Big[\|F\|_{-1}^2 + |G|_{0.5}^2 + \epsilon_s \|\tilde{v}\|_1^2\Big].$$

The estimate of w_1 follows from (3.16):

$$\begin{aligned}\|w_1\|_1^2 + \|q_1\|_0^2 &\leq C\Big\| \Big[(A_\ell^j A_\ell^k - \text{Id}_\ell^j \text{Id}_\ell^k)\tilde{v}^i_{,j}\Big]_{,k} + (A_i^k - \text{Id}_i^k)\tilde{q}_{,k} - \nabla\Delta\varphi\Big\|_{H^{-1}(\Omega)}^2 \\ &\leq C\Big[\|A^t A - \text{Id}^t \text{Id}\|_2^2 \|\tilde{v}\|_1^2 + \|A - \text{Id}\|_2^2 \|\tilde{q}\|_0^2 + \|\Delta\varphi\|_0^2\Big] \\ &\leq C\epsilon_1\Big[\|\tilde{v}\|_1^2 + \|\tilde{q}\|_0^2\Big],\end{aligned}$$

where we have estimated the $H^{-1}(\Omega)$-norm using a test function in $H_0^1(\Omega)$ which removes potential boundary contributions. Combining the estimates for w_1, w_2, q_1, and q_2, we find that

$$\|w\|_1^2 + \|q\|_0^2 \leq C\Big[\|F\|_{-1}^2 + |G|_{0.5}^2 + \epsilon_1 M\Big].$$

For $s \geq 1$, let $0 < \epsilon_s \leq \frac{1}{2C}$ and

$$M = 2C\Big[\|F\|_{s-2}^2 + |G|_{s-0.5}^2\Big]. \tag{3.22}$$

We find that

$$\|v\|_s^2 + \|q\|_{s-1}^2 \le M\,.$$

The map $(\tilde{v}, \tilde{q}) \mapsto (v, q)$ is easily seen to be weakly continuous, and by the Tychonoff fixed-point theorem (see [10]), there is a fixed point (v, q) to (3.19) with estimates (3.17) and (3.18). The uniqueness follows from the linearity of the equation and (3.17) or (3.18). ◻

3.3.2. The case of time-dependent domains $\boldsymbol{\eta(t)(\Omega)}$. In the time-dependent setting, we have a material velocity field $v \in L^2(0,T;H^3(\Omega))$ and the associated Lagrangian coordinate

$$\eta(x,t) = x + \int_0^t v(x,s)ds\,.$$

Since $A_t = -A\nabla v A$,

$$A(t) = \mathrm{Id} - \int_0^t A\nabla v A ds\,.$$

Therefore, by Hölder's inequality,

$$\begin{aligned}\sup_{t\in[0,T]} \|A(t)\|_2 &\le \mathfrak{n}|\Omega| + C\int_0^t \|A\|_2^2 \|\nabla v\|_2 ds \le \mathfrak{n}|\Omega| + C\Big[\sup_{t\in[0,T]} \|A(t)\|_2^2\Big] \int_0^t \|v\|_3 ds \\ &\le \mathfrak{n}|\Omega| + C\Big[\sup_{t\in[0,T]} \|A(t)\|_2^2\Big]\sqrt{t}\|v\|_{L^2(0,T;H^3(\Omega))}\,.\end{aligned}$$

Choosing $T > 0$ so that $\sqrt{T}\|v\|_{L^2(0,T;H^3(\Omega))} \le \frac{1}{4CM_0}$, where C is the constant from the inequality above, we conclude that

$$\sup_{t\in[0,T]} \|A(t)\|_2 \le 2M_0\,.$$

Therefore,

$$\|A(t) - \mathrm{Id}\|_2 \le C\int_0^t \|A(s)\|_2^2 \|v(s)\|_3 ds \le 4CM_0^2\sqrt{t}\|v\|_{L^2(0,T;H^3(\Omega))}.$$

If $T > 0$ is chosen sufficiently small such that $\sqrt{T}\|v\|_{L^2(0,T;H^3(\Omega))} \ll 1$, then the basic assumption 4 holds and (3.17) and (3.18) are verified for $(v(t), q(t))$ for $t \in [0,T]$.

3.4. Pressure as a Lagrange multiplier. In the following discussion, we define

$$H^{1;2}(\Omega;\Gamma) = \Big\{ w \in H^1(\Omega) \;\Big|\; w \in H^2(\Gamma) \Big\}\,.$$

With the norm

$$\|u\|_{H^{1;2}(\Omega;\Gamma)}^2 = \|u\|_1^2 + |u|_2^2\,,$$

the space $H^{1;2}(\Omega;\Gamma)$ is a Hilbert space.

LEMMA 3.2. *Let η be given as in section 3.3.2. There exists a $T>0$ depending on $\|\eta_t\|^2_{L^2(0,T;H^3(\Omega))}$ so that for all $q(t)\in L^2(\Omega)$, $t\in[0,T]$, there exists a constant $C>0$ and a vector field $\phi(t)\in H^{1;2}(\Omega;\Gamma)$ such that $A^j_i(t)\phi^i_{,j}=q$ and*

$$\|\phi\|_{H^{1;2}(\Omega;\Gamma)} \le C\|q\|_0 \,. \tag{3.23}$$

Proof. We write $\phi(x,t)=\varphi(x,t)+\eta(x,t)\bar{q}(t)$, where

$$\begin{aligned} A^j_i(t)\varphi^i_{,j} &= q-\bar{q} && \text{in } \Omega\,,\\ \varphi^i &= 0 && \text{on } \Gamma\,, \end{aligned}$$

and $\bar{q}(t)=\frac{1}{|\Omega|}\int_\Omega q(x,t)dx$. Let $\varphi^i=\psi^i+A^k_i\Phi_{,k}$, where Φ satisfies

$$\begin{aligned} (A^k_iA^j_i\Phi_{,k})_{,j} &= q-\bar{q} && \text{in } \Omega\,,\\ A^k_i\Phi_{,k}N_i &= 0 && \text{on } \Gamma\,, \end{aligned}$$

and ψ satisfies

$$A^j_i\psi^i_{,j} = 0 \qquad \text{in } \Omega\,, \tag{3.24a}$$

$$\psi^i = -A^k_i\Phi_{,k} \qquad \text{on } \Gamma\,. \tag{3.24b}$$

By standard elliptic estimates and the Sobolev embedding theorem we find that

$$\|\Phi\|_2 \le C\mathcal{P}(\|A\|_2)\|q(t)\|_0$$

for some polynomial function $\mathcal{P}$.

A solution of (3.24) can be obtained by considering the following Stokes problem:

$$\begin{aligned} -(A^k_iA^j_i\psi^i{}_{,k})_{,j} + A^k_ir_{,k} &= F && \text{in } \Omega\,.\\ A^k_i\psi^i{}_{,k} &= 0 && \text{in } \Omega\,.\\ \psi &= -A^k_i\Phi_{,k} && \text{on } \Gamma\,, \end{aligned}$$

where F is a smooth vector field. By (3.17),

$$\|\psi\|_1^2+\|r\|_0^2 \le C\Big[\|F\|^2_{-1}+|\nabla\Phi|^2_{0.5}\Big]\,.$$

Estimate (3.23) follows immediately from all the estimates above with $F=0$. ∎

Define a linear functional on $H^{1;2}(\Omega;\Gamma)$ by $L(\varphi)=(p,A^j_i(t)\varphi^i_{,j})_{L^2(\Omega)}$ for $\varphi\in H^{1;2}(\Omega;\Gamma)$. By the Riesz representation theorem, there is a bounded linear operator $Q(t):L^2(\Omega)\to H^{1;2}(\Omega;\Gamma)$ such that for all $\varphi\in H^{1;2}(\Omega;\Gamma)$,

$$(p,A^j_i(t)\varphi^i_{,j})_{L^2(\Omega)} = (Q(t)p,\varphi)_{H^{1;2}(\Omega;\Gamma)} := (Q(t)p,\varphi)_{H^1(\Omega)}+(Q(t)p,\varphi)_{H^2(\Gamma)}.$$

Letting $\varphi=Q(t)p$ shows that

$$\|Q(t)p\|_{H^{1;2}(\Omega;\Gamma)} \le C\|p\|_0$$

for some constant $C>0$. By Lemma 3.2, for every $p\in L^2(\Omega)$, there exists $\varphi\in H^{1;2}(\Omega;\Gamma)$ such that $A^j_i\varphi^i_{,j}=p$ with estimate $\|\varphi\|_{H^{1;2}(\Omega;\Gamma)}\le C\|p\|_0$; hence

$$\|p\|_0^2 \le \|Q(t)p\|_{H^{1;2}(\Omega;\Gamma)}\|\varphi\|_{H^{1;2}(\Omega;\Gamma)} \le C\|Q(t)p\|_{H^{1;2}(\Omega;\Gamma)}\|p\|_0,$$

which shows that $R(Q(t))$ is closed in $H^{1;2}(\Omega;\Gamma)$. Since $\bar{\mathcal{V}}_v(t) \subseteq R(Q(t))^\perp$ and $R(Q(t))^\perp \subseteq \bar{\mathcal{V}}_v(t)$, it follows that for each $t > 0$,

$$H^{1;2}(\Omega;\Gamma) = R(Q(t)) \oplus_{H^{1;2}(\Omega;\Gamma)} \bar{\mathcal{V}}_v(t). \tag{3.25}$$

We can now introduce our Lagrange multiplier.

LEMMA 3.3. *Let $\mathcal{L}(t) \in H^{1;2}(\Omega;\Gamma)'$ be such that $\mathcal{L}(t)\varphi = 0$ for any $\varphi \in \bar{\mathcal{V}}_v(t)$. Then there exists a unique $q(t) \in L^2(\Omega)$, which is termed the pressure function, satisfying*

$$\forall\ \varphi \in H^{1;2}(\Omega;\Gamma), \quad \mathcal{L}(t)(\varphi) = (q(t), A_i^j(t)\varphi^i_{,j})_{L^2(\Omega)}\,.$$

Moreover, there is a $C > 0$ (which does not depend on $t \in [0,T]$ and on the choice of $v \in C_T(M)$) such that

$$\|q(t)\|_0 \le C\|\mathcal{L}(t)\|_{H^{1;2}(\Omega;\Gamma)'}.$$

Proof. By the decomposition (3.25), for given A, let $\varphi = v_1 + v_2$, where $v_1 \in \mathcal{V}_v(t)$ and $v_2 \in R(Q(t))$. It follows that

$$\mathcal{L}(t)(\varphi) = \mathcal{L}(t)(v_2) = (\psi(t), v_2)_{H^{1;2}(\Omega;\Gamma)} = (\psi(t), \varphi)_{H^{1;2}(\Omega;\Gamma)}$$

for a unique $\psi(t) \in R(Q(t))$.

From the definition of $Q(t)$ we then get the existence of a unique $q(t) \in L^2(\Omega)$ such that

$$\forall\ \varphi \in H^{1;2}(\Omega;\Gamma), \quad \mathcal{L}(t)(\varphi) = (q(t), A_i^j(t)\varphi^i_{,j})_{L^2(\Omega)}.$$

The estimate stated in the lemma is then a simple consequence of (3.23). ∎

3.5. A polynomial-type inequality. For a constant $M \ge 0$, suppose that $f(t) \ge 0$, $t \mapsto f(t)$ is continuous, and

$$f(t) \le M + C\,t\,P(f(t))\,, \tag{3.26}$$

where P denotes a polynomial function and C is a generic constant. Then for t taken sufficiently small, we have the bound

$$f(t) \le 2M\,.$$

This type of inequality, introduced in [8], can be viewed as a generalization of standard nonlinear Gronwall inequalities.

4. Statement of the main results. In order to state our main theorems, it is necessary to explain the compatibility conditions which we require the initial data to satisfy. Because these conditions are somewhat novel, relative to compatibility conditions for standard parabolic evolution equations, we include the first section of our *user's guide* to explain the derivation of the compatibility conditions in a simpler linear setting.

4.1. User's guide, part I: The specification of compatibility conditions for parabolic equations.

4.1.1. A toy linear model. In this first section of our user's guide, we propose a relatively simple linear parabolic problem to illustrate the issue of compatibility conditions. Suppose that $\Omega = \mathbb{T}^1 \times (0,1)$. Given $\mathbf{a}_i^j(t)$ a smooth matrix that only depends on time with $\mathbf{a}_i^j(0) = \mathrm{Id}_i^j$, let $u \in \mathbb{R}^2$, $p \in \mathbb{R}$ satisfy

$$u_t^i - \nu \mathbf{a}_\ell^k (\mathbf{a}_\ell^j u^i_{,j})_{,k} + \mathbf{a}_i^k p_{,k} = 0 \qquad \text{in} \quad \Omega \times (0,T)\,, \tag{4.1a}$$

$$\mathbf{a}_i^j u^i_{,j} = 0 \qquad \text{in} \quad \Omega \times (0,T)\,, \tag{4.1b}$$

$$-\Big[\nu(\mathbf{a}_i^j u^k_{,j} + \mathbf{a}_k^j u^i_{,j}) - p\mathrm{Id}_i^k\Big]\mathbf{a}_k^\ell N_\ell = h(t) \qquad \text{on} \quad \partial\Omega \times (0,T)\,, \tag{4.1c}$$

$$u = u_0 \qquad \text{on} \quad \Omega \times \{t=0\}, \tag{4.1d}$$

where $h(t)$ is a vector-valued boundary forcing function. Let $u_1 = u_t(0)$; then

$$\operatorname{div} u_1 = (\mathbf{a}_i^j u^i_{t,j})(0) = (\mathbf{a}_i^j u^i_{,j})_t(0) - (\mathbf{a}_t)_i^j(0) u^i_{0,j} = -\mathbf{b}_i^j u^i_{0,j}\,, \tag{4.2}$$

where the notation $\mathbf{b} = \mathbf{a}_t(0)$ is used. Letting $t = 0$ in (4.1a) and then letting div act on both sides of the resulting equation, we find that $p_0 = p(0)$ satisfies

$$-\Delta p_0 = -\mathbf{b}_i^j u^i_{0,j} \qquad \text{in} \quad \Omega\,.$$

In order to invert $-\Delta$, a boundary condition is required. By (4.1c),

$$p_0 N = \nu \operatorname{Def} u_0 N + h(0) \qquad \text{on} \quad \partial\Omega\,. \tag{4.3}$$

Therefore, p_0 satisfies

$$-\Delta p_0 = -\mathbf{b}_i^j u^i_{0,j} \qquad \text{in} \quad \Omega\,, \tag{4.4a}$$

$$p_0 N = \nu \operatorname{Def} u_0 N + h(0) \qquad \text{on} \quad \partial\Omega\,. \tag{4.4b}$$

Equation (4.4) in general is not solvable unless $\nu \operatorname{Def} u_0 N - h(0)$ points in the normal direction; nevertheless, if there is a solution to (4.4), then p_0 must satisfy a standard elliptic equation:

$$-\Delta p_0 = -\mathbf{b}_i^j u^i_{0,j} \qquad \text{in} \quad \Omega\,, \tag{4.5a}$$

$$p_0 = \nu(\operatorname{Def} u_0 N)\cdot N + h(0)\cdot N \qquad \text{on} \quad \partial\Omega. \tag{4.5b}$$

Therefore, in order for problem (4.1) to have a strong solution (so that the boundary condition (4.1c) makes sense), u_0 must at least satisfy a constraint—the so-called first order compatibility condition (to (4.1))—that

$$\nu \operatorname{Def} u_0 N + h(0) \text{ is parallel to } N\,,$$

or, equivalently,

$$\mathbf{P}_{\tan}\Big[\nu \operatorname{Def} u_0 N + h(0)\Big] = 0\,, \tag{4.6}$$

where $\mathbf{P}_{\tan}$ is the orthogonal projection onto the tangent plane of Γ, that is, for a vector w defined on Γ,

$$\mathbf{P}_{\tan} w = w - (w\cdot N)N\,.$$

Now suppose that $u_0 \in H^4(\Omega)$ satisfies the first compatibility condition. Then by elliptic regularity, $p_0 \in H^3(\Omega)$. Suppose that a strong solution u is continuously differentiable in time. Then it makes sense to talk about u_1 and, by (4.1a), $u_1 \in H^2(\Omega)$. Let $u_2 = u_{tt}(0)$. Similarly to (4.2),

$$\begin{aligned}\operatorname{div} u_2 = (\mathbf{a}_i^j u^i_{tt,j})(0) &= (\mathbf{a}_i^j u^i_{,j})_{tt}(0) - (\mathbf{a}_{tt})_i^j(0) u^i_{0,j} - 2(\mathbf{a}_t)_i^j(0) u^i_{1,j} \\ &= -\mathbf{c}_i^j u^i_{0,j} - 2\mathbf{b}_i^j u^i_{1,j}\,,\end{aligned}$$

where the notation $\mathbf{c} = \mathbf{a}_{tt}(0)$ is used. Therefore, time differentiating (4.1a) and letting $t = 0$ and then letting div act on both sides of the resulting equation, we find that $p_1 \equiv p_t(0)$ satisfies

$$\begin{aligned}-\Delta p_1 &= \operatorname{div} u_2 - \nu\Delta \operatorname{div} u_1 + \mathbf{b}_i^k p_{0,ki} \\ &= -\mathbf{c}_i^j u^i_{0,j} - 2\mathbf{b}_i^j u^i_{1,j} + \nu \mathbf{b}_i^j u^i_{0,jkk} + \mathbf{b}_i^k p_{0,ki} \qquad \text{in} \quad \Omega\,,\end{aligned}$$

while time differentiating (4.1c) and letting $t = 0$ lead to

$$-\nu\Big[(\operatorname{Def} u_1)^i_k + \mathbf{b}_i^j u^k_{0,j} + \mathbf{b}_k^j u^i_{0,j} - p_1 \operatorname{Id}^k_i\Big] N_k - \nu\Big[(\operatorname{Def} u_0)^i_k - p_0 \operatorname{Id}^i_k\Big]\mathbf{b}_k^\ell N_\ell = h_t(0)^i\,.$$

As a consequence, p_1 satisfies

$$\begin{aligned}&\text{(4.7a)} \quad -\Delta p_1 = -\mathbf{c}_i^j u^i_{0,j} - 2\mathbf{b}_i^j u^i_{1,j} - 2\mathbf{b}_i^j u^i_{1,j} - \nu \mathbf{b}_i^j u^i_{0,jkk} + \mathbf{b}_i^k p_{0,ki} \qquad \text{in} \quad \Omega\,, \\ &\qquad p_1 N_i = h_t(0)^i + \nu[(\operatorname{Def} u_1)^i_k + \mathbf{b}_i^j u^k_{0,j} + \mathbf{b}_k^j u^i_{0,j}] N_k \\ &\text{(4.7b)} \qquad\qquad + [\nu(\operatorname{Def} u_0)^i_k - p_0 \operatorname{Id}^i_k]\mathbf{b}_k^\ell N_\ell \qquad \text{on} \quad \partial\Omega\,.\end{aligned}$$

Therefore, in order to have a strong solution with better regularity, that is, a solution that is continuously differentiable in time, the identity

(4.8)
$$\mathbf{P}_{\tan}\Big[h_t(0)^i + \nu[(\operatorname{Def} u_1)^i_k + \mathbf{b}_i^j u^k_{0,j} + \mathbf{b}_k^j u^i_{0,j}] N_k + [\nu(\operatorname{Def} u_0)^i_k - p_0 \operatorname{Id}^i_k]\mathbf{b}_k^\ell N_\ell\Big] = 0$$

must hold. Equation (4.8) is called the second order compatibility condition to (4.1). We emphasize that *the second compatibility condition is needed only when looking for solutions that are continuously differentiable in time.*

4.1.2. A regularization of our toy linear model. Now suppose that we want to add an artificial viscosity on the boundary $\partial\Omega$. An intuitive way of adding the viscosity on the boundary is to modify the boundary condition as

$$-(\nu \operatorname{Def} u - p\operatorname{Id})N = f(t) + \kappa\Delta_0^2 u\,,$$

where Δ_0 denotes the surface Laplacian (and in our example, $\Delta_0 = \frac{\partial^2}{\partial x_1^2}$); however, for this kind of artificial viscosity, even though u_0 satisfies (4.6), in general it does not satisfy the corresponding compatibility condition

$$\mathbf{P}_{\tan}\Big[\nu \operatorname{Def} u_0 N + f(0) + \kappa\Delta_0^2 u_0\Big] = 0\,.$$

Moreover, adding this artificial viscosity requires more regularity on the initial data.

Suppose that u_0 is smooth enough and satisfies (4.6). To overcome the issue of compatibility, we instead consider

$$u_t - \nu \mathbf{a}_\ell^k (\mathbf{a}_\ell^j u^i_{,j})_{,k} + \mathbf{a}_i^k p_{,k} = 0 \quad \text{in } \Omega \times (0,T)\,, \tag{4.9a}$$

$$\mathbf{a}_i^j u^i_{,j} = 0 \quad \text{in } \Omega \times (0,T)\,, \tag{4.9b}$$

$$-\Big[\nu(\mathbf{a}_i^j u^k_{,j} + \mathbf{a}_k^j u^i_{,j}) - p\mathrm{Id}_i^k\Big]\mathbf{a}_k^\ell N_\ell = f(t) + \kappa\Delta_0^2(u-u_0) \quad \text{on } \partial\Omega \times (0,T)\,, \tag{4.9c}$$

$$u = u_0 \quad \text{on } \Omega \times \{t=0\}\,. \tag{4.9d}$$

Then at time $t = 0$, we essentially add nothing on the boundary, and the first order compatibility condition to (4.9) is the same as (4.6).

However, in general u_0 does not satisfy the second order compatibility condition to (4.9); that is, in general the quantity

$$\begin{aligned}\mathbf{P}_{\tan}\Big[& f_t(0)^i + \kappa\Delta_0^2 u_1 + \nu[(\mathrm{Def}\, u_1)^i_k + \mathbf{b}_i^j u^k_{0,j} + \mathbf{b}_k^j u^i_{0,j}]N_k \\ &+ [\nu(\mathrm{Def}\, u_0)^i_k - p_0 \mathrm{Id}^i_k]\mathbf{b}_k^\ell N_\ell\Big]\end{aligned}$$

does not vanish. In order to introduce an artificial viscosity in which the first and the second order compatibility conditions (of the approximated problem) both hold, we modify the boundary condition (4.9c) as

$$-(\nu\,\mathrm{Def}\, u - p\mathrm{Id})N = f(t) + \kappa\Delta_0^2(u - u_0 - tu_1). \tag{4.9c$'$}$$

Note that the first and the second order compatibility conditions to (4.9′) (that is, (4.9) with (4.9c′) replacing (4.9c)) is the same as (4.6) and (4.8).

4.2. Compatibility conditions for the real fluid-shell problem (2.5). As discussed in section 4.1, we now state the first and the second order compatibility conditions to (2.5).

4.2.1. The case $\mathfrak{n} = 2$. When $\mathfrak{n} = 2$, only the first order compatibility condition is needed for the existence of the solution to (2.5). As discussed in section 4.1, the first order compatibility condition is equivalent to the validity of the boundary condition (2.5d) at time $t = 0$. Since $\mathcal{L}_m(\eta)$ and $\mathcal{L}_b(\eta)$ both vanish at time $t = 0$, the first order compatibility condition to (2.5) is then

$$\mathbf{P}_{\tan}\Big[\nu\,\mathrm{Def}\, u_0 + \bar\sigma\varepsilon(\nu\Delta u_0 - \nabla q_0 + f(0))\Big] = 0\,, \tag{4.10}$$

where q_0 is the solution to (3.3) or (3.7).

4.2.2. The case $\mathfrak{n} = 3$. When $\mathfrak{n} = 3$, the first and the second order compatibility conditions are both needed for the existence of a solution to (2.5). The first order compatibility for the case $\mathfrak{n} = 3$ is the same as (4.10). To obtain the second order compatibility condition, suppose that the solution is continuously differentiable in time; then time differentiating (2.5d) and setting $t = 0$ in the resulting equation lead to

$$\begin{aligned} q_1 N_i = \bar\sigma\varepsilon \mathrm{w}_2 + \Big[&\nu(\mathrm{Def}\,\mathrm{w}_1)^j_i - \nu(u^k_{0,j}u^i_{0,k} + 2u^k_{0,i}u^j_{0,k} + u^i_{0,k}u^j_{0,k}) + q_0 u^j_{0,i}\Big]N_j \\ &+ \Big[\varepsilon\mathcal{L}_m(\eta) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\eta)\Big]^i_t(0) \qquad \text{on } \Gamma\,,\end{aligned}$$

which in turn implies that

$$\mathbf{P}_{\tan}\Big[\nu(\mathrm{Def\,w}_1)^j_i N_j - \nu(u^k_{0,j}u^i_{0,k} + 2u^k_{0,i}u^j_{0,k} + u^i_{0,k}u^j_{0,k})N_j + q_0 u^j_{0,i}N_j$$

$$+\Big(\varepsilon\mathcal{L}_m(\eta) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\eta)\Big)^i_t(0) + \bar{\sigma}\mathrm{w}_2\Big] = 0\,, \tag{4.11}$$

where w_2 is defined in (3.10). Equation (4.11) is the second order compatibility condition to (2.5).

4.3. Existence of initial data satisfying the compatibility conditions. In this subsection, we show that the set of velocity fields satisfying the first (and the second) order compatibility conditions is not empty. In fact, there are many such vector fields satisfying these two compatibility conditions.

4.3.1. The case $\mathfrak{n} = 2$. We will construct a vector u_0 satisfying the first order compatibility condition (4.10). In fact, a vector u_0 satisfies (4.10) if and only if

$$-\nu\Delta u_0 + \nabla r_0 = h + f(0) \qquad \text{in} \quad \Omega\,, \tag{4.12a}$$

$$\mathrm{div}\, u_0 = 0 \qquad \text{in} \quad \Omega\,, \tag{4.12b}$$

$$(-\nu\,\mathrm{Def}\, u_0 + r_0\mathrm{Id})N = \bar{\sigma}\varepsilon h \qquad \text{on} \quad \Gamma \tag{4.12c}$$

for some h. Since (4.12) is always solvable as long as h and $f(0)$ are smooth enough, we conclude that there exists u_0 satisfying (4.10) for any external forcing $f(0) \in H^1(\Omega)'$ and equilibrium state Γ.

4.3.2. The case $\mathfrak{n} = 3$. For $\mathfrak{n} = 3$, we consider only the case without inertia. We first claim that u_0 satisfies the first order compatibility condition (4.10) if and only if

$$-\nu\Delta u_0 + \nabla r_0 = h_1 \qquad \text{in} \quad \Omega\,, \tag{4.13a}$$

$$\mathrm{div}\, u_0 = 0 \qquad \text{in} \quad \Omega\,, \tag{4.13b}$$

$$(-\nu\,\mathrm{Def}\, u_0 + r_0\mathrm{Id})N = 0 \qquad \text{on} \quad \Gamma \tag{4.13c}$$

for some r_0 and h_1. The "if" part is straightforward, and we only need to prove the "only if" part. Suppose that u_0 satisfies the first order compatibility condition (4.10). Let r_0 be the scalar function solving

$$-\Delta r_0 = 0 \qquad \text{in} \quad \Omega\,,$$

$$r_0 = \nu(\mathrm{Def}\, u_0 N)\cdot N \qquad \text{on} \quad \Gamma\,.$$

Then $h_1 = -\nu\Delta u_0 + \nabla r_0$ in (4.13) will produce this desired u_0. Given such a u_0, we will use $\Delta_s u_0$ to denote this h_1, and given h_1, the solution u_0 to (4.13) will be denoted by $\Delta_s^{-1}h_1$.

Similarly, the second order compatibility condition (4.11) holds if and only if (w_1, r_1) solves

$$-\nu\Delta\mathrm{w}_1 + \nabla r_1 = h_2 \qquad \text{in}\ \Omega\,, \tag{4.14a}$$

$$\mathrm{div\,w}_1 = -u^j_{0,i}u^i_{0,j} \qquad \text{in}\ \Omega\,, \tag{4.14b}$$

$$(-\nu\,\mathrm{Def\,w}_1 + r_1\mathrm{Id})N_i = \nu\Big[u^k_{0,j}u^i_{0,k} + 2u^k_{0,i}u^j_{0,k} + u^i_{0,k}u^j_{0,k}\Big]N_j$$

$$+ r_0 u^j_{0,i}N_j + \Big[\varepsilon\mathcal{L}_m(\eta) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\eta)\Big]^i_t(0) \quad \text{on}\ \Gamma \tag{4.14c}$$

for some h_2. Note that the last term in (4.14c) involves only the derivatives of u_0 and Γ, and r_0 on Γ depends only on u_0. By the superposition principle, the solution w_1 to (4.14) can be expressed as $\mathrm{w}_1 = \Delta_s^{-1} h_2 + \mathcal{F}_1(u_0)$ for some map $\mathcal{F}_1$.

Let $\mathcal{F}$ be a map defined by

$$\mathcal{F}(h_1, h_2) = h_1 + \mathcal{F}_1(\Delta_s^{-1} h_1) + \Delta_s^{-1} h_2 \,.$$

If $f(0)$ is in the image of $\mathcal{F}$, then there exist h_1 and h_2 so that the solution u_0 and w_1 to (4.13) and (4.14) satisfies the first and the second order compatibility conditions. In particular, $f(0)$ satisfying $\mathbf{P}_{\tan}(\mathrm{Def}\, f(0) N) = 0$ is in the image of $\mathcal{F}$.

Conversely, if the first and the second order compatibility conditions hold for the initial data u_0, $f(0)$ must be in the image of $\mathcal{F}$.

4.4. The main theorems. Having defined the compatibility conditions for the fluid-shell interaction problem, we now state the main theorems.

THEOREM 4.1 (the case $\mathfrak{n} = 2$, $\bar{\sigma} = 0$ or 1). *Let $\nu > 0$, $\varepsilon > 0$, $f \in \mathcal{F}^2(T)$, and Γ be of class $H^{4.5}$. If $u_0 \in H^{2+\bar{\sigma}}(\Omega)$ is divergence-free and satisfies the first order compatibility condition* (4.10)*, then there exists a solution of* (2.5) *with $v \in \mathcal{V}^3(T)$ for some $T > 0$ depending on ε, ν, $\|u_0\|_{2+\bar{\sigma}}$, and $\|f\|_{\mathcal{F}^2(T)}$, and $\Gamma(t)$ is of class $H^{4.5}$ for almost all $t \in (0, T)$.*

Furthermore, if in addition Γ is of class $H^{6.5}$, $f \in \mathcal{F}^4(T)$, and $u_0 \in H^5(\Omega)$ and satisfies the second order compatibility condition (4.11)*, then the solution is unique, with $v \in \mathcal{V}^5(T)$ and $\Gamma(t)$ of class $H^{6.5}$ for almost all $t \in (0, T)$.*

THEOREM 4.2 (the case $\mathfrak{n} = 3$, $\bar{\sigma} = 0$). *Let $\nu > 0$ so that $\frac{\varepsilon}{\nu} \ll 1$, $\varepsilon \ll 1$, $f \in \mathcal{F}^4(T)$, and Γ is of class $H^{6.5}$. If $u_0 \in H^5(\Omega)$ is divergence-free so that the first and the second order compatibility conditions* (4.10), (4.11) *hold, then there exists a unique solution $v \in \mathcal{V}^5(T)$ for some $T > 0$ depending on ε, ν, $\|u_0\|_5$, and $\|f\|_{\mathcal{F}^4(T)}$, and $\Gamma(t)$ is of class $H^{6.5}$ for almost all $t \in (0, T)$.*

Remark 5. Surprisingly, we are unable to establish continuity-in-time for solutions to (2.5). The derivative loss can clearly be seen in estimates for the time derivatives of the initial data. To be precise, the fundamental new feature of this particular Navier–Stokes systems is the effect on the pressure (and its time derivatives) by the shell traction; specifically, the reader should observe that

$$\mathcal{L}_b(\eta)|_{t=0} = 0 \quad \text{and} \quad \mathcal{L}_m(\eta)|_{t=0} = 0 \,, \tag{4.15}$$

while

$$\partial_t \mathcal{L}_b(\eta)|_{t=0} \neq 0 \quad \text{and} \quad \partial_t \mathcal{L}_m(\eta)|_{t=0} \neq 0 \,. \tag{4.16}$$

For the 3D fluid, this means that if $u(0) \in H^5(\Omega)$, because of (4.15) and the elliptic estimate, the initial pressure $q(0) \in H^4(\Omega)$; on the other hand, from the Navier–Stokes equations at time $t = 0$, we then see that $v_t(0) \in H^3(\Omega)$, but due to (4.16), elliptic estimates show that $q_t(0) \in H^1(\Omega)$. We hence observe that while $q(0)$ loses one derivative with respect to $v(0)$, we see that $q_t(0)$ loses two derivatives with respect to $v_t(0)$.

Again, the Navier–Stokes equations at time $t = 0$ show that $v_{tt}(0) \in L^2(\Omega)$, and hence derivative loss ensues. We have the following initial regularity:

$$v(0) \in H^5(\Omega) \longrightarrow v_t(0) \in H^3(\Omega) \longrightarrow v_{tt}(0) \in L^2(\Omega) \,.$$

The three-derivative loss of $v_{tt}(0)$ implies that $v_{tt}(t)$ cannot be better than $L^2(\Omega)$, and we are consequently unable to establish the type of parabolic regularity expected of parabolic systems, from which continuity-in-time of solutions would follow.

Remark 6. For the 2D fluid, we require more regularity on the initial data to establish uniqueness. The uniqueness argument relies on closing energy estimates in the space $\mathcal{V}^3(T)$, but in order to control certain error terms, we require more regularity on the supposed solutions whose difference we are estimating. Due to the nonlinearities of the bending energy of the Koiter shell, we are unable to close estimates in $\mathcal{V}^k(T)$ for any even integer $k \geq 4$. As such, for the uniqueness proof, we are forced to assume that the two supposed solutions are indeed in $\mathcal{V}^5(T)$, and thus require the extra regularity on the initial data.

For the 3D fluid, the Sobolev embedding theorem forces us to prove existence in $\mathcal{V}^5(T)$, but we are able to prove our uniqueness assertion, by comparing two solutions in the larger space $\mathcal{V}^3(T)$, and thus control all error terms in our energy estimates thanks to the added spatial regularity of the two supposed solutions, already imposed for the existence theory.

Remark 7. When $\mathfrak{n} = 3$ and $\bar{\sigma} = 1$, Theorem 4.2 is also valid if the initial data u_0 is more regular, or to be more precise, $u_0 \in H^8(\Omega)$; however, we do not know if in general there are u_0 and w_1 satisfying the first and the second order compatibility conditions. Nevertheless, the first and the second order compatibility conditions hold for some specific cases. For example, if $u_0 = 0$ and $f(t) = f(0) + t^2 g$ for some smooth g, where $f(0)$ is the solution to

$$\begin{aligned} \nu\Delta f(0) + \nabla r_0 &= -h \qquad \text{in} \quad \Omega\,, \\ \operatorname{div} f(0) &= 0 \qquad \text{in} \quad \Omega\,, \\ (\nu \operatorname{Def} f(0) + r_0 \mathrm{Id})N &= \varepsilon h \qquad \text{in} \quad \Gamma \end{aligned}$$

for some function h, then $\mathrm{w}_1 = -f(0)$ and $\mathrm{w}_2 = h$ so that the first and the second order compatibility conditions are valid.

Remark 8. For the case $\mathfrak{n} = 2$, the existence of a solution is guaranteed independent of the thickness ε; however, when $\mathfrak{n} = 3$, ν has to be much larger than ε in order to develop the existence theory.

Remark 9. For the case $\mathfrak{n} = 2$, the first order compatibility condition guarantees only the existence of a solution. In order to obtain the uniqueness of the solution, the initial data u_0 must satisfy the second order compatibility condition, as for the case $\mathfrak{n} = 3$.

5. The linearized and regularized problem. One potential solution strategy for (2.5) might be to linearize the nonlinear equation and apply some fixed-point arguments; however, linearizing $\mathcal{L}_m(\eta)$ and $\mathcal{L}_b(\eta)$ destroys the structure (see Remark 15 in section 7.2 for reference) and prevents the normal vector from gaining regularity. Therefore, we introduce a regularized version of the linearized problem by adding an extra artificial viscosity on the boundary for the purpose of smoothing the normal vector, and we prove that the estimate is independent of this artificial viscosity.

5.1. Regularizing the initial data. As discussed in section 4.1, in order to introduce the artificial viscosity of the type $\kappa\Delta_0^2 v$ on the boundary, the initial data has to be more regular than the regularity stated in Theorem 4.1 or 4.2; moreover, the initial data has to be regularized in a specific fashion to ensure the compatibility conditions. In this section, we regularize the initial data in a way that the first and second order compatibility conditions still hold.

We first regularize the boundary. There are many ways of regularizing the boundary, and we choose the following approach. Let $\Omega_{\tilde{\kappa}_j} \subseteq \Omega$ denote a (nested) sequence of monotone increasing open sets with smooth boundary $\Gamma_{\tilde{\kappa}_j}$, that is, $\Omega_{\tilde{\kappa}_i} \subseteq \Omega_{\tilde{\kappa}_j}$ if

$i \leq j$, and $\bigcup_j \Omega_{\tilde{\kappa}_j} = \Omega$ (which is conceptually the same as $\lim_{j\to\infty} \Omega_{\tilde{\kappa}_j} = \Omega$). We will then study (2.5) with Ω and Γ replaced by $\Omega_{\tilde{\kappa}_j}$ and $\Gamma_{\tilde{\kappa}_j}$ for some fixed j.

To regularize the initial velocity, we need to introduce the mollifiers and an extension operator. Let χ be a nonnegative, smooth function supported in the unit ball so that $\|\chi\|_{L^1(\mathbb{R}^{\mathfrak{n}})} = 1$, and the sequence of mollifiers is defined by $\chi_{\hat{\epsilon}}(x) = \hat{\epsilon}^{-\mathfrak{n}}\chi(\hat{\epsilon}^{-1}x)$. We use $\mathcal{E}_j$ to denote a Sobolev extension operator extending a vector field $H^s(\Omega_{\tilde{\kappa}_j})$ to a vector field in $H^s(\mathbb{R}^{\mathfrak{n}})$ for some $s > 0$.

For the sake of notational simplicity, from sections 5.1.1 through 8.7 we continue to use Ω and Γ to denote $\Omega_{\tilde{\kappa}_j}$ and $\Gamma_{\tilde{\kappa}_j}$, and use $\mathcal{E}$ to denote $\mathcal{E}_j$. Beginning with section 8.8, having already passed to the limit $\lim_{j\to\infty} \Omega_{\tilde{\kappa}_j} = \Omega$, the notation Ω and Γ once again refers to the actual domain and boundary.

5.1.1. The regularization of the velocity for the case $\mathfrak{n} = 2$. Suppose that the first order compatibility condition (4.10) holds for the initial data u_0. Compute q_0 and w_1 accordingly. Let $\tilde{f}_0 = \chi_{\hat{\epsilon}} * f(0)$ and $\tilde{\mathrm{w}}_1 = \chi_{\hat{\epsilon}} * \mathcal{E}\mathrm{w}_1$, then define a smooth function $h_1 = -\tilde{\mathrm{w}}_1 + \tilde{f}_0$. Let $\tilde{u}_0 = \Delta_s^{-1} h_1$ be the unique solution to (4.13). It is easy to see that $\tilde{u}_0 \to u_0$ in $H^3(\Omega)$ as $\tilde{\epsilon} \to 0$, and

$$\|\tilde{u}_0\|_{3+s} \leq C\tilde{\epsilon}^{-s}\|u_0\|_3\,. \tag{5.1}$$

The regularized forcing $\tilde{F}$ is defined by $\chi_{\hat{\epsilon}} * F$.

5.1.2. The regularization of the velocity for the case $\mathfrak{n} = 3$. When taking the second compatibility condition into account, we cannot regularize $f(0)$ in the way we did for the case $\mathfrak{n} = 2$ since $\chi_{\hat{\epsilon}} * f(0)$ in general is not in the image of $\mathcal{F}$.

Suppose that the compatibility conditions (4.10) and (4.11) hold for the initial data u_0. Compute q_0, w_1, and q_1 accordingly. Let $h_1 = \chi_{\hat{\epsilon}} * \mathcal{E}(-\nu\Delta u_0 + \nabla q_0)$ and $h_2 = \chi_{\hat{\epsilon}} * \mathcal{E}(-\nu\Delta \mathrm{w}_1 + \nabla q_1)$; then h_1 and h_2 are smooth functions. Let $\tilde{u}_0 = \Delta_s^{-1} h_1$, $\tilde{q}_0$ and $\tilde{\mathrm{w}}_1 = \Delta_s^{-1} h_2 + \mathcal{F}(\Delta_s^{-1} h_1)$, $\tilde{q}_1$ be the unique solution to (4.13) and (4.14), respectively. By the virtue of the mollification,

$$\|\tilde{u}_0\|_{5+s} \leq C\tilde{\epsilon}^{-s}\Big[\|u_0\|_5 + \|q_0\|_4\Big] \leq C\tilde{\epsilon}^{-s}\Big[1 + \|u_0\|_5^2\Big]\,, \tag{5.2}$$

and $\tilde{u}_0 \to u_0$ in $H^5(\Omega)$ and $\tilde{\mathrm{w}}_1 \to \mathrm{w}_1$ in $H^{3-\bar{\sigma}}(\Omega)$ as $\hat{\epsilon} \to 0$.

Define $\tilde{f}_0 = -\nu\Delta\tilde{u}_0 + \nabla\tilde{q}_0 + \tilde{\mathrm{w}}_1$,

$$\tilde{F} = \tilde{f}_0 + \chi_{\hat{\epsilon}} * \mathcal{E}\left(\int_0^t F_t(s)ds\right). \tag{5.3}$$

Finally, define $\tilde{\mathrm{w}}_2^i = h_2 + \tilde{u}^k_{0,i}\tilde{q}_{0,k} - \nu\big[\tilde{u}^j_{0,k}\tilde{u}^i_{0,kj} + \tilde{u}^k_{0,jj}\tilde{u}^i_{0,k} + \tilde{u}^k_{0,j}\tilde{u}^i_{0,kj}\big] + \tilde{F}^i_t(0)$. Then $\tilde{F}$ is smooth, $\tilde{F} \to F$ in $\mathcal{F}^4(T)$, and $\tilde{\mathrm{w}}_2 \to \mathrm{w}_2$ in $H^{\bar{\sigma}}(\Omega)$ as $\hat{\epsilon} \to 0$. Note that $\tilde{F}(0) = \tilde{f}_0$.

Remark 10. All the regularized variables, such as $\tilde{u}_0$ and $\tilde{f}_0$, depend on two parameters, $\tilde{\kappa}$ and $\hat{\epsilon}$. The dependence on $\hat{\epsilon}$ is through the convolution, while the dependence on $\tilde{\kappa}$ is through the regularization of the geometry of the domain, such as the outer normal $\tilde{N}$ to $\Omega_{\tilde{\kappa}_j}$ in the boundary condition (4.13c) or (4.14c).

5.2. The linearized problem with artificial viscosity. For initial data u_0, we introduce smooth vectors $\tilde{u}_0$, $\tilde{\mathrm{w}}_1$, $\tilde{\mathrm{w}}_2$, and $\tilde{F}$ as above. Given $\bar{v} \in \mathcal{V}^{2\mathfrak{n}-1}(T)$ with $\bar{v}(0) = \tilde{u}_0$, $\partial_t^k\bar{v}(0) = \tilde{\mathrm{w}}_k$, $k = 1, \mathfrak{n}-1$ so that

$$\|\bar{v}\|^2_{\mathcal{V}^{2\mathfrak{n}-1}(T)} + \sum_{k=0}^{\mathfrak{n}-2}\int_0^T |\partial_t^k\bar{v}(t)|^2_{2\mathfrak{n}+1.5-2k}dt \leq M$$

for some M determined later, let $\tilde{v} = \chi_{\hat{\epsilon}} * \mathcal{E}\bar{v}$ if $\mathfrak{n} = 2$ or

$$\tilde{v} = \tilde{u}_0 + t\tilde{\mathrm{w}}_1 + \chi_{\hat{\epsilon}} * \mathcal{E}\left(\int_0^t \int_0^s \bar{v}_{tt}(s')ds'ds\right) \tag{5.4}$$

if $\mathfrak{n} = 3$ with associate $\tilde{\eta}$, let $\tilde{A}$ be the inverse matrix of $\nabla\tilde{\eta}$, let $\tilde{n}$ be the associate normal vector, let $\tilde{g}_{\alpha\beta}$ be the associate metric tensor defined by $\tilde{g}_{\alpha\beta} = \tilde{\eta}_{,\alpha} \cdot \tilde{\eta}_{,\beta}$, and let $\tilde{b}_{\alpha\beta}$ be the associate covariant component of the second fundamental form defined by $\tilde{b}_{\alpha\beta} = \tilde{\eta}_{,\alpha\beta} \cdot \tilde{n}$. We consider the following linear problem:

$$\eta_t = v \quad \text{in} \quad (0,T)\times\Omega\,, \tag{5.5a}$$

$$v^i_t - \nu\tilde{A}^j_\ell(\tilde{A}^k_\ell v^i_{,k})_{,j} + \tilde{A}^k_i q_{,k} = \tilde{F}^i \quad \text{in} \quad (0,T)\times\Omega\,, \tag{5.5b}$$

$$\tilde{A}^j_i v^i_{,j} = 0 \quad \text{in} \quad (0,T)\times\Omega\,, \tag{5.5c}$$

$$-\Big[\nu(D_{\tilde{A}}v)^j_i - q\mathrm{Id}^i_j\Big]\tilde{A}^\ell_j N_\ell = \left[\varepsilon\mathcal{L}_m(\tilde{\eta}) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\tilde{\eta})\right]^i + \bar{\sigma}\varepsilon\eta^i_{tt} + \kappa\varepsilon^3\mathcal{L}(v)^i \quad \text{on} \quad (0,T)\times\Gamma\,, \tag{5.5d}$$

$$v(0) = \tilde{u}_0 \quad \text{on} \quad \{t=0\}\times\Omega\,, \tag{5.5e}$$

$$\eta(0) = \mathbf{e} \quad \text{on} \quad \{t=0\}\times\Omega\,, \tag{5.5f}$$

where the smooth parameter $\hat{\epsilon}$ is fixed to be $\kappa^{1/4}$, $\tilde{F} = \tilde{f}\circ\tilde{\eta}$, $\mathcal{L}_m(\tilde{\eta})$ and $\mathcal{L}_b(\tilde{\eta})$ are treated as given forcing terms defined by

$$\mathcal{L}_m(\tilde{\eta}) = -\frac{1}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{g}_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\tilde{\eta}_{,\gamma}\Big]_{,\delta}\,,$$

$$\mathcal{L}_b(\tilde{\eta}) = \frac{2}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{b}_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\tilde{n}\Big]_{,\gamma\delta} + \frac{2}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{b}_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\tilde{\Gamma}^\tau_{\gamma\delta}\tilde{n}\Big]_{,\tau}\,,$$

and the operator $\mathcal{L}$, following [9], is defined as

$$\begin{aligned}\mathcal{L}(v) &= \frac{1}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}v_{,\alpha\beta}\Big]_{,\gamma\delta} - \frac{1}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{u}_0 + (\mathfrak{n}-2)t\tilde{\mathrm{w}}_1)_{,\alpha\beta}\Big]_{,\gamma\delta}\\ &:= \frac{1}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}v_{,\alpha\beta}\Big]_{,\gamma\delta} - L_0\,.\end{aligned}$$

Note that $\mathcal{L}(v)$ satisfies

$$\mathcal{L}(v)\Big|_{t=0} = 0 \;\text{ if }\; \mathfrak{n} = 2,3 \;\text{ and }\; [\mathcal{L}(v)]_t\Big|_{t=0} = 0 \;\text{ if }\; \mathfrak{n} = 3,$$

and $\mathcal{L}_b(\tilde{\eta})$ satisfies

$$\sum_{k=0}^{\mathfrak{n}-1}\int_0^T |\partial_t^k\mathcal{L}_b(\tilde{\eta})|^2_{2\mathfrak{n}-2.5-2k}dt \le C\left[\|\tilde{v}\|^2_{\mathcal{V}^{2\mathfrak{n}-1}(T)} + \sum_{k=0}^{\mathfrak{n}-1}\int_0^T |\partial_t^k\tilde{\eta}|^2_{2\mathfrak{n}+1.5-2k}dt\right]\,, \tag{5.6}$$

where we assume T (depending on M) is chosen small enough so that $\|\tilde{\eta} - \mathbf{e}\|^2_{2\mathfrak{n}-1} \le C\sqrt{T}$ for some constant C independent of M.

Remark 11. As we explain in section 4.1, $\mathcal{L}(v)$ is designed in such a way that (5.5d) and its time derivative hold at time $t = 0$. Therefore, $\tilde{u}_0$ satisfied the corresponding compatibility conditions to (5.5). Note that (5.3) and (5.4) are also used in order to guarantee the compatibility conditions.

Remark 12. An elliptic-type estimate from the study of (5.5d) is important to close the estimates (see section 7 for details). Therefore, the extra forcing term κL_0 appearing in (5.5d) have to be as smooth as the normal traction. Since we are looking for solutions $v \in \mathcal{V}^{2\mathfrak{n}-1}(T)$, κL_0 must be an $L^2(0,T;H^{2\mathfrak{n}-2.5}(\Gamma))$-vector, which is not the case if $u_0 \in H^r(\Omega)$ for some $2\mathfrak{n}-1 \le r < 4\mathfrak{n}-2$. Therefore, u_0 has to be regularized.

Suppose that $\hat{\epsilon} = \kappa^\alpha$. By (5.1) and (5.2), we find that for the case $\mathfrak{n} = 2$,

$$\|\tilde{u}_0\|_{2\mathfrak{n}+2} \le C\kappa^{(1-2\mathfrak{n})\alpha}\|u_0\|_3 \le C\kappa^{-3\alpha}\|u_0\|_3\,,$$

while for the case $\mathfrak{n} = 3$,

$$\|\tilde{u}_0\|_{2\mathfrak{n}+4} \le C\kappa^{(1-2\mathfrak{n})\alpha}\Big[1 + \|u_0\|_5^2\Big] \le C\kappa^{-5\alpha}\Big[1 + \|u_0\|_5^2\Big]\,.$$

Therefore, $|\kappa L_0|_{2\mathfrak{n}-2.5} \to 0$ as $\kappa \to 0$ for $\alpha = 1/4$ since

$$\begin{aligned} &|\kappa L_0|_{2\mathfrak{n}-2.5} \le C\kappa\Big[\|\tilde{u}_0\|_{2\mathfrak{n}+2} + (\mathfrak{n}-2)t\|\tilde{\mathrm{w}}_1\|_{2\mathfrak{n}+2}\Big] \\ &\le C\kappa\Big[\|\tilde{u}_0\|_{2\mathfrak{n}+2} + (\mathfrak{n}-2)t\|\tilde{u}_0\|^2_{2\mathfrak{n}+4}\Big] \le \begin{cases} C\kappa^{-3\alpha}\|u_0\|_3 & \text{if } \ \mathfrak{n}=2\,, \\ C\kappa^{1-5\alpha}\Big[1+\|u_0\|_5^2\Big] & \text{if } \ \mathfrak{n}=3\,. \end{cases} \end{aligned}$$

5.3. User's guide, part II: The penalization for our toy model. In this subsection, we again use our toy model to illustrate the methodology of constructing a solution to (5.5). We first remind the reader that for the case $\mathbf{a}_i^j = \mathrm{Id}_i^j$ for all $t > 0$, (4.1) is a standard Stokes problem, and a weak solution to the problem can be constructed by the Galerkin scheme. For simplicity, let (4.1′) denote (4.1) with $\mathbf{a} = \mathrm{Id}$ for all $t > 0$. A vector-valued function u is called a weak solution to (4.1′) if, for all $\varphi \in H^1_{\mathrm{div}}(\Omega)$,

$$\langle u_t, \varphi\rangle + \frac{\nu}{2}\int_\Omega \mathrm{Def}\, u : \mathrm{Def}\,\varphi dx - \int_\Gamma f(t)\cdot\varphi dS = 0\,, \tag{5.7}$$

where $H^1_{\mathrm{div}}(\Omega)$ is a subspace of $H^1(\Omega)$ with zero divergence. Let $\{e_\ell\}_{\ell=1}^\infty$ be a basis of $H^1_{\mathrm{div}}(\Omega)$ that is orthonormal in $L^2(\Omega)$. The Galerkin scheme states that the weak solution to (4.1′) can be approximated by $u_\ell(x,t) = d_\ell^i(t)e_i(x)$, where d_ℓ^i solve the ODE

$$\frac{d}{dt}d_\ell^i(t) + \frac{\nu}{2}d_\ell^j(t)\int_\Omega \mathrm{Def}\, e_j(x) : \mathrm{Def}\, e_i(x)dx = \int_\Gamma f(t)\cdot e_i dS \quad \forall\ t \in [0,T]$$

with initial condition $d_\ell^i(0) = \int_\Omega \tilde{u}_0(x)e_\ell(x)dx$. We note that the time independence of e_j is important in obtaining the ODE above.

Another way of solving (4.1′) is to introduce a penalized problem. Letting θ be the penalization parameter, we look for $u_\theta \in L^2(0,T;H^1(\Omega))$ satisfying that for all $\varphi \in H^1(\Omega)$,

$$\langle u_{\theta t}, \varphi\rangle + \frac{\nu}{2}\int_\Omega \mathrm{Def}\, u_\theta : \mathrm{Def}\,\varphi dx + \frac{1}{\theta}\int_\Omega \mathrm{div}\, u_\theta\, \mathrm{div}\,\varphi dx = 0\,. \tag{5.8}$$

Note that in this formulation, the divergence-free constraint is removed from the space of test functions. A u_θ satisfying (5.8) can be obtained by the Galerkin scheme, and one can show that $\frac{1}{\theta}\operatorname{div} u_\theta$ has θ-independent estimate, and hence possesses the weak limit q. On the other hand, u_θ has a weak limit u, and by the existence of the weak limit of $\frac{1}{\theta}\operatorname{div} u_\theta$, $\operatorname{div} u$ has to vanish. Therefore, the solution of (4.1′) can be obtained by finding the (weak) limit of weak solutions to (5.8) as the penalization parameter θ approaches zero. Detailed arguments can be found, for example, in [18].

When $\mathbf{a}_i^j$ is time dependent, the Galerkin scheme stated for solving (5.7) fails to work. The main reason is that in order to satisfy the condition $\mathbf{a}_i^j u^i_{,j} = 0$, one needs to impose the same condition on the basis functions e_ℓ, which is generally not possible (unless e_ℓ is time dependent). Therefore, we consider the following penalized scheme. Let θ denote the penalization parameter, and let $u_\theta \in L^2(0,T;H^1(\Omega))$ satisfy that for all $\varphi \in H^1(\Omega)$,

$$\langle u_{\theta t}, \varphi\rangle + \frac{\nu}{2}\int_\Omega (\mathbf{a}_i^j u^k_{\theta,j} + \mathbf{a}_k^j u^i_{\theta,j})(\mathbf{a}_i^j \varphi^k_{,j} + \mathbf{a}_k^j \varphi^i_{,j})dx + \frac{1}{\theta}\int_\Omega \mathbf{a}_i^j u^i_{,\theta,j}\mathbf{a}_\ell^k \varphi^\ell_{,k} dx = 0\,.$$

Similarly, one can show that $\frac{1}{\theta}a_i^j u^i_{\theta,j}$ has θ-independent bound, and the limit of u_θ satisfies the constraint $\mathbf{a}_i^j u^i_{,j} = 0$.

5.4. Weak solutions to (5.5).

DEFINITION 5.1. *A vector $v \in \mathcal{V}_{\tilde v}(T)$ with $v_t \in \mathcal{V}_{\tilde v}(T)'$ is a weak solution to* (5.5)*, provided that*

$$\text{(5.9a)}\qquad \text{(i)}\ \ \langle v_t, \varphi\rangle + \bar\sigma\varepsilon\langle v_t, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v : D_{\tilde A}\varphi dx + \kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{,\alpha\beta}\varphi^i_{,\gamma\delta} dS$$
$$= \langle \tilde F, \varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta), \varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta), \varphi\rangle_\Gamma + \int_\Gamma \kappa\varepsilon^3 {L_0}^i \varphi^i dS\,,$$

$$\text{(5.9b)}\qquad \text{(ii)}\ \ v(0,\cdot) = \tilde u_0$$

for almost all $t \in [0,T]$, where $\langle\cdot,\cdot\rangle$ denotes the duality product between $\mathcal{V}_{\tilde v}(t)$ and its dual $\mathcal{V}_{\tilde v}(t)'$, and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality product between $H^2(\Gamma)$ and $H^{-2}(\Gamma)$.

As discussed in section 5.3, the nonlinear divergence-free constraint in Lagrangian coordinate creates technical difficulties in the construction of a weak solution to (5.5). Therefore, we introduce the following penalization approach to the problem.

DEFINITION 5.2 (the penalized problem). *Letting $\theta > 0$ denote the penalization parameter, a vector $v_\theta \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$ with $v_{\theta t} \in L^2(0,T;H^{1;2}(\Omega;\Gamma)')$ is a weak solution to the penalized problem if, for all $\varphi \in H^{1;2}(\Omega;\Gamma)$,*

$$\text{(i)}\ \ \langle v_{\theta t}, \varphi\rangle + \bar\sigma\varepsilon\langle v_{\theta t}, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v_\theta : D_{\tilde A}\varphi dx + \kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\theta,\alpha\beta}\varphi^i_{,\gamma\delta} dS$$
$$\text{(5.10a)}\qquad - (q_\theta, \tilde A^\ell_k \varphi^k_{,\ell})_{L^2(\Omega)}$$
$$= \langle \tilde F, \varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta), \varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta), \varphi\rangle_\Gamma + \int_\Gamma \kappa\varepsilon^3 {L_0}^i \varphi^i dS$$

$$\text{(5.10b)}\quad \text{(ii)}\ \ v_\theta(0,\cdot) = \tilde u_0$$

for almost all $t \in [0,T]$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between $H^{1;2}(\Omega;\Gamma)$ and its dual, and $q_\theta = \tilde q_0 - \frac{1}{\theta}\tilde A_i^j v^i_{\theta,j}$ if $\mathfrak{n} = 2$ or $q_\theta = \tilde q_0 + t\tilde q_1 - \frac{1}{\theta}\tilde A_i^j v^i_{\theta,j}$ if $\mathfrak{n} = 3$.

The goal of the following four sections is to establish the existence of a weak solution v to the problem (5.5) (or the weak formulation (5.9)), as well as the energy inequality satisfied by v and v_t via the study of (5.10). It is done by first finding a solution to the penalized problem (5.10) for all θ and proving that the solution has

θ-independent estimates. This θ-independent bound enables us to pass to a weak limit which we show satisfies (5.9).

5.5. Weak solutions for the penalization of (5.5). In the following, we assume that $\mathfrak{n} = 3$. The argument for the case $\mathfrak{n} = 2$ is similar, and it requires the study of only one time derivative of (5.10).

We start with finding $v_{\ell tt}$. By introducing a (smooth) basis $(e_\ell)_{\ell=1}^\infty$ of $H^{1;2}(\Omega;\Gamma)$, and taking the approximation at rank $m \geq 2$ under the form $v_\ell(t,x) = \sum_{k=1}^\ell d_k(t)e_k(x)$ and satisfying on $[0,T]$, for all $\varphi \in \operatorname{span}(e_1,\dots,e_\ell)$,

(5.11a)

$$\text{(i)}\quad \langle v_{\ell ttt},\varphi\rangle + \bar\sigma\varepsilon\langle v_{\ell ttt},\varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A}v_{\ell tt} : D_{\tilde A}\varphi dx + \kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{\ell tt,\alpha\beta}\varphi^i_{,\gamma\delta}dS$$
$$- ((\tilde A^j_i q_\ell)_{tt},\varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F_{tt},\varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta)_{tt},\varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta)_{tt},\varphi\rangle_\Gamma$$
$$- \nu\int_\Omega \Big[(\tilde A^m_i\tilde A^k_i)_{tt}v^j_{\ell,m} + (\tilde A^m_i\tilde A^k_j)_{tt}v^i_{\ell,m}\Big]\varphi^j_{,k}dx$$
$$- 2\nu\int_\Omega \Big[(\tilde A^m_i\tilde A^k_i)_{t}v^j_{\ell t,m} + (\tilde A^m_i\tilde A^k_j)_{t}v^i_{\ell t,m}\Big]\varphi^j_{,k}dx\,,$$

(5.11b)

$$\text{(ii)}\ v_{\ell tt}(0) = (\tilde{\mathrm{w}}_2)_\ell\,,\quad v_{\ell t}(0) = (\tilde{\mathrm{w}}_1)_\ell\,,\quad v_\ell(0) = (\tilde u_0)_\ell \ \text{ in } \Omega\,,$$

where $q_\ell = \tilde q_0 + t\tilde q_1 - \frac{1}{\theta}\tilde A^j_i v^i_{\ell,j}$, and $(\tilde u_0)_\ell$, $(\tilde{\mathrm{w}}_1)_\ell$, and $(\tilde{\mathrm{w}}_2)_\ell$ denote the $H^{1;2}(\Omega;\Gamma)$ projections of $\tilde u_0$, $\tilde{\mathrm{w}}_1$, and $\tilde{\mathrm{w}}_2$ on $\operatorname{span}(e_1,e_2,\dots,e_\ell)$, respectively. Note that $d(t)$ satisfies an ODE of the type

$$d'''(t) + C_1(t)d''(t) + C_2(t)d'(t) + C_3(t)d(t) = \mathfrak{F}(t) \quad t\in(0,T)\,,$$
$$d_k(0) = \langle(\tilde u_0)_\ell,e_k\rangle\,,\quad d'_k(0) = \langle(\tilde{\mathrm{w}}_1)_\ell,e_k\rangle\,,\quad d''_k(0) = \langle(\tilde{\mathrm{w}}_2)_\ell,e_k\rangle\,,$$

where C_i's $\in \mathcal{C}^1([0,T])$ and $\mathfrak{F} \in L^2(0,T)$. The existence and uniqueness of the solution to this ODE can be proved by Picard's iteration scheme as the proof for the fundamental theorem of ODE.

Remark 13. The process above is used to obtain a weak solution $v_{\theta tt}$ to the second time differentiated problem of the original penalized problem (5.10), that is, to obtain a weak solution to $(5.10)_{tt}$. In fact, a weak solution v_θ to (5.10) can be obtained by a similar Galerkin approximation, and as suggested by the notation,

$$v_\theta(t) = \tilde u_0 + \int_0^t v_{\theta t}(s)ds = \tilde u_0 + t\tilde{\mathrm{w}}_1 + \int_0^t\int_0^s v_{\theta tt}(s')ds'ds\,.$$

The advantage of constructing the solution $v_{\theta tt}$ prior to v_θ is that it clarifies why compatibility conditions are required, and it is also clear how the estimates are obtained.

Since

$$-((\tilde A^j_i q_\ell)_{tt},v^i_{\ell tt,j})_{L^2(\Omega)} = \theta\|q_{\ell tt}\|^2_0 - (2(\tilde A^j_i)_t q_{\ell t} + (\tilde A^j_i)_{tt}q_\ell, v^i_{\ell tt,j})_{L^2(\Omega)}$$
$$- (q_{\ell tt},(\tilde A^j_i)_{tt}v^i_{\ell,j} - 2(\tilde A^j_i)_t v^i_{\ell t,j})_{L^2(\Omega)}\,,$$

the use of the test function $\varphi = v_{\ell tt}$ in (5.11a) leads to the inequality

(5.12)
$$\begin{aligned}
&\|v_{\ell tt}(t)\|_0^2 + \bar\sigma\varepsilon|v_{\ell tt}(t)|_0^2 + \int_0^t \Big[\|\nabla v_{\ell tt}\|_0^2 + \kappa\varepsilon^3|v_{\ell tt}|_2^2 + \theta\|q_{\ell tt}\|_0^2\Big]ds \\
&\le C\left[\|\tilde{\mathrm{w}}_2\|_0^2 + \bar\sigma\varepsilon|\tilde{\mathrm{w}}_2|_0^2 + \int_0^t \|\tilde F_{tt}\|_0^2 ds\right] + C\int_0^t \Big[\frac{1}{\kappa\varepsilon^2}|\mathcal{L}_m(\tilde\eta)_{tt}|_{-2}^2 + \frac{\varepsilon^3}{\kappa}|\mathcal{L}_b(\tilde\eta)_{tt}|_{-2}^2 \\
&+ \|(\tilde A\tilde A)_{tt}\|_{L^\infty(\Omega)}^2\|\nabla v_\ell\|_0^2 + \|(\tilde A\tilde A)_t\|_{L^\infty(\Omega)}^2\|\nabla v_{\ell t}\|_0^2 + (q_{\ell tt}, (\tilde A_i^j)_{tt} v^i_{\ell,j})_{L^2(\Omega)} \\
&+ 2(q_{\ell tt}, (\tilde A_i^j)_t v^i_{\ell t,j})_{L^2(\Omega)} + 2((\tilde A_i^j)_t q_{\ell t}, v^i_{\ell tt,j})_{L^2(\Omega)} + ((\tilde A_i^j)_{tt} q_\ell, v^i_{\ell tt,j})_{L^2(\Omega)}\Big]ds\,.
\end{aligned}$$

By the inequality

$$\|f(t)\|_X^2 \le C\Big[\|f(0)\|_X^2 + t\int_0^t \|f_t\|_X^2 ds\Big]\,, \tag{5.13}$$

we find that

$$|\partial_t^k\partial^m\eta_\ell|_0^2 \le C\Big[|(\partial_t^k\partial^m\eta_\ell)(0)|_2^2 + t\int_0^t |\partial_t^k\partial^m v_\ell|_0^2 ds\Big]\,, \tag{5.14a}$$

$$\|\partial_t^k\nabla^m v_\ell\|_0^2 \le C\Big[\|(\partial_t^k\nabla^m v_\ell)(0)\|_0^2 + t\int_0^t \|\partial_t^{k+1}\nabla^m v_\ell\|_0^2 ds\Big]\,. \tag{5.14b}$$

As a result,

$$\|\tilde A_{tt}\|_{L^\infty(\Omega)}^2 \le C(1 + \|\tilde v_t\|_3^2)\,, \quad \|\tilde A_t\|_3^2 \le C\|\tilde v\|_3^2 \le C(\|\tilde u_0\|_3^2 + tM)\,.$$

Therefore, using $q_\ell = \tilde q_0 + t\tilde q_1 - \frac{1}{\theta}\tilde A_i^j v^i_{\ell,j}$ in the last two terms of (5.12), Young's inequality and (5.14) imply that

$$\begin{aligned}
&\|v_{\ell tt}(t)\|_0^2 + \bar\sigma\varepsilon|v_{\ell tt}(t)|_0^2 + \int_0^t \Big[\|\nabla v_{\ell tt}\|_0^2 + \kappa\varepsilon^3|v_{\ell tt}|_2^2 + \theta\|q_{\ell tt}\|_0^2\Big]ds \\
&\le C_\kappa\mathcal{M}(\varepsilon, u_0, f) + C\int_0^t \Big[\|(\tilde A\tilde A)_{tt}\|_{L^\infty(\Omega)}^2\|\nabla v_\ell\|_0^2 + \|(\tilde A\tilde A)_t\|_{L^\infty(\Omega)}^2\|\nabla v_{\ell t}\|_0^2\Big]ds \\
&+ \bar\delta\int_0^t \theta\|q_{\ell tt}\|_0^2 ds + C_{\bar\delta,\theta}\int_0^t \Big[\|\tilde A_{tt}\|_{L^\infty(\Omega)}^2\|\nabla v_\ell\|_0^2 + \|\tilde A_t\|_{L^\infty(\Omega)}^2\|\nabla v_{\ell t}\|_0^2\Big]ds \\
&+ \bar\delta\int_0^t \|\nabla v_{\ell tt}\|_0^2 ds + C_{\bar\delta}\int_0^t \Big[\|\tilde A_t\|_{L^\infty(\Omega)}^2\|q_{\ell t}\|_0^2 + \|\tilde A_{tt}\|_{L^\infty(\Omega)}^2\|q_\ell\|_0^2\Big]ds \\
&\le C_{\bar\delta,\theta,\kappa}\mathcal{M}(\varepsilon, u_0, f) + C_{\theta,\bar\delta}t^2 M\int_0^t \|\nabla v_{\ell tt}\|_0^2 ds + \bar\delta\int_0^t \Big[\|\nabla v_{\ell tt}\|_0^2 + \theta\|q_{\ell tt}\|_0^2\Big]ds\,,
\end{aligned}$$

and consequently, for $T = T(\theta, \kappa, \varepsilon, M)$, we find that the quantity

$$\sup_{t\in[0,T]}\Big[\|v_{\ell tt}(t)\|_0^2 + \bar\sigma\varepsilon|v_{\ell tt}(t)|_0^2\Big] + \int_0^T \Big[\|\nabla v_{\ell tt}\|_0^2 + \kappa\varepsilon^3|v_{\ell tt}|_2^2 + \theta\|q_{\ell tt}\|_0^2\Big]dt$$

is bounded uniformly in ℓ, and this implies the weak compactness of the sequences $v_{\ell tt}$ and $q_{\ell tt}$. By (5.13), $\partial_t^k\eta_\ell$ is also uniformly bounded in $L^2(0,T;H^{1;2}(\Omega;\Gamma))$ for

$k = 0, 1, 2$ and $\partial_t^k q_\ell$ is uniformly bounded in $L^2(0, T; L^2(\Omega))$ for $k = 0, 1$ as well. Therefore, there exists a subsequence of ℓ, still denoted by ℓ, so that

(5.15a) $$\partial_t^k \eta_\ell \rightharpoonup \partial_t^k \eta_\theta \quad \text{in} \quad L^2(0, T; H^{1;2}(\Omega; \Gamma)) \quad \text{for } k = 0, 1, 2, 3\,,$$

(5.15b) $$\partial_t^k q_\ell \rightharpoonup \partial_t^k q_\theta \quad \text{in} \quad L^2(0, T; L^2(\Omega)) \qquad \text{for } k = 0, 1, 2\,.$$

From the standard procedure for weak solutions, we can now infer from the weak convergence and the definition of v_ℓ that $v_{\theta ttt} \in L^2(0, T; H^1(\Omega)')$ (and $v_{\theta ttt} \in L^2(0, T; H^{-2}(\Gamma))$ if $\bar\sigma = 1$), which in turn implies that $v_{\theta tt} \in \mathcal{C}^0([0, T]; H^1(\Omega)')$, $v_{\theta t} \in \mathcal{C}^0([0, T]; L^2(\Omega))$, $v_\theta \in \mathcal{C}^0([0, T]; H^1(\Omega))$, with $v_\theta(0) = \tilde u_0$, $v_{\theta t}(0) = \tilde{\mathrm{w}}_1$, $v_{\theta tt}(0) = \tilde{\mathrm{w}}_2$, and

$$\int_0^T \Big[\|v_{\theta tt}\|_1^2 + \bar\sigma\varepsilon |v_{\theta tt}|_0^2 + \kappa\varepsilon^3 |v_{\theta tt}|_2^2 + \theta \|q_{\theta tt}\|_0^2\Big] dt \le C_{\theta,\kappa} \mathcal{M}(\varepsilon, u_0, f)\,.$$

Moreover, time integrating (5.11a) from 0 to t, we have for $v_{\ell t}$,

(5.16)
$$\begin{aligned}
&\langle v_{\ell tt}, \varphi\rangle + \bar\sigma\varepsilon \langle v_{\ell tt}, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v_{\ell t} : D_{\tilde A}\varphi dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\ell t,\alpha\beta} \varphi^i_{,\gamma\delta} dS \\
&- ((\tilde A^j_i q_\ell)_t, \varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F_t, \varphi\rangle - \varepsilon \langle \mathcal{L}_m(\tilde\eta)_t, \varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta)_t, \varphi\rangle_\Gamma \\
&- \nu \int_\Omega \Big[(\tilde A^m_i \tilde A^j_k)_t v^k_{\ell,m} + (\tilde A^m_k \tilde A^j_k)_t v^i_{\ell,m}\Big] \varphi^i_{,j} dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} \tilde{\mathrm{w}}^i_{1,\alpha\beta} \varphi^i_{,\gamma\delta} dS + c_\ell(\varphi)\,,
\end{aligned}$$

where

$$\begin{aligned}
c_\ell(\varphi) = {} & \langle (\tilde{\mathrm{w}}_2)_\ell, \varphi\rangle + \bar\sigma\varepsilon \langle (\tilde{\mathrm{w}}_2)_\ell, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega \operatorname{Def}(\tilde{\mathrm{w}}_1)_\ell : \operatorname{Def}\varphi dx \\
& - ((\tilde q_1)_\ell, \operatorname{div}\varphi)_{L^2(\Omega)} + \frac{1}{\theta}(\tilde u^j_{0,i} \tilde u^i_{0\ell,j} - \operatorname{div}(\tilde{\mathrm{w}}_1)_\ell, \operatorname{div}\varphi)_{L^2(\Omega)} \\
& + ((\tilde q_0)_\ell, \tilde u^j_{0,i}\varphi^i_{,j})_{L^2(\Omega)} - \frac{1}{\theta}(\operatorname{div}(\tilde u_0)_\ell, \tilde u^j_{0,i}\varphi^i_{,j})_{L^2(\Omega)} \\
& - \langle \tilde F_t(0), \varphi\rangle + \varepsilon \langle \mathcal{L}_m(\tilde\eta)_t(0), \varphi\rangle_\Gamma + \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta)_t(0), \varphi\rangle_\Gamma \\
& + \nu \int_\Omega \Big[\tilde u^k_{0,j} \tilde u^i_{0,k} + \tilde u^k_{0,i} \tilde u^j_{0,k} + (\operatorname{Def}\tilde u_0)^k_i \tilde u^j_{0,k}\Big] \varphi^i_{,j} dx \\
& + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} \Big[(\tilde{\mathrm{w}}^i_1)_{\ell,\alpha\beta} - \tilde{\mathrm{w}}^i_{1,\alpha\beta}\Big] \varphi^i_{,\gamma\delta} dS\,.
\end{aligned}$$

Note that as $\ell \to \infty$, $\operatorname{div}(\tilde u_0)_\ell \to 0$, $\operatorname{div}(\tilde{\mathrm{w}}_1)_\ell - \tilde u^j_{0,i} \tilde u^i_{0\ell,j} \to 0$. Therefore, by the definition of $\tilde q_k$ and $\tilde{\mathrm{w}}_k$ and the second order compatibility condition, $c_\ell(\varphi) \to 0$ as $\ell \to \infty$.

Time integrating (5.16), we find that

(5.17)
$$\begin{aligned}
&\langle v_{\ell t}, \varphi\rangle + \bar\sigma\varepsilon \langle v_{\ell t}, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v_\ell : D_{\tilde A}\varphi dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\ell,\alpha\beta} \varphi^i_{,\gamma\delta} dS \\
&- (\tilde A^j_i q_\ell, \varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F, \varphi\rangle - \varepsilon \langle \mathcal{L}_m(\tilde\eta), \varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta), \varphi\rangle_\Gamma \\
&+ \kappa\varepsilon^3 \int_\Gamma {L_0}^i \varphi^i dS + c_\ell(\varphi) t + d_\ell(\varphi)\,,
\end{aligned}$$

where

$$d_\ell(\varphi) = \langle (\tilde{\mathrm{w}}_1)_\ell, \varphi\rangle + \bar\sigma\varepsilon\langle (\tilde{\mathrm{w}}_1)_\ell, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega \mathrm{Def}(\tilde u_0)_\ell : \mathrm{Def}\,\varphi dx$$
$$- ((\tilde q_0)_\ell, \mathrm{div}\,\varphi)_{L^2(\Omega)} + \frac{1}{\theta}(\mathrm{div}(\tilde u_0)_\ell, \mathrm{div}\,\varphi)_{L^2(\Omega)} - \langle f(0), \varphi\rangle$$
$$+ \kappa\varepsilon^3 \int_\Gamma \Big[(\tilde u_0^i + t\tilde{\mathrm{w}}_1^i)_{\ell,\alpha\beta} - (\tilde u_0^i + t\tilde{\mathrm{w}}_1^i)_{,\alpha\beta}\Big]\varphi^i_{,\gamma\delta} dS$$

and again by the definition of $\tilde q_k$ and $\tilde{\mathrm{w}}_k$ and the first order compatibility condition, $d_\ell(\varphi) \to 0$.

Time integrating (5.16) and (5.17), and passing $\ell \to \infty$, by virtue of weak convergence we find that

$$\int_0^T \Big[\langle v_{\theta tt}, \varphi\rangle + \bar\sigma\varepsilon\langle v_{\theta tt}, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v_{\theta t} : D_{\tilde A}\varphi dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\theta t,\alpha\beta}\varphi^i_{,\gamma\delta} dS$$
$$- ((\tilde A^j_i q_\theta)_t, \varphi^i_{,j})_{L^2(\Omega)}\Big] dt$$
$$= \int_0^T \Big[\langle \tilde F_t, \varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta)_t, \varphi\rangle_\Gamma - \nu\int_\Omega \Big((\tilde A^m_i \tilde A^k_i)_t v^j_{\theta,m} + (\tilde A^m_i \tilde A^k_j)_t v^i_{\theta,m}\Big)\varphi^j_{,k} dx$$
$$- \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta)_t, \varphi\rangle_\Gamma + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta}\tilde{\mathrm{w}}^i_{1,\alpha\beta}\varphi^i_{,\gamma\delta} dS\Big] dt$$

and

$$\int_0^T \Big[\langle v_{\theta t}, \varphi\rangle + \bar\sigma\varepsilon\langle v_{\theta t}, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v_\theta : D_{\tilde A}\varphi dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\theta,\alpha\beta}\varphi^i_{,\gamma\delta} dS$$
$$- (\tilde A^j_i q_\theta, \varphi^i_{,j})_{L^2(\Omega)}\Big] dt$$
$$= \int_0^T \Big[\langle \tilde F, \varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta), \varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta), \varphi\rangle_\Gamma + \int_\Gamma \kappa\varepsilon^3 {L_0}^i \varphi^i dS\Big] dt\,.$$

Choosing φ to be independent of time, then for almost all $t \in [0, T]$ and $\varphi \in H^{1;2}(\Omega; \Gamma)$,

$$\langle v_{\theta tt}, \varphi\rangle + \bar\sigma\varepsilon\langle v_{\theta tt}, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v_{\theta t} : D_{\tilde A}\varphi dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\theta t,\alpha\beta}\varphi^i_{,\gamma\delta} dS$$
$$- ((\tilde A^j_i q_\theta)_t, \varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F_t, \varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta)_t, \varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta)_t, \varphi\rangle_\Gamma \tag{5.18}$$
$$- \nu\int_\Omega \Big[(\tilde A^m_i \tilde A^k_i)_t v^j_{\theta,m} + (\tilde A^m_i \tilde A^k_j)_t v^i_{\theta,m}\Big]\varphi^j_{,k} dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta}\tilde{\mathrm{w}}^i_{1,\alpha\beta}\varphi^i_{,\gamma\delta} dS$$

and

(5.19)
$$\langle v_{\theta t}, \varphi\rangle + \bar\sigma\varepsilon\langle v_{\theta t}, \varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A} v_\theta : D_{\tilde A}\varphi dx + \kappa\varepsilon^3 \int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\theta,\alpha\beta}\varphi^i_{,\gamma\delta} dS$$
$$- (\tilde A^j_i q_\theta, \varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F, \varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta), \varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta), \varphi\rangle_\Gamma + \int_\Gamma \kappa\varepsilon^3 {L_0}^i \varphi^i dS\,.$$

5.6. Strong convergence of v_ℓ, $v_{\ell t}$, and $v_{\ell tt}$. Since $v_\theta \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$, we can use it as a test function in (5.19) and obtain (after time integration)

$$\begin{aligned}
&\frac{1}{2}\Big[\|v_\theta(t)\|_0^2 + \bar\sigma\varepsilon|v_\theta(t)|_0^2\Big] + \frac{\nu}{2}\int_0^t \|D_{\tilde A} v_\theta(s)\|_0^2 ds \\
&+ \kappa\varepsilon^3\int_0^t\int_\Gamma a^{\alpha\beta\gamma\delta} v^i_{\theta,\alpha\beta} v^i_{\theta,\gamma\delta} dS ds + \theta\int_0^t (q_\theta(s), q_\theta(s) - \tilde q_0 - s\tilde q_1)_{L^2(\Omega)} ds \\
&= \frac{1}{2}\Big[\|\tilde u_0\|_0^2 + \bar\sigma\varepsilon|\tilde u_0|_0^2\Big] + \int_0^t \Big[\langle \tilde F, v_\theta\rangle - \varepsilon\langle \tilde{\mathcal L}_m(\tilde\eta), v_\theta\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal L_b(\tilde\eta), v_\theta\rangle_\Gamma \\
&\qquad + \int_\Gamma \kappa\varepsilon^3 {L_0}^i v_\theta^i dS\Big] ds\,.
\end{aligned} \tag{5.20}$$

Consequently, for $T = T(\kappa,\varepsilon,M)$,

$$\begin{aligned}
&\sup_{t\in[0,T]}\Big[\|v_\theta(t)\|_0^2 + \bar\sigma\varepsilon|v_\theta(t)|_0^2\Big] + \int_0^T \|\nabla v_\theta(t)\|_0^2 dt \\
&+ \int_0^T \Big[\kappa\varepsilon^3|v_\theta(t)|_2^2 + \theta\|q_\theta(t)\|_0^2\Big] dt \le C\mathcal M(\varepsilon,u_0,f)
\end{aligned} \tag{5.21}$$

for some C independent of M, κ, and θ.

Similarly, since $v_\ell \in \operatorname{span}(e_1,\dots,e_\ell)$ for all $t\in[0,T]$, we can use it as a test function in (5.17a) and obtain

$$\begin{aligned}
&\frac{1}{2}\Big[\|v_\ell(t)\|_0^2 + \bar\sigma\varepsilon|v_\ell(t)|_0^2\Big] + \frac{\nu}{2}\int_0^t \|D_{\tilde A} v_\ell(s)\|_0^2 ds \\
&+ \kappa\int_0^t \varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta} \eta^i_{\ell,\alpha\beta}\eta^i_{\ell,\gamma\delta} dS ds + \theta\int_0^t (q_\ell(s), q_\ell(s) - \tilde q_0 - s\tilde q_1)_{L^2(\Omega)} ds \\
&= \frac{1}{2}\Big[\|\tilde u_0\|_0^2 + \bar\sigma\varepsilon|\tilde u_0|_0^2\Big] + \int_0^t \Big[\langle \tilde F, v_\ell\rangle - \varepsilon\langle \tilde{\mathcal L}_m(\tilde\eta), v_\ell\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal L_b(\tilde\eta), v_\ell\rangle_\Gamma \\
&\quad + \kappa\varepsilon^3\int_\Gamma {L_0}^i v_\ell^i dS + tc_\ell(v_\ell) + d_\ell(v_\ell)\Big] ds\,.
\end{aligned} \tag{5.22}$$

By the compatibility conditions,

$$tc_\ell(v_\ell) + d_\ell(v_\ell) \to 0 \qquad \text{as} \quad \ell\to\infty\,.$$

Therefore, the right-hand side of (5.22) converges to the right-hand side of (5.20), and by the weak convergence of v_ℓ in $L^2(0,T;H^{1;2}(\Omega;\Gamma))$ (which implies that $q_\ell \rightharpoonup q_\theta$ in $L^2(0,T;L^2(\Omega))$) we find that

$$\begin{aligned}
&\lim_{\ell\to\infty}\Big[\frac{1}{2}\|v_\ell(t)\|_0^2 + \frac{1}{2}\bar\sigma\varepsilon|v_\ell(t)|_0^2 + \frac{1}{2}\kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta}\eta^i_{\ell,\alpha\beta}(t)\eta^i_{\ell,\gamma\delta}(t) dS \\
&\quad + \frac{\nu}{2}\int_0^t \|D_{\tilde A} v_\ell(s)\|_0^2 ds + \theta\int_0^t \|q_\ell(s)\|_0^2 ds\Big] = \frac{1}{2}\|v_\theta(t)\|_0^2 + \frac{1}{2}\bar\sigma\varepsilon|v_\theta(t)|_0^2 \\
&\quad + \frac{1}{2}\kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta}\eta^i_{\theta,\alpha\beta}(t)\eta^i_{\theta,\gamma\delta}(t) dS + \frac{\nu}{2}\int_0^t \|D_{\tilde A} v_\theta(s)\|_0^2 ds + \theta\int_0^t \|q_\theta(s)\|_0^2 ds\,.
\end{aligned}$$

As a result, $v_\ell \to v_\theta$ in $L^2(0,T;H^{1;2}(\Omega;\Gamma))$. A similar argument can be used to show that $v_{\ell t}\to v_{\theta t}$ in $L^2(0,T;H^{1;2}(\Omega;\Gamma))$.

As for the strong convergence of $v_{\ell tt}$, we use $v_{\ell tt}$ as a test function in (5.16) and $v_{\theta tt}$ as a test function in (5.19) to show that

$$\lim_{\ell\to\infty}\left[\int_0^t(\|v_{\ell tt}\|_0^2+\bar\sigma\varepsilon|v_{\ell tt}|_0^2)ds+\frac{\nu}{2}\|D_{\tilde A}v_{\ell t}(t)\|_0^2+\frac{\kappa\varepsilon^3}{2}\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{\ell t,\alpha\beta}(t)v^i_{\ell t,\gamma\delta}(t)dS\right]$$
$$=\int_0^t\Big[\|v_{\theta tt}\|_0^2+\bar\sigma\varepsilon|v_{\theta tt}|_0^2\Big]ds+\frac{\nu}{2}\|D_{\tilde A}v_{\theta t}(t)\|_0^2+\frac{\kappa\varepsilon^3}{2}\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{\theta t,\alpha\beta}(t)v^i_{\theta t,\gamma\delta}(t)dS\,.$$

Therefore, $v_{\ell tt}\to v_{\theta tt}$ in $L^2(0,T;L^2(\Omega))$ (and $v_{\ell tt}\to v_{\theta tt}$ also in $L^2(0,T;L^2(\Gamma))$ if $\bar\sigma=1$).

5.7. Improved pressure estimates. By Lemma 3.3 (the Lagrange multiplier lemma), (5.19) implies that

$$\|q_\theta\|_0^2\le C\Big[\|v_{\theta t}\|_0^2+\bar\sigma\varepsilon|v_{\theta t}|_0^2+\nu\|v_\theta\|_1^2+\kappa\varepsilon^3|v_\theta|_2^2+\|\tilde F\|_0^2$$
$$+\varepsilon|\mathcal{L}_m(\tilde\eta)|_0^2+\varepsilon^3|\mathcal{L}_b(\tilde\eta)|_0^2+\kappa\varepsilon^3t|\tilde u_0|_2^2\Big]$$
$$\le C\mathcal{M}(\varepsilon,u_0,f)+C\Big[\|v_{\theta t}\|_0^2+\bar\sigma\varepsilon|v_{\theta t}|_0^2+\|v_\theta\|_1^2+\kappa\varepsilon^3|v_\theta|_2^2\Big]\,. \tag{5.23}$$

Similarly, since $(\tilde A^j_i q_\theta)_t=\tilde A^j_i q_{\theta t}+(\tilde A^j_i)_t q_\theta$, by (5.13) and (5.23) we apply Lemma 3.3 to (5.18) and find that

$$\|q_{\theta t}\|_0^2\le C\Big[\|v_{\theta tt}\|_0^2+\bar\sigma\varepsilon|v_{\theta tt}|_0^2+\nu\|v_{\theta t}\|_1^2+\kappa\varepsilon^3|v_{\theta t}|_2^2+\|q_\theta\|_0^2+\|\tilde F_t\|_0^2$$
$$+\nu\|v_\theta\|_1^2+\varepsilon|\mathcal{L}_m(\tilde\eta)_t|_0^2+\varepsilon^3|\mathcal{L}_b(\tilde\eta)_t|_0^2+\kappa\varepsilon^3|\tilde u_0|_2^2\Big]$$
$$\le C\mathcal{M}(\varepsilon,u_0,f)+C\Big[\|v_{\theta tt}\|_0^2+\bar\sigma\varepsilon|v_{\theta tt}|_0^2+\|v_{\theta t}\|_1^2+\kappa\varepsilon^3|v_{\theta t}|_2^2\Big]\,. \tag{5.24}$$

5.8. Weak limits as $\boldsymbol{\theta\to 0}$. Since $v_{\theta t}\in L^2(0,T;H^{1;2}(\Omega;\Gamma))$, by using it as a test function in (5.18),

(5.25)
$$\frac12\frac{d}{dt}\Big[\|v_{\theta t}(t)\|_0^2+\bar\sigma\varepsilon|v_{\theta t}(t)|_0^2\Big]+\frac{\nu}{2}\int_\Omega\|D_{\tilde A}v_{\theta t}\|_0^2dx+\kappa\varepsilon^3\int_\Omega a^{\alpha\beta\gamma\delta}v^i_{\theta t,\alpha\beta}v^i_{\theta t,\gamma\delta}dS$$
$$-((\tilde A^j_i q_\theta)_t,v^i_{\theta t,j})_{L^2(\Omega)}=\langle\tilde F_t,v_{\theta t}\rangle-\varepsilon\langle\mathcal{L}_m(\tilde\eta)_t,v_{\theta t}\rangle_\Gamma-\frac{\varepsilon^3}{3}\langle\mathcal{L}_b(\tilde\eta)_t,v_{\theta t}\rangle_\Gamma$$
$$-\nu\int_\Omega\Big[(\tilde A^m_i\tilde A^k_i)_t v^j_{\theta,m}+(\tilde A^m_i\tilde A^k_j)_t v^i_{\theta,m}\Big]v^j_{\theta t,k}dx-\kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta}\tilde{\mathrm{w}}^i_{1,\alpha\beta}v^i_{\theta t,\gamma\delta}dS\,.$$

For a θ-independent estimate, we need to estimate only the term $((\tilde A^j_i q_\theta)_t,v^i_{\theta t,j})_{L^2(\Omega)}$. By the definition of q_θ,

$$-((\tilde A^j_i q_\theta)_t,v^i_{\theta t,j})_{L^2(\Omega)}=-((\tilde A^j_i)_t q_\theta,v^i_{\theta t,j})_{L^2(\Omega)}+(q_{\theta t},-(\tilde A^i_j v^i_{\theta,j})_t+(\tilde A^j_i)_t v^i_{\theta,j})_{L^2(\Omega)}$$
$$=\theta\|q_{\theta t}\|_0^2-((\tilde A^j_i)_t q_\theta,v^i_{\theta t,j})_{L^2(\Omega)}+(q_{\theta t},\theta\tilde q_1+(\tilde A^j_i)_t v^i_{\theta,j})_{L^2(\Omega)}\,.$$

For the last term, we study the time integral of it, and integration by parts in time implies that

$$\int_0^t(q_{\theta t},(\tilde A^j_i)_t v^i_{\theta,j})_{L^2(\Omega)}ds$$
$$=(q_\theta,(\tilde A^j_i)_t v^i_{\theta,j})_{L^2(\Omega)}\Big|_{s=0}^{s=t}-\int_0^t(q_\theta,(\tilde A^j_i)_{tt}v^i_{\theta,j}+(\tilde A^j_i)_t v^i_{\theta t,j})_{L^2(\Omega)}ds\,. \tag{5.26}$$

By the definition of q_θ, we find that

$$(\tilde{A}_i^j)_t v^i_{\theta,j} = \theta(\tilde{q}_1 - q_{\theta t}) - \tilde{A}_i^j v^i_{\theta t,j} \,.$$

Using this identity in the left-hand side of (5.26), by (5.13) and estimate (5.21) we find that

$$\begin{aligned}
\frac{\theta}{2}\int_0^t \|q_{\theta t}\|_0^2 ds &\le -\int_0^t ((\tilde{A}_i^j q_\theta)_t, v^i_{\theta t,j})_{L^2(\Omega)} ds + \theta t\|\tilde{q}_1\|_0^2 + C_{\bar\delta}\|\nabla v_\theta\|_0^2 + \bar\delta\|q_\theta\|_0^2 \\
&\qquad + C_{\bar\delta}\int_0^t \|q_\theta\|_0^2 ds + \bar\delta\int_0^t \|v_{\theta t}\|_1^2 ds + \int_0^t \|\tilde{A}_{tt}\|^2_{L^\infty(\Omega)}\|v\|_1^2 ds \\
&\le -\int_0^t ((\tilde{A}_i^j q_\theta)_t, v^i_{\theta t,j})_{L^2(\Omega)} ds + C_{\bar\delta}\mathcal{M}(\varepsilon,u_0,f) + C_{\bar\delta}\int_0^t \Big[\|v_{\theta t}\|_0^2 + \bar\sigma\varepsilon|v_{\theta t}|_0^2\Big] ds \\
&\quad + (Ct+\bar\delta)\int_0^t \|v_{\theta t}\|_1^2 ds + \bar\delta\Big[\|v_{\theta t}\|_0^2 + \bar\sigma\varepsilon|v_{\theta t}|_0^2 + \kappa\varepsilon^3\int_0^t |v_{\theta t}|_2^2 ds\Big] \,.
\end{aligned}$$

Therefore, time integrating (5.25) and choosing $\bar\delta > 0$ and $T > 0$ small enough, for $T = T(\kappa,\varepsilon,M)$ small enough,

$$\begin{aligned}
&\|v_{\theta t}(t)\|_0^2 + \bar\sigma\varepsilon|v_{\theta t}(t)|_0^2 + \int_0^t \Big[\|\nabla v_{\theta t}\|_0^2 + \kappa\varepsilon^3|v_{\theta t}|_2^2 + \theta\|q_{\theta t}\|_0^2\Big] ds \\
&\le C\mathcal{M}(\varepsilon,u_0,f) + C\int_0^t \Big[\|v_{\theta t}\|_0^2 + \bar\sigma\varepsilon|v_{\theta t}|_0^2\Big] ds
\end{aligned}$$

and by the Gronwall inequality,

$$\begin{aligned}
&\sup_{t\in[0,T]}\Big[\|v_{\theta t}(t)\|_0^2 + \bar\sigma\varepsilon|v_{\theta t}(t)|_0^2\Big] + \int_0^T \|\nabla v_{\theta t}\|_0^2 ds \\
&\quad + \int_0^T \Big[\kappa\varepsilon^3|v_{\theta t}|_2^2 + \theta\|q_{\theta t}\|_0^2\Big] dt \le C\mathcal{M}(\varepsilon,u_0,f)
\end{aligned} \tag{5.27}$$

for some constant C independent of M, κ, and θ.

Since (5.21) and (5.27) are θ-independent, we can extend v_θ to an interval $[0,T]$ for some $T = T(\kappa,\varepsilon,M)$ and conclude that as $\theta \to 0$,

$$\partial_t^k v_\theta \rightharpoonup \partial_t^k v_\kappa \quad \text{in} \quad L^2(0,T;H^{1;2}(\Omega;\Gamma)) \quad \text{for } k = 0,1\,, \tag{5.28a}$$

$$q_\theta \rightharpoonup q_\kappa \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \tag{5.28b}$$

for some vectors v_κ and $v_{\kappa t} \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$ and scalar $q_\kappa \in L^2(0,T;L^2(\Omega))$; moreover, (5.21) also shows that $\|\tilde{A}_i^j v^i_{\theta,j}\|^2_{L^2(0,T;L^2(\Omega))} \to 0$ as $\theta \to 0$. Therefore the weak limit v_κ satisfies the constraint (5.5c).

The last step in this subsection is to show that $v_{\kappa tt} \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$. By the strong convergence of $\partial_t^k v_\ell$ for $k = 0,1,2$, the weak convergence of $q_{\ell tt}$, and the property of lower semicontinuity of norms, (5.12) holds with ℓ replaced by θ (by passing $\ell \to \infty$ in (5.12)) so that

$$\begin{aligned}
&\|v_{\theta tt}(t)\|_0^2 + \bar\sigma\varepsilon|v_{\theta tt}(t)|_0^2 + \int_0^t \Big[\|\nabla v_{\theta tt}\|_0^2 + \kappa\varepsilon^3|v_{\theta tt}|_2^2 + \theta\|q_{\theta tt}\|_0^2\Big] ds \\
&\le C_\kappa\mathcal{M}(\varepsilon,u_0,f) + C\int_0^t \|\nabla v_{\theta t}\|_0^2 ds + C_{\bar\delta}\int_0^t \|q_{\theta t}\|_0^2 ds + \bar\delta\int_0^t \|\nabla v_{\theta tt}\|_0^2 ds \\
&\quad + C\int_0^t (q_{\theta tt}, (\tilde{A}_i^j)_{tt} v^i_{\theta,j} + 2(\tilde{A}_i^j)_t v^i_{\theta t,j})_{L^2(\Omega)} ds \,.
\end{aligned}$$

For the last integral, similarly to (5.26), we integrate by parts and find that we need to estimate the term

$$\int_0^t (q_{\theta t}, (\tilde{A}_i^j)_{ttt} v_{\theta,j}^i)_{L^2(\Omega)} ds\,.$$

By the $\hat{\epsilon}$ regularization, $\|\tilde{A}_{ttt}\|_{L^\infty(\Omega)} \le C_\kappa$. As a consequence,

$$\begin{aligned} &\Big| \int_0^t (q_{\theta t}, (\tilde{A}_i^j)_{ttt} v_{\theta,j}^i)_{L^2(\Omega)} ds \Big| \le C_\kappa \int_0^t \|v_\theta\|_1^2 ds + C \int_0^t \|q_{\theta t}\|_0^2 ds \\ &\le C_\kappa \mathcal{M}(\varepsilon, u_0, f) + C \int_0^t \Big[\|v_{\theta tt}\|_0^2 + \bar{\sigma}\varepsilon |v_{\theta tt}|_0^2 + \kappa \varepsilon^3 |v_{\theta t}|_2^2 \Big] ds. \end{aligned} \tag{5.29}$$

Therefore, (5.13) together with estimates (5.21) and (5.27) implies that

$$\begin{aligned} &\Big| \int_0^t (q_{\theta tt}, (\tilde{A}_i^j)_{tt} v_{\theta,j}^i + 2(\tilde{A}_i^j)_t v_{\theta t,j}^i)_{L^2(\Omega)} ds \Big| \\ &\le C_{\bar{\delta},\kappa} \mathcal{M}(\varepsilon, u_0, f) + C \int_0^t \Big[\|v_{\theta tt}\|_0^2 + \bar{\sigma}\varepsilon |v_{\theta tt}|_0^2 \Big] ds + \bar{\delta} \Big[\|v_{\theta tt}(t)\|_0^2 + \bar{\sigma}\varepsilon |v_{\theta tt}(t)|_0^2 \Big] \\ &\quad + (Ct + \bar{\delta}) \int_0^t \Big[\|\nabla v_{\theta tt}\|_0^2 + \varepsilon^3 |v_{\theta tt}|_2^2 \Big] ds, \end{aligned}$$

which further implies that

$$\begin{aligned} &\|v_{\theta tt}(t)\|_0^2 + \bar{\sigma}\varepsilon |v_{\theta tt}(t)|_0^2 + \int_0^t \Big[\|\nabla v_{\theta tt}\|_0^2 + \kappa\varepsilon^3 |v_{\theta tt}|_2^2 + \theta \|q_{\theta tt}\|_0^2 \Big] ds \\ &\le C_{\bar{\delta},\kappa} \mathcal{M}(\varepsilon, u_0, f) + C_{\bar{\delta}} \int_0^t \Big[\|v_{\theta tt}\|_0^2 + \bar{\sigma}\varepsilon |v_{\theta tt}|_0^2 \Big] ds + \bar{\delta} \int_0^t \|\nabla v_{\theta tt}\|_0^2 ds \\ &\quad + \bar{\delta} \Big[\|v_{\theta tt}(t)\|_0^2 + \bar{\sigma}\varepsilon |v_{\theta tt}(t)|_0^2 \Big] + (Ct + \bar{\delta}) \int_0^t \Big[\|\nabla v_{\theta tt}\|_0^2 + \varepsilon^3 |v_{\theta tt}|_2^2 \Big] ds\,. \end{aligned}$$

By choosing $\bar{\delta} > 0$ and $T = T(\kappa, \varepsilon, M) > 0$ small enough, with the help of the Gronwall inequality we find that

$$\begin{aligned} &\sup_{t \in [0,T]} \Big[\|v_{\theta tt}(t)\|_0^2 + \bar{\sigma}\varepsilon |v_{\theta tt}(t)|_0^2 \Big] + \int_0^T \|\nabla v_{\theta tt}\|_0^2 dt \\ &\quad + \int_0^T \Big[\kappa\varepsilon^3 |v_{\theta tt}|_2^2 + \theta \|q_{\theta tt}\|_0^2 \Big] ds \le C_\kappa \mathcal{M}(\varepsilon, u_0, f) \end{aligned} \tag{5.30}$$

for some constants C and C_κ independent of θ and M.

Remark 14. At the stage of constructing a solution to the penalized problem, we do not know higher regularity of v_θ and $q_{\theta t}$, so we need $\tilde{A}_{ttt} \in L^\infty(\Omega)$ to estimate the left-hand side of (5.29), and this is why the input $\bar{v}$ has to be regularized; however, the limit (v_κ, q_κ) of (v_θ, q_θ) as $\theta \to 0$ is more regular, so the left-hand side of (5.29) with $(v_\kappa,\, q_\kappa)$ replacing $(v_\theta,\, q_\theta)$ can be estimated by the L^2-L^2-L^∞-type Hölder inequality:

$$\begin{aligned} \Big| \int_0^t (q_{\kappa t}, (\tilde{A}_i^j)_{ttt} v_{\kappa,j}^i)_{L^2(\Omega)} ds \Big| &\le \int_0^t \|q_{\kappa t}\|_0 \|\tilde{A}_{ttt}\|_0 \|\nabla v_\kappa\|_{L^\infty(\Omega)} ds \\ &\le C \int_0^t (\|\bar{v}_{tt}\|_1 + \|\bar{v}\|_2^3 + \|\bar{v}_t\|_{1.5} \|\bar{v}\|_2) \|q_{\kappa t}\|_0 \|v_\kappa\|_3 ds \\ &\le C \int_0^t \Big[\|\bar{v}_{tt}\|_1 + C(M) \Big] \|q_{\kappa t}\|_0 \|v_\kappa\|_3 ds\,. \end{aligned}$$

The right-hand side of the inequality above is bounded for the energy estimates for v_κ, so no regularization is needed to close the energy estimates for v_κ.

Since estimates (5.21), (5.27), and (5.30) are θ-independent, by the property of lower semicontinuity of norms,

$$\sum_{k=0}^{2}\int_0^T \Big[\|\partial_t^k v_\kappa\|_1^2 + \kappa\varepsilon^3|\partial_t^k v_\kappa|_2^2\Big]dt \le C_\kappa \mathcal{M}(\varepsilon, u_0, f)\,. \tag{5.31}$$

As a summary, $v_\kappa \in \mathcal{V}^1(T)$ is a weak solution to (5.5) satisfying estimate (5.31); moreover, similarly to (5.18) and (5.19), for all $\varphi \in H^{1;2}(\Omega;\Gamma)$,

$$\begin{aligned}&\langle v_{\kappa tt},\varphi\rangle + \bar\sigma\varepsilon\langle v_{\kappa tt},\varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A}v_{\kappa t} : D_{\tilde A}\varphi dx + \kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{\kappa t,\alpha\beta}\varphi^i_{,\gamma\delta}dS\\ &- ((\tilde A^j_i q_\kappa)_t, \varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F_t,\varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta)_t,\varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta)_t,\varphi\rangle_\Gamma\\ &- \nu\int_\Omega\Big[(\tilde A^m_i\tilde A^k_i)_t v^j_{\kappa,m} + (\tilde A^m_i\tilde A^k_j)_t v^i_{\kappa,m}\Big]\varphi^j_{,k}dx - \kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta}\tilde{\mathrm{w}}^i_{1,\alpha\beta}\varphi^i_{,\gamma\delta}dS\end{aligned} \tag{5.32}$$

and

(5.33)

$$\begin{aligned}&\langle v_{\kappa t},\varphi\rangle + \bar\sigma\varepsilon\langle v_{\kappa t},\varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_{\tilde A}v_\kappa : D_{\tilde A}\varphi dx + \kappa\varepsilon^3\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{\kappa,\alpha\beta}\varphi^i_{,\gamma\delta}dS\\ &- (\tilde A^j_i q, \varphi^i_{,j})_{L^2(\Omega)} = \langle \tilde F,\varphi\rangle - \varepsilon\langle \mathcal{L}_m(\tilde\eta),\varphi\rangle_\Gamma - \frac{\varepsilon^3}{3}\langle \mathcal{L}_b(\tilde\eta),\varphi\rangle_\Gamma + \int_\Gamma \kappa\varepsilon^3 {L_0}^i\varphi^i dS\,.\end{aligned}$$

6. $\boldsymbol{\kappa}$-dependent energy estimates.

The energy estimates for the linear problem are essentially the same as the nonlinear estimates, so we briefly state the computations and results.

6.1. Partition of unity.

Since Ω is compact, by partition of unity, we can choose two nonnegative smooth functions ζ_0 and ζ so that

$$\begin{aligned}\zeta_0 + \zeta = 1 \quad &\text{in } \Omega\ ;\\ \operatorname{supp}(\zeta_0) \subset\subset \Omega\ &;\\ \operatorname{supp}(\zeta) \subset\subset \Omega_1 &:= \{x\in\mathbb{R}^{\mathrm{n}} \mid \operatorname{dist}(x,\Gamma) < \epsilon_0\}\end{aligned}$$

for some ϵ_0. Note that then $\zeta = 1$ while $\zeta_0 = 0$ on Γ.

6.2. Higher regularity for $\boldsymbol{v_\kappa}$, $\boldsymbol{q_\kappa}$, $\boldsymbol{v_{\kappa t}}$, and $\boldsymbol{q_{\kappa t}}$.

By (3.18), for all $2 \le s \le 5$,

$$\|v_\kappa\|_s^2 + \|q_\kappa\|_{s-1}^2 \le C\Big[\|v_{\kappa t}\|_{s-2}^2 + \|\tilde F\|_{s-2}^2 + |v_\kappa|_{s-0.5}^2\Big]\,. \tag{6.1}$$

For the regularity of $v_{\kappa t}$ and $q_{\kappa t}$, we time differentiate (5.5b) and obtain

$$\begin{aligned}-\nu\tilde A^j_\ell(\tilde A^k_\ell v^i_{\kappa t,k})_{,j} + \tilde A^k_i q_{\kappa t,k} = &- v^i_{\kappa tt} - (\tilde w^j v^i_{\kappa,j})_t + F^i_t + \nu(\tilde A^j_\ell)_t(\tilde A^k_\ell v^i_{\kappa,k})_{,j}\\ &+ \nu\tilde A^j_\ell[(\tilde A^k_\ell)_t v^i_{\kappa,k}]_{,j}\,.\end{aligned}$$

Therefore, by (3.18), for $s \ge 2$,

$$\begin{aligned}\|v_{\kappa t}\|_s^2 + \|q_{\kappa t}\|_{s-1}^2 \le C\Big[&\|v_{\kappa tt}\|_{s-2}^2 + \|\tilde F_t\|_{s-2}^2 + \|\tilde A_t \operatorname{div}(\tilde A^T\nabla v_\kappa)\|_{s-2}^2\\ &+ \|\tilde A\operatorname{div}(\tilde A_t^T\nabla v_\kappa)\|_{s-2}^2 + |v_{\kappa t}|_{s-0.5}^2\Big]\,.\end{aligned}$$

We then conclude from (6.1) that

$$
\begin{aligned}
\|v_{\kappa t}\|_2^2 + \|q_{\kappa t}\|_1^2 &\le C\Big[\|v_{\kappa tt}\|_0^2 + \|\tilde{F}_t\|_0^2 + \|\nabla^2 v_\kappa\|_0^2 + |v_{\kappa t}|_{1.5}^2\Big] \\
&\le C\Big[\|v_{\kappa tt}\|_0^2 + \|v_{\kappa t}\|_0^2 + \|\tilde{F}_t\|_0^2 + \|\tilde{F}\|_0^2 + |v_\kappa|_{1.5}^2 + |v_{\kappa t}|_{1.5}^2\Big]
\end{aligned} \tag{6.2}
$$

and

$$
\begin{aligned}
\|v_{\kappa t}\|_3^2 + \|q_{\kappa t}\|_2^2 &\le C\Big[\|v_{\kappa tt}\|_1^2 + \|\tilde{F}_t\|_1^2 + \|\nabla^2 v_\kappa\|_1^2 + |v_{\kappa t}|_{2.5}^2\Big] \\
&\le C\Big[\|v_{\kappa tt}\|_1^2 + \|v_{\kappa t}\|_1^2 + \|\tilde{F}_t\|_1^2 + \|\tilde{F}\|_1^2 + |v_{\kappa t}|_{2.5}^2 + |v_\kappa|_{2.5}^2\Big] .
\end{aligned} \tag{6.3}
$$

Combining (6.1) (with $s = 5$) and (6.3), we find that

$$
\begin{aligned}
\int_0^t \Big[\|v_\kappa\|_5^2 + \|v_{\kappa t}\|_3^2 + \|q_\kappa\|_4^2 + \|q_{\kappa t}\|_2^2\Big] ds &\le C\mathcal{M}(\varepsilon, u_0, f) \\
&\quad + C\int_0^t \Big[\|v_{\kappa tt}\|_1^2 + \|v_{\kappa t}\|_1^2 + |v_{\kappa t}|_{2.5}^2 + |v_\kappa|_{4.5}^2\Big] ds
\end{aligned} \tag{6.4}
$$

for some constant C independent of M.

6.3. Energy estimates. Let $\varphi = \zeta\bar{\partial}^8(\zeta v_\kappa)$ be a test function in (5.33) and $\varphi = \zeta\bar{\partial}^4(\zeta v_{\kappa t})$ be a test function in (5.32), where $\bar{\partial}$ denotes the tangential differentiation or the tangential difference quotient. The interior terms can be estimated in the same way as in [5], and we find that

$$
\begin{aligned}
&\|\bar{\partial}^4(\zeta v_\kappa(t))\|_0^2 + \bar{\sigma}\varepsilon|\bar{\partial}^4 v_\kappa(t)|_0^2 + \int_0^t \Big[\|\nabla\bar{\partial}^4(\zeta v_\kappa)\|_0^2 + \kappa\varepsilon^3|\bar{\partial}^6 v_\kappa|_0^2\Big] ds \\
&\le C_\kappa\mathcal{M}(\varepsilon, u_0, f) + C\int_0^t \Big[\|v_{\kappa tt}\|_0^2 + \|v_{\kappa t}\|_0^2 + |v_\kappa|_{3.5}^2 + |v_{\kappa t}|_{1.5}^2 + \kappa\varepsilon^3|v_\kappa|_5^2\Big] ds \\
&\quad + C\varepsilon^3\int_0^t\int_\Gamma \frac{1}{\sqrt{a}}\bar{\partial}^2\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{b}_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\tilde{n}^i\Big]_{,\gamma\delta}\bar{\partial}^6 v_\kappa^i\, dSds \\
&\quad + C\varepsilon^3\int_0^t\int_\Gamma \frac{1}{\sqrt{a}}\bar{\partial}^2\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{b}_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\tilde{\Gamma}^\tau_{\gamma\delta}\tilde{n}^i\Big]_{,\tau}\bar{\partial}^6 v_\kappa^i\, dSds
\end{aligned} \tag{6.5}
$$

and

$$
\begin{aligned}
&\|\bar{\partial}^2(\zeta v_{\kappa t}(t))\|_0^2 + \bar{\sigma}\varepsilon|\bar{\partial}^2 v_{\kappa t}(t)|_0^2 + \int_0^t \Big[\|\nabla\bar{\partial}^2(\zeta v_{\kappa t})\|_0^2 + \kappa\varepsilon^3|\bar{\partial}^4 v_{\kappa t}|_0^2\Big] ds \\
&\le C_\kappa\mathcal{M}(\varepsilon, u_0, f) + C\int_0^t \Big[\|v_{\kappa tt}\|_0^2 ds + |v_{\kappa t}|_{1.5}^2 + \kappa\varepsilon^3|v_{\kappa t}|_3^2\Big] ds \\
&\quad + C\varepsilon^3\int_0^t\int_\Gamma \frac{1}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{b}_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\tilde{n}^i\Big]_{t,\gamma\delta}\bar{\partial}^4 v_{\kappa t}^i\, dSds \\
&\quad + C\varepsilon\int_0^t\int_\Gamma \frac{1}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(\tilde{b}_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\tilde{\Gamma}^\tau_{\gamma\delta}\tilde{n}^i\Big]_{t,\tau}\bar{\partial}^4 v_{\kappa t}^i\, dSds .
\end{aligned} \tag{6.6}
$$

Passing $\theta \to 0$ in (5.30), we find that

$$
\sup_{t\in[0,T]}\Big[\|v_{\kappa tt}(t)\|_0^2 + \bar{\sigma}\varepsilon|v_{\kappa tt}(t)|_0^2\Big] + \int_0^T \Big[\|\nabla v_{\kappa tt}\|_0^2 + \kappa\varepsilon^3|v_{\kappa tt}|_2^2\Big] ds \le C_\kappa\mathcal{M}(\varepsilon, u_0, f) . \tag{6.7}
$$

Define $\mathcal{E}_\kappa(t)$ by

$$\mathcal{E}_\kappa(t) = \sup_{s\in[0,t]} \Big[\|v_\kappa\|_4^2 + \|v_{\kappa t}\|_2^2 + \|v_{\kappa tt}\|_0^2 + \bar\sigma\varepsilon(|v_\kappa|_4^2 + |v_{\kappa t}|_2^2 + |v_{\kappa tt}|_0^2)\Big](s) + \int_0^t \Big[\|v_\kappa\|_5^2 + \|v_{\kappa t}\|_3^2 + \|v_{\kappa tt}\|_1^2 + \kappa\varepsilon^3(|v_\kappa|_6^2 + |v_{\kappa t}|_4^2 + |v_{\kappa tt}|_2^2)\Big] ds\,.$$

Then (6.5) and (6.6) imply that

$$\begin{aligned} &\int_0^t \Big[|v_\kappa|_{4.5}^2 + |v_{\kappa t}|_{2.5}^2 + \kappa\varepsilon^3(|v_\kappa|_6^2 + |v_{\kappa t}|_4^2)\Big] ds \\ &\le C_{\bar\delta,\kappa}\mathcal{M}(\varepsilon,u_0,f) + (Ct+\bar\delta)\mathcal{E}_\kappa(t) + C_\kappa\varepsilon^3 \int_0^t \Big[|\tilde\eta|_6^2 + |\tilde v|_4^2\Big] ds\,. \end{aligned} \tag{6.8}$$

Using (6.7) and (6.8), we conclude from (6.4) that

$$\mathcal{E}_\kappa(t) \le C_{\bar\delta,\kappa}\mathcal{M}(\varepsilon,u_0,f) + (Ct + C_\kappa\varepsilon^3 t + \bar\delta)\mathcal{E}_\kappa(t)$$

and by choosing $\bar\delta > 0$ and $T = T(\kappa,\varepsilon,M) > 0$ small enough,

$$\mathcal{E}_\kappa(t) \le C_\kappa\mathcal{M}(\varepsilon,u_0,f)\,. \tag{6.9}$$

6.4. Elliptic estimates. In order to close the iteration scheme, we need to have controls on $\|v_\kappa\|^2_{L^2(0,T;H^{7.5}(\Gamma))}$ and $\|v_{\kappa t}\|^2_{L^2(0,T;H^{5.5}(\Gamma))}$. By (6.9), v_κ is a strong solution to (5.5) and hence (5.5d) holds for almost all $t\in[0,T]$, or

$$\begin{aligned} k\varepsilon^3\frac{1}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}v^i_{\kappa,\alpha\beta}\Big]_{,\gamma\delta} &= -\Big[\nu(D_{\tilde A}v)^j_i - q\mathrm{Id}^i_j\Big]\tilde A^\ell_j N_\ell - \Big[\varepsilon\mathcal{L}_m(\tilde\eta) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\tilde\eta)\Big]^i \\ &\quad - \bar\sigma\varepsilon v^i_{\kappa t} - \kappa\varepsilon L_0 \qquad \text{on } (0,T)\times\Gamma\,. \end{aligned}$$

By the elliptic regularity,

$$\kappa\varepsilon^3|v_\kappa|^2_{7.5} \le C\Big[\|v_\kappa\|_5^2 + \|q_\kappa\|_4^2 + \varepsilon|\mathcal{L}_m(\tilde\eta)|^2_{3.5} + \varepsilon^3|\mathcal{L}_b(\tilde\eta)|^2_{3.5} + \bar\sigma\varepsilon|v_t|^2_{3.5} + \kappa\varepsilon|L_0|^2_{3.5}\Big] \tag{6.10}$$

for some constant C depending on $|a|_{5.5}$. Since

$$|\tilde\eta(t)|^2_{7.5} \le C\Big[|\mathbf{e}|_{7.5} + t\int_0^t |\tilde v|^2_{7.5}ds\Big] \le C\Big[|\Gamma|_{7.5} + tM\Big]\,,$$

choosing $T = T(\kappa,\varepsilon,M) > 0$ small enough, by (6.1), (6.2), and (6.7) we find that

$$\kappa\varepsilon^3\int_0^T |v_\kappa|^2_{7.5}dt \le C_\kappa\mathcal{M}(\varepsilon,u_0,f)\,. \tag{6.11}$$

Similarly, time differentiating (5.5d) and applying elliptic regularity, we find that

$$\kappa\varepsilon^3\int_0^T |v_{\kappa t}|_{5.5}dt \le C_\kappa\mathcal{M}(\varepsilon,u_0,f)\,. \tag{6.12}$$

Combining (6.9), (6.11), and (6.12), we conclude that

$$\|v_\kappa\|^2_{\mathcal{V}^5(T)} + \int_0^T \Big[|v_\kappa|^2_{7.5} + |v_{\kappa t}|^2_{5.5}\Big]dt \le C_\kappa\mathcal{M}(\varepsilon,u_0,f)\,. \tag{6.13}$$

6.5. Fixed-point arguments. Let $M = C_\kappa \mathcal{M}(\varepsilon, u_0, f)$ from (6.13). Then for $T = T(\kappa, \varepsilon) > 0$ small enough, we conclude from (6.13) that

$$\|v_\kappa\|^2_{\mathcal{V}^5(T)} + \int_0^T \Big[|v_\kappa|^2_{7.5} + |v_{\kappa t}|^2_{5.5}\Big] dt \le C_\kappa \mathcal{M}(\varepsilon, u_0, f)\,.$$

Therefore, we established a map $\Phi : \bar v \mapsto v$ mapping from the closed convex set

$$C_T(M) := \left\{ v \in \mathcal{V}^5(T) \Big| \|v\|^2_{\mathcal{V}^5(T)} + \int_0^T \Big[|v|^2_{7.5} + |v_t|^2_{5.5}\Big] dt \le M \right\}$$

into itself. As in [5], the map Φ is weakly continuous defined on $C_T(M)$. Therefore, by the Tychonoff fixed-point theorem, there exists a fixed point $v \in C_T(M)$ to the map Φ.

7. Improved regularity for b and g.

7.1. Estimates without considering the artificial viscosity. We want to study the equation

$$\frac{\varepsilon^3}{3}\mathcal{L}_b(\eta) + \varepsilon \mathcal{L}_m(\eta) = h - \bar\sigma \varepsilon \eta_{tt} \qquad \text{on} \quad \Gamma \tag{7.1}$$

given h in $L^2(0,T;H^{3.5}(\Gamma))$, where we remind the readers again that

$$\begin{aligned} \mathcal{L}_b(\eta) = {} & \frac{2}{\sqrt{a}}\Big[\sqrt{a} a^{\alpha\beta\gamma\delta}(\eta_{,\alpha\beta}\cdot n - \mathfrak{b}_{\alpha\beta}) n\Big]_{,\gamma\delta} && (\equiv \mathrm{A}) \\ & + \frac{2}{\sqrt{a}}\Big[\sqrt{a} a^{\alpha\beta\gamma\delta} g^{\sigma\tau}(\eta_{,\alpha\beta}\cdot n - \mathfrak{b}_{\alpha\beta})(\eta_{,\gamma\delta}\cdot \eta_{,\sigma}) n\Big]_{,\tau} && (\equiv \mathrm{B}) \end{aligned}$$

and

$$\mathcal{L}_m(\eta) = -\frac{1}{\sqrt{a}}\Big[\sqrt{a} a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta_{,\gamma}\Big]_{,\delta} \qquad (\equiv \mathrm{C})\,.$$

7.2. Some identities and inequalities. If $\Gamma^{\cdot}_{\cdot\cdot}$ denotes the Christoffel symbol with respect to the metric g,

$$\eta_{,\alpha\beta} = b_{\alpha\beta} n + \Gamma^\gamma_{\alpha\beta}\eta_{,\gamma}\,, \tag{7.2a}$$

$$n_{,\gamma} = -g^{\alpha\beta} b_{\alpha\gamma}\eta_{,\beta}\,. \tag{7.2b}$$

Define the energy $\mathcal{E}_{\bar\sigma}(T)$ by

$$\begin{aligned} \mathcal{E}_{\bar\sigma}(T) = \sup_{t\in[0,T]} & \Big[\|v_{tt}(t)\|^2_0 + \bar\sigma\varepsilon\Big(|v(t)|^2_4 + |v_t(t)|^2_2 + |v_{tt}(t)|^2_0\Big) + \varepsilon|g(t)|^2_4 + \varepsilon^3|b(t)|^2_4\Big] \\ & + \int_0^T \Big[\|v(t)\|^2_5 + \|v_t(t)\|^2_3 + \|v_{tt}(t)\|^2_1 + \varepsilon^3|b(t)|^2_{4.5}\Big] dt\,. \end{aligned}$$

Then $\sup_{t\in[0,T]} \|\eta(t) - \mathbf{e}\|^2_5 \le T\mathcal{E}_{\bar\sigma}(T)$ and

$$\sup_{t\in[0,T]} \Big[|g(t)|_{3.5} + |b(t)|_{2.5} + |\eta(t)|_{4.5} + |n(t)|_{3.5}\Big] \le C\Big[1 + T\mathcal{P}(\mathcal{E}_{\bar\sigma}(T))\Big]\,,$$

$$\sup_{t\in[0,T]} \Big[|g(t) - \mathfrak{g}|^2_{3.5} + |b(t) - \mathfrak{b}|^2_{2.5}\Big] \le CT\mathcal{P}(\mathcal{E}_{\bar\sigma}(T))\,,$$

$$\sup_{t\in[0,T]} |n(t)|_5 \le \frac{C}{\varepsilon^3}\Big[\varepsilon^3 + \mathcal{E}_{\bar\sigma}(T)\Big]\Big[1 + T\mathcal{P}(\mathcal{E}_{\bar\sigma}(T))\Big]\,,$$

where the constant $C = C(|\Gamma|_{4.5})$. Moreover, by interpolation,

$$|b-\mathfrak{b}|^2_{3.5} \le C|b-\mathfrak{b}|_4^{\frac{4}{3}}|b-\mathfrak{b}|_{2.5}^{\frac{2}{3}} \le \frac{CT^{\frac{1}{3}}}{\varepsilon^2}\Big[1+\mathcal{P}(\mathcal{E}_{\bar\sigma}(T))\Big],$$

and by (7.2), for $1 \le s \le 6$,

$$|\eta(t)|_{s+1} \le C\Big[1+|b(t)|_{s-1}+|g(t)|_s\Big]\Big[1+T\mathcal{P}(\mathcal{E}_{\bar\sigma}(T))\Big], \tag{7.3}$$

$$|n(t)|_{s+1} \le C\Big[1+|b(t)|_{s}+|g(t)|_s\Big]\Big[1+T\mathcal{P}(\mathcal{E}_{\bar\sigma}(T))\Big]. \tag{7.4}$$

In the following discussion, we use C (or $C_{\bar\delta}$) to denote a constant independent of $\mathcal{E}_{\bar\sigma}$. From the κ-problem, $\mathcal{E}_\kappa(T_\kappa) \le M_\kappa$ for some constant M_κ. We further choose $T_\kappa > 0$ so that $T_\kappa^{1/6}\mathcal{P}(\mathcal{E}_\kappa(T_\kappa)) \le 1$. Therefore, the inequalities above become

$$\sup_{t\in[0,T]}\Big[|g(t)|_{3.5}+|b(t)|_{2.5}+|\eta(t)|_{4.5}+|n(t)|_{3.5}\Big] \le C\,, \tag{7.5a}$$

$$\sup_{t\in[0,T]}\Big[|g(t)-\mathfrak{g}|^2_{3.5}+|b(t)-\mathfrak{b}|^2_{2.5}\Big] \le C\sqrt{T}\,, \tag{7.5b}$$

$$|b-\mathfrak{b}|^2_{3.5} \le \frac{CT^{1/6}}{\varepsilon^2}\,, \tag{7.5c}$$

$$|\eta(t)|_{s+1} \le C\Big[1+|b(t)|_{s-1}+|g(t)|_s\Big]\,, \tag{7.5d}$$

$$|n(t)|_{s+1} \le C\Big[1+|b(t)|_{s}+|g(t)|_s\Big]\,. \tag{7.5e}$$

Remark 15. By (7.5), $n \in H^{s+1}(\Gamma)$, provided that $b\ (= \bar\partial^2\eta\cdot n)$ and $g\ (= \bar\partial\eta\cdot\bar\partial\eta)$ are both $H^s(\Gamma)$-functions. On the contrary, $\bar\partial^2\eta\cdot\tilde n \in H^s(\Gamma)$ and $\bar\partial\eta\cdot\bar\partial\tilde\eta \in H^s(\Gamma)$ at best imply that $\eta \in H^{s+1}(\Gamma)$, and as a result, $n \in H^s(\Gamma)$. The nonlinear structure ensures that n is as regular as η and $n \in H^{s+1}(\Gamma)$ is crucial to close the estimates.

7.3. User's guide, part III: An illustration of the elliptic-type estimates on the boundary.

7.3.1. An illustration of the 3D case wherein ε must be taken small. In this subsection, we use a relatively simple problem to illustrate how the elliptic-type estimate on the boundary for the 3D case can be obtained with ε taken sufficiently small. Note that the 2D case does not require smallness of ε.

We will use a one-dimensional boundary to illustrate the need for smallness of ε. We consider the case $\Gamma = \mathbb{T}^1$ so that at the initial time $t = 0$, $\mathfrak{b} = 0$, and $\mathfrak{g} = 1$, and $a^{\alpha\beta\gamma\delta} = 1$. With this assumption on the initial data, the left-hand side of (2.5d) reduces to $-\varepsilon\big[(g-1)\eta'\big]' + \varepsilon^3\big[(bn)'' + (g^{-1}bg'n)'\big]$; denoting the forcing function by h, we obtain

$$-\varepsilon\Big[(g-1)\eta'\Big]' + \varepsilon^3\Big[(bn)'' + (g^{-1}bg'n)'\Big] = h \quad \text{on } \mathbb{T}^1\times(0,T]\,, \tag{7.6}$$

where we assume that $h \in L^2(0,T;H^1(\Gamma))$. Before proceeding, we assume that $|g|_1$, $|b|_1$, and $|n|_1$ are bounded by a generic constant C. Twice differentiating (7.6) and then testing the resulting equation against η'', we find that

$$\varepsilon\int_\Gamma\Big[\underline{g''\eta'} + 2g'\eta'' + (g-1)\eta'''\Big]\cdot\eta'''dS + \varepsilon^3\int_\Gamma\Big[\underline{b''n} + 2b'n' + b_{\alpha\beta}n''\Big]\cdot\eta''''dS$$
$$= \varepsilon^3\int_\Gamma\Big[g^{-1}bg'n\Big]''\cdot\eta'''dS + \int_\Gamma h''\cdot\eta''dS\,.$$

The underlined terms will produce the energy contribution

$$\mathcal{E}_s \equiv \varepsilon |g|_2^2 + \varepsilon^3 |b|_2^2,$$

which, by (7.5d) and (7.5e), implies that $\varepsilon|\eta|_3^2$ and $\varepsilon^3|n|_3^2$ are bounded, and satisfy the inequality

$$\varepsilon|\eta|_3^2 + \varepsilon^3|n|_3^2 \le C(1+\mathcal{E}_s)\,. \tag{7.7}$$

The energy method produces error terms, the worst of which is given by

$$W_1 \equiv \varepsilon \int_\Gamma (g-1)\eta''' \cdot \eta''' dS, \quad W_2 \equiv \varepsilon^3 \int_\Gamma b n'' \cdot \eta'''' dS, \quad W_3 \equiv \varepsilon^3 \int_\Gamma \left[g^{-1} b g' n\right]'' \eta''' dS\,.$$

It is easy to see that

$$|W_1| \le \varepsilon \|g-1\|_{L^\infty(\Gamma)} |\eta|_3^2 \le C\|g-1\|_{L^\infty(\Gamma)}(1+\mathcal{E}_s)\,.$$

To estimate W_2, we integrate by parts and use the embedding $H^1(\Gamma) \subseteq L^\infty(\Gamma)$ to find that

$$|W_2| \le \varepsilon^3 \|b\|_{L^\infty(\Gamma)} |n|_3 |\eta|_3 \le C\varepsilon(1+\mathcal{E}_s)\,.$$

For W_3, when two derivatives hit g^{-1} or n, the estimates are similar to the estimate for W_2, so we concentrate on the terms

$$W_{31} = \varepsilon^3 \int_\Gamma g^{-1} b'' g' n \cdot \eta''' dS\,, \quad W_{32} = \varepsilon^3 \int_\Gamma g^{-1} b g''' n \cdot \eta''' dS\,.$$

We make use of the important geometric identity

$$n \cdot \eta''' = b' - \eta'' \cdot n' = b' + \frac{1}{2} g^{-1} b g'$$

to write the inequality

$$|n\cdot \eta'''|_1 \le C\left[|b|_2 + |g|_{1.5}^2 + |g|_2 + |b|_{1.5}|g|_{1.5}\right] \le C\left[1 + |b|_2 + |g|_2\right]. \tag{7.8}$$

Integrating by parts to move one derivative off of g''' in W_{32}, we obtain that

$$|W_{31}| + |W_{32}| \le C\varepsilon^3 |g|_2 |n\cdot\eta'''|_1 \le C\varepsilon(1+\mathcal{E}_s)\,.$$

Collecting all of the above inequalities and using Young's inequality for the integral containing the forcing function h, we find that

$$\mathcal{E}_s \le \frac{C}{\varepsilon}|h|_1^2 + C\left[\varepsilon + \|g-1\|_{L^\infty(\Gamma)}\right](1+\mathcal{E}_s)\,. \tag{7.9}$$

As can be seen from the right-hand side of (7.9), the error term W_1 produces the coefficient $\|g-1\|_{L^\infty(\Gamma)}$, which *must be made small* in order to obtain an elliptic-type estimate which bounds $\mathcal{E}_s$ by the forcing function h.

Clearly, whenever $g(t)$ is continuous in time, the term $\|g-1\|_{L^\infty(\Gamma)}$ can be made arbitrarily small by taking $T>0$ as small as necessary. This occurs when the forcing function $h(t)$ is continuous in time. For us, the forcing $h(t)$ is representing the fluid traction and, because of the regularity of the fluid velocity, is necessarily continuous in time.

As a consequence, if $\|g-1\|_{L^\infty(\Gamma)}$ and ε can be made small enough, then

$$\mathcal{E}_s \le \frac{C}{\varepsilon}|h|_1^2 + C\varepsilon\,.$$

7.3.2. An illustration of the 2D case wherein the elliptic-type estimate is independent of ε. For the 2D case, we obtain two new identities that lead to the ε-independent elliptic-type estimates.

By computing the normal and tangential components of (7.6), we find that

$$\varepsilon^3 b'' = h\cdot n + \varepsilon(g-1)b + \varepsilon^3 g^{-1}b^2 - \varepsilon^3(g^{-1}g'b)'\,, \tag{7.10a}$$

$$\varepsilon g' = h\cdot\eta' + \frac{3}{2}\varepsilon(g-1)g + 3\varepsilon^3 b'b - \frac{1}{2}\varepsilon^3 g^{-1}g'b^2 + \varepsilon^3(g^{-1}bg')'b\,. \tag{7.10b}$$

Thus, using (7.10a) and (7.10b) we see that b and g verify the inequalities

$$\varepsilon^3|b|_2^2 \le C(\varepsilon)\Big[|h|_0^2 + 1 + |g|_{1.5}^4 + \|b\|^2_{L^\infty(\Gamma)}|g|_2^2\Big]\,, \tag{7.11a}$$

$$\varepsilon|g|_2^2 \le C(\varepsilon)\Big[|h|_1^2 + 1 + |b|_{1.5}^4 + (\|g-1\|^2_{L^\infty(\Gamma)} + \|b\|^2_{L^\infty(\Gamma)})(|b|_2^2 + |g|_2^2)\Big]\,. \tag{7.11b}$$

Again, by continuity-in-time of $\eta(t)$ (which is due to the regularizing effects of the Navier–Stokes fluid), if we choose $T>0$ sufficiently small, then $\|g-1\|_{L^\infty(\Gamma)}$ and $\|b\|_{L^\infty(\Gamma)}$ can be made arbitrarily small, so that (7.11) implies the estimate

$$\varepsilon|g|_2^2 + \varepsilon^3|b|_2^2 \le C(\varepsilon)(|h|_1^2 + 1)\,.$$

7.4. Estimates for b and g without the artificial viscosity. Now suppose that $h\in H^4(\Gamma)$ and $\Gamma\in H^7$ (thus $a\in H^6(\Gamma)$ and $\mathfrak{b}\in H^5(\Gamma)$). By the identity $\mathcal{L}_b(\eta) = \mathcal{L}_b(\eta) - [\mathcal{L}_b(\eta)\cdot n]n + [\mathcal{L}_b(\eta)\cdot n]n$,

$$\begin{aligned}\mathcal{L}_b(\eta) = {} & \frac{4}{\sqrt{a}}\Big[\sqrt{a}a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\Big]_{,\gamma} n_{,\delta} + 2a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\Big[n_{,\gamma\delta} - (n_{,\gamma\delta}\cdot n)n\Big] \\ & + 2a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\Gamma^\tau_{\gamma\delta}n_{,\tau} + \frac{3}{\varepsilon^2}a^{\alpha\beta\gamma\delta}(g_{\alpha\beta}-\mathfrak{g}_{\alpha\beta})b_{\gamma\delta}n \\ & + \frac{3}{\varepsilon^3}(h\cdot n)n - \frac{3}{\varepsilon^2}(v_t\cdot n)n\,.\end{aligned} \tag{7.12}$$

Test (7.12) against $\frac{1}{2}\bar\partial^{10}\eta$. First, we note that

$$\begin{aligned}&\frac{1}{2}\int_\Gamma \mathcal{L}_b(\eta)\cdot\bar\partial^{10}\eta dS \\ &= -\int_\Gamma a^{\alpha\beta\gamma\delta}\bar\partial^5(b_{\alpha\beta}-\mathfrak{b}_{\gamma\delta})\bar\partial^5 b_{\gamma\delta}dS - \int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\sum_{k=0}^{9}C^{10}_k\bar\partial^k\eta_{,\gamma\delta}\bar\partial^{10-k}n dS \\ &\quad - \int_\Gamma \frac{1}{\sqrt{a}}\sum_{k=0}^{4}\Big[C^5_k\bar\partial^{5-k}(\sqrt{a}a^{\alpha\beta\gamma\delta})\bar\partial^k(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\Big]\bar\partial^5 b_{\gamma\delta}dS \\ &\quad - \int_\Gamma a^{\alpha\beta\gamma\delta}g^{\sigma\tau}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})(\eta_{,\gamma\delta}\cdot\eta_{,\sigma})\Big[\bar\partial^9(\bar\partial\eta_{,\tau}\cdot n) - \sum_{k=0}^{8}C^9_k\bar\partial^{k+1}\eta_{,\tau}\bar\partial^{9-k}n\Big]dS\,.\end{aligned}$$

The first term on the right-hand side will produce the energy $|b|_5^2$ with error term $\int_\Gamma a^{\alpha\beta\gamma\delta}\bar\partial^5\mathfrak{b}_{\alpha\beta}\bar\partial^5 b_{\alpha\beta}dS$, which can be bounded by $\bar\delta|b|_5^2 + C_{\bar\delta}|\mathfrak{b}|_5^2$ for some constant $C_{\bar\delta}$ depending on $|a|_{1.5}$ for all $\bar\delta>0$. For the second term (on the right-hand side), integrating by parts if necessary, with (7.3) and (7.4) we find that

$$\begin{aligned}
&\left|\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\sum_{k=0}^{9} C_k^{10}\bar\partial^k\eta_{,\gamma\delta}\bar\partial^{10-k}n dS\right| \\
&\le C\Big[\Big(|a|_{1.5}|b-\mathfrak{b}|_5+|a|_5|b-\mathfrak{b}|_{1.5}+|a|_4|b-\mathfrak{b}|_{2.5}+|a|_{2.5}|b-\mathfrak{b}|_4\Big)|n|_{3.5} \\
&\qquad +|a|_{1.5}|b-\mathfrak{b}|_{1.5}|n|_6\Big]|\eta|_6+C|a|_4|b-\mathfrak{b}|_4||\eta|_{4.5}|n|_6 \\
&\le C_{\bar\delta}\Big[1+|a|_5^2+|\mathfrak{b}|_5^2\Big]+\bar\delta|b|_5^2+C_{\bar\delta}|g|_5^2+C\sqrt{T}\Big[|b|_5^2+|g|_5^2\Big]
\end{aligned}$$

for some constant C and $C_{\bar\delta}$ depending on $|a|_4$. Similarly, for the last term we integrate by parts and apply Young's inequality to obtain

$$\begin{aligned}
&\left|\int_\Gamma a^{\alpha\beta\gamma\delta}g^{\sigma\tau}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})(\eta_{,\gamma\delta}\cdot\eta_{,\sigma})\Big[\bar\partial^9(\bar\partial\eta_{,\tau}\cdot n)-\sum_{k=0}^{8}C_k^9\bar\partial^{k+1}\eta_{,\tau}\bar\partial^{9-k}n\Big]dS\right| \\
&\le C\Big[|a|_3\Big(|b-\mathfrak{b}|_4|\Gamma^{\cdot}_{\cdot\cdot}|_{1.5}+|b-\mathfrak{b}|_{1.5}|\Gamma^{\cdot}_{\cdot\cdot}|_4+|b-\mathfrak{b}|_3|\Gamma^{\cdot}_{\cdot\cdot}|_{2.5}+|b-\mathfrak{b}|_{2.5}|\Gamma^{\cdot}_{\cdot\cdot}|_3\Big) \\
&\qquad +|a|_4|b-\mathfrak{b}|_{1.5}|\Gamma^{\cdot}_{\cdot\cdot}|_{1.5}\Big](|b|_5+|\eta|_6)+C\Big[|a|_{1.5}|b-\mathfrak{b}|_{1.5}|\Gamma^{\cdot}_{\cdot\cdot}|_{1.5}|\eta|_6 \\
&\qquad +|a|_4\Big(|b-\mathfrak{b}|_{1.5}|\Gamma^{\cdot}_{\cdot\cdot}|_4+|b-\mathfrak{b}|_4|\Gamma^{\cdot}_{\cdot\cdot}|_{1.5}+|b-\mathfrak{b}|_{2.5}|\Gamma^{\cdot}_{\cdot\cdot}|_3+|b-\mathfrak{b}|_3|\Gamma^{\cdot}_{\cdot\cdot}|_{2.5}\Big) \\
&\qquad +|a|_{2.5}|b-\mathfrak{b}|_{2.5}|\Gamma^{\cdot}_{\cdot\cdot}|_{2.5}|\eta|_5\Big]|n|_5 \\
&\le C_{\bar\delta}\Big[1+|\mathfrak{b}|_5^2+|g|_5^2\Big]+\bar\delta|b|_5^2+C\sqrt{T}\Big[|b|_5^2+|g|_5^2\Big]\,.
\end{aligned}$$

As for the third term, Young's inequality directly implies that

$$\left|\int_\Gamma\frac{1}{\sqrt{a}}\sum_{k=0}^{4}\Big[C_k^5\bar\partial^{5-k}(\sqrt{a}a^{\alpha\beta\gamma\delta})\bar\partial^k(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\Big]\bar\partial^5 b_{\gamma\delta}dS\right|\le C_{\bar\delta}\Big[1+|\mathfrak{b}|_4^2\Big]+\bar\delta|b|_5^2 \tag{7.13}$$

for some $C_{\bar\delta}$ depending on $|a|_4$. Combining all the estimates above, by choosing $T>0$ and $\bar\delta>0$ small enough, for some constant $\lambda_0>0$ we find that

$$\lambda_0|b|_5^2\le-\int_\Gamma\mathcal{L}_b(\eta)\cdot\bar\partial^{10}\eta dS+C\Big[1+|a|_5^2+|\mathfrak{b}|_5^2\Big]+C|g|_5^2\,, \tag{7.14}$$

where C and $C_{\bar\delta}$ depend on $|a|_4$.

Test the right-hand side of (7.12) against $\bar\partial^{10}\eta$. For the terms due to the membrane energy and forcing h, similarly to the argument above, rewriting $n\cdot\bar\partial^{10}\eta$ as $\bar\partial^8 b-\sum_{k=1}^{8}C_k^8\bar\partial^{10-k}\eta\cdot\bar\partial^k n$ and integrating by parts imply that

$$\left|\int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta}-\mathfrak{g}_{\alpha\beta})b_{\gamma\delta}n\cdot\bar\partial^{10}\eta dS\right|\le\frac{C_{\bar\delta}\sqrt{T}}{\varepsilon^2}\Big[1+|a|_3^2\Big]+\bar\delta\varepsilon^2\Big[|b|_5^2+|g|_5^2\Big]$$

and

$$\left|\int_\Gamma(h\cdot n)n\cdot\bar\partial^{10}\eta dS\right|\le C|h|_3\Big[1+|b|_5+|g|_5\Big].$$

For the term due to the inertia, we study the time integral and integrate by parts in time to obtain

$$\int_0^T \int_\Gamma (v_t \cdot n) n \cdot \bar\partial^{10} \eta dS dt = \int_\Gamma (v \cdot n)(n \cdot \bar\partial^{10}\eta) dS \Big|_{t=0}^{t=T} - \int_0^T \int_\Gamma v^i (n^i n^j \bar\partial^{10} \eta^j)_t dS dt\,.$$

By the identity $n \cdot \bar\partial^{10}\eta = \bar\partial^8 b - \sum_{k=0}^7 C_k^8 \bar\partial^{9-k}\eta \bar\partial^{k+1} n$ again,

$$\begin{aligned}\Big|\int_0^T \int_\Gamma (v_t \cdot n) n \cdot \bar\partial^{10} \eta dS dt\Big| \le C\Big[|u_0|_4^2 + |\Gamma|_6^2\Big] + C\int_0^T \Big[|v|_5^2 + |g|_5^2 + |b|_4^2\Big] dt \\ + \Big[\frac{C_{\bar\delta_1}}{\varepsilon^2} |v|_{3.5}^2 + \bar\delta_1 \varepsilon^2 (|b|_{4.5}^2 + |g|_{4.5}^2)\Big](T)\,.\end{aligned} \tag{7.15}$$

Integrating by parts and applying Young's inequality for the remaining terms on the right-hand side, we find that

$$\begin{aligned}\int_0^T \int_\Gamma \mathrm{RHS} \cdot \bar\partial^{10}\eta dS dt \le \frac{C_{\bar\delta}}{\varepsilon^4}\Big[1 + |a|_3^2 + |\mathfrak{b}|_5^2 + \varepsilon^2(|u_0|_4^2 + |\Gamma|_6^2)\Big] + \bar\delta \int_0^T |b|_5^2 dt \\ + (C_{\bar\delta} + CT^{1/4}) \int_0^T |g|_5^2 dt + \frac{C}{\varepsilon^2}\int_0^T |v|_5^2 dt + \frac{C_{\bar\delta}}{\varepsilon^6}\int_0^T |h|_3^2 dt \\ + \bar\sigma \Big[\frac{C_{\bar\delta_1}}{\varepsilon^4}|v|_{3.5}^2 + \bar\delta_1 (|b|_{4.5}^2 + |g|_{4.5}^2)\Big](T)\,,\end{aligned}$$

where $C_{\bar\delta_1} = \frac{C}{\bar\delta_1}$. Therefore, by choosing $\bar\delta > 0$ small enough,

$$\begin{aligned}\int_0^T |b|_5^2 dt \le \frac{C}{\varepsilon^4}\Big[1 + |a|_5^2 + |\mathfrak{b}|_5^2 + \varepsilon^2 (|u_0|_4^2 + |\Gamma|_6^2)\Big] + C\int_0^T |g|_5^2 dt + \frac{C}{\varepsilon^2}\int_0^T |v|_5^2 dt \\ + \frac{C}{\varepsilon^6}\int_0^T |h|_3^2 dt + \bar\sigma\Big[C_{\bar\delta_1}|v|_{4.5}^2 + \bar\delta_1(|b|_{4.5}^2 + |g|_{4.5}^2)\Big](T)\end{aligned} \tag{7.16}$$

for some constant C depending on $|a|_4$.

Remark 16. The above estimate can be obtained by testing (7.1) against $(n \cdot \bar\partial^{10}\eta)n$.

Testing (7.1) against $\bar\partial^{10}\eta$, we find that

$$\begin{aligned}&\varepsilon \int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^{10}\eta^i_{,\delta} dS + \frac{\varepsilon^3}{3}\int_\Gamma \mathcal{L}_b(\eta)\cdot \bar\partial^{10}\eta dS \\ &= \int_\Gamma h^i \bar\partial^{10}\eta^i dS - \bar\sigma\varepsilon \int_\Gamma \eta^i_{tt}\bar\partial^{10}\eta^i dS\,.\end{aligned}$$

Since

$$\begin{aligned}&2\int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^{10}\eta^i_{,\delta} dS \\ &= -\int_\Gamma a^{\alpha\beta\gamma\delta}\bar\partial^5 (g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\bar\partial^5 g_{\gamma\delta} dS + \int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\sum_{k=1}^9 C_k^{10}\bar\partial^{10-k}\eta^i_{,\gamma}\bar\partial^k \eta^i_{,\delta} dS \\ &\quad - \int_\Gamma \frac{1}{\sqrt{a}}\sum_{k=0}^4 C_k^5 \bar\partial^{5-k}(\sqrt{a}a^{\alpha\beta\gamma\delta})\bar\partial^k (g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\bar\partial^5 g_{\gamma\delta} dS\,,\end{aligned}$$

integrating by parts and Young's inequality imply that for some $\lambda_0 > 0$,

$$\lambda_0 |g|_5^2 \le -\int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^{10}\eta^i_{,\delta}dS + C|a|_4\Big[|g-\mathfrak{g}|_3|\eta|_5|\eta|_6 + |g-\mathfrak{g}|_4|g|_5\Big]$$
$$\le -\int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^{10}\eta^i_{,\delta}dS + C_{\bar\delta}(1+\sqrt{T}) + C_{\bar\delta}|\mathfrak{g}|_4^2 + \bar\delta|b|_4^2 + \bar\delta|g|_5^2\,.$$

Integrating by parts, the last two terms due to the forcing and the inertia satisfy the following estimates:

$$\left|\int_\Gamma h\cdot\bar\partial^{10}\eta dS\right| \le |h|_4|\eta|_6 \le \frac{C_{\bar\delta}}{\varepsilon}|h|_4^2 + \bar\delta\varepsilon\Big[1+|b|_4^2+|g|_5^2\Big]$$

and

$$\left|\int_0^T\int_\Gamma v_t\cdot\bar\partial^{10}\eta dSdt\right| \le C\Big[1+|u_0|_4^2+|\Gamma|_6^2\Big] + C\Big[|g(T)|_{4.5}^2+|v(T)|_{4.5}^2\Big] + \int_0^T|v|_5^2dt\,.$$

Combining (7.14) and the estimates above, by choosing $T>0$ and $\bar\delta>0$ small enough we find that

$$\int_0^T\Big[\varepsilon|g|_5^2+\varepsilon^3|b|_5^2\Big]dt \le C\varepsilon\Big[\frac{1}{\varepsilon^2}+|\mathfrak{g}|_4^2+\bar\sigma(|u_0|_4^2+|\Gamma|_6^2)+\varepsilon^2(|a|_5^2+|\mathfrak{b}|_5^2)\Big]$$
$$+ C\bar\sigma\varepsilon\Big[|g(T)|_{4.5}^2+|v(T)|_{4.5}^2\Big] + \frac{C}{\varepsilon}\int_0^T|h|_4^2dt + C\bar\sigma\varepsilon\int_0^T|v|_5^2dt. \tag{7.17}$$

Using (7.17) in (7.16), for $\varepsilon \ll 1$,

$$\varepsilon^3\int_0^T|b|_5^2dt \le \frac{C}{\varepsilon}\Big[1+|a|_5^2+|\mathfrak{b}|_5^2+\varepsilon^4|\mathfrak{g}|_4^2+\varepsilon^2(|u_0|_4^2+|\Gamma|_6^2)\Big]$$
$$+ C\varepsilon\int_0^T|v|_5^2dt + \frac{C}{\varepsilon^3}\int_0^T|h|_3^2dt + C\varepsilon\int_0^T|h|_4^2dt$$
$$+ \bar\sigma\Big[C_{\bar\delta_1}\varepsilon^3|v(T)|_{3.5}^2 + \bar\delta_1\varepsilon^3|b(T)|_{4.5}^2 + (C+\bar\delta_1)\varepsilon^3|g(T)|_{4.5}^2\Big]$$

for some constant C depending on $|a|_4$. Since $|a|_{4.5}+|\mathfrak{g}|_{4.5} \le \mathcal{P}(|\Gamma|_{5.5})$, by interpolation we conclude that

$$\varepsilon^3\int_0^T|b|_{4.5}^2dt \le M(\varepsilon,|u_0|_{3.5},|\Gamma|_{6.5}) + C\varepsilon\int_0^T|v|_{4.5}^2dt + \frac{C}{\varepsilon^3}\int_0^T|h|_{2.5}^2dt$$
$$+ C\varepsilon\int_0^T|h|_{3.5}^2dt + \bar\sigma\Big[C_{\bar\delta_1}\varepsilon^3|v(T)|_4^2 + \bar\delta_1\varepsilon^3|b(T)|_4^2 + (C+\bar\delta_1)\varepsilon^3|g(T)|_4^2\Big]\,.$$

Having $h = -\big[\nu(A^k_i v^j_{,k} + A^k_j v^i_{,k}) - q\mathrm{Id}^j_i\big]A^\ell_j N_\ell - \kappa\varepsilon^3 L_0$ in mind, since $|\kappa L_0|_{3.5}\le 1$ for κ small enough, we find that

$$\varepsilon^3\int_0^T|b|_{4.5}^2dt \le \mathcal{M}(\varepsilon,\|u_0\|_4,|\Gamma|_{6.5},\|f\|_{\mathcal{F}^4(T)}) + \Big[C\varepsilon+\frac{CT}{\varepsilon^3}\Big]\mathcal{E}_{\bar\sigma}(T)$$
$$+ \bar\sigma\Big[C_{\bar\delta_1}\varepsilon^2+\bar\delta_1+(C+\bar\delta_1)\varepsilon^2\Big]\mathcal{E}_{\bar\sigma}(T), \tag{7.18a}$$

$$\varepsilon\int_0^T|g|_{4.5}^2dt \le \mathcal{M}(\varepsilon,\|u_0\|_4,|\Gamma|_{6.5},\|f\|_{\mathcal{F}^4(T)}) + \left(\frac{C}{\varepsilon}+C\bar\sigma\varepsilon+C\right)\mathcal{E}_{\bar\sigma}(T) \tag{7.18b}$$

for some constants C and $C_{\bar\delta_1}$ depending on $|\Gamma|_5$.

Remark 17. Γ is regularized in order to obtain elliptic estimate (6.10).

7.5. Estimates of b and g with the artificial viscosity. With the artificial viscosity, we study the equation

$$\varepsilon\mathcal{L}_m(\eta) + \frac{\varepsilon^3}{3}\mathcal{L}_b(\eta) + \kappa\varepsilon^3\frac{1}{\sqrt{a}}(\sqrt{a}a^{\alpha\beta\gamma\delta}v_{,\alpha\beta})_{,\gamma\delta} = h - \bar{\sigma}\varepsilon\eta_{tt}\,. \tag{7.19}$$

Testing the equation above against $\bar{\partial}^{10}\eta$, it is easy to see that for $T > 0$ small enough (but independent of κ)

$$\begin{aligned}\kappa\varepsilon^3\sup_{t\in[0,T]}|\eta(t)|_7^2 + \int_0^T \Big[\varepsilon|g|_5^2 + \varepsilon^3|b|_5^2\Big]dt &\le C_{\bar{\delta}}\varepsilon\Big[\frac{1}{\varepsilon^2} + |\mathfrak{g}|_4^2 + \bar{\sigma}(|u_0|_4^2 + |\Gamma|_6^2)\Big]\\ &\quad + C\varepsilon^3(|a|_5^2 + |\mathfrak{b}|_5^2 + \kappa|\Gamma|_7^2)\Big] + C\bar{\sigma}\varepsilon\Big[|g(T)|_{4.5}^2 + |v(T)|_{4.5}^2\Big]\\ &\quad + \frac{C}{\varepsilon}\int_0^T|h|_4^2dt + C\bar{\sigma}\varepsilon\int_0^T|v|_5^2dt + \bar{\delta}\kappa\varepsilon^3\int_0^T|v|_6^2dt\end{aligned} \tag{7.20}$$

for some C depending on $|a|_4$, since

$$\begin{aligned}\frac{1}{2}\int_\Gamma a^{\alpha\beta\gamma\delta}\bar{\partial}^5\eta^i_{,\alpha\beta}(s)\bar{\partial}^5\eta^i_{,\gamma\delta}(s)dS\Big|_{s=0}^{s=t} &= -\int_0^t\int_\Gamma\frac{1}{\sqrt{a}}(\sqrt{a}a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta})_{,\gamma\delta}\bar{\partial}^{10}\eta^i dSds\\ &\quad + \int_0^t\int_\Gamma\frac{1}{\sqrt{a}}\sum_{k=0}^4 C_k^5\bar{\partial}^{5-k}(\sqrt{a}a^{\alpha\beta\gamma\delta})\bar{\partial}^k v^i_{,\alpha\beta}\bar{\partial}^5\eta_{,\gamma\delta}dSds\end{aligned}$$

and by Young's inequality the second integral is bounded by

$$C_{\bar{\delta}}\int_0^t|\eta|_{6.5}^2ds + \bar{\delta}\int_0^t|v|_{6.5}^2ds \le C_{\bar{\delta}} + \bar{\delta}\int_0^t\Big[|\eta(s)|_7^2 + |v(s)|_{6.5}^2\Big]ds\,.$$

Testing (7.19) against $(n\cdot\bar{\partial}^{10}\eta)n$, it remains to estimate

$$\int_\Gamma\frac{1}{\sqrt{a}}(\sqrt{a}a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta})_{,\gamma\delta}n^in^j\bar{\partial}^{10}\eta^j dS\,.$$

Integrating by parts,

$$\begin{aligned}&\int_\Gamma\frac{1}{\sqrt{a}}(\sqrt{a}a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta})_{,\gamma\delta}n^in^j\bar{\partial}^{10}\eta^j dS = \int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta}(n^in^j\bar{\partial}^{10}\eta^j)_{,\gamma\delta}dS\\ &= \int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta}\Big[n^i_{,\gamma\delta}n^j\bar{\partial}^{10}\eta^j + n^in^j_{,\gamma\delta}\bar{\partial}^{10}\eta^j + 2n^i_{,\gamma}n^j_{,\delta}\bar{\partial}^{10}\eta^j + 2n^i_{,\gamma}n^j\bar{\partial}^{10}\eta^j_{,\delta}\\ &\qquad + 2n^in^j_{,\gamma}\bar{\partial}^{10}\eta^j_{,\delta}\Big]dS + \int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(b_{\alpha\beta})_t - \eta^i_{,\alpha\beta}n^i_t\Big]n^j\bar{\partial}^{10}\eta^j_{,\gamma\delta}dS\,.\end{aligned}$$

The worst term in the first integral is the last piece. Similarly to estimate (7.20), by Young's inequality we find that

$$\begin{aligned}\left|\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta}n^in^j_{,\gamma}\bar{\partial}^{10}\eta^j_{,\delta}dS\right| &\le C\Big[|v|_6 + |v|_{3.5}|n|_5\Big]|\eta|_7\\ &\le C_{\bar{\delta}}|\eta|_7^2 + \bar{\delta}|v|_6^2 + C|v|_{3.5}^2\Big[1 + |b|_4^2 + |g|_4^2\Big]\,,\end{aligned}$$

and hence the first integral has the same upper bound. Similarly,

$$\left|\int_\Gamma a^{\alpha\beta\gamma\delta}\eta^i_{,\alpha\beta}n^i_t n^j\bar\partial^{10}\eta^j_{,\gamma\delta}dS\right| \le C\Big[|\eta|_7+|n_tn|_5+|\eta|_6|n_tn|_{2.5}+|\eta|_{4.5}|n_tn|_4\Big]|\eta|_7$$
$$\le C_{\bar\delta}|\eta|_7^2+\bar\delta|v|_6^2+C\Big[1+|\eta|_5^2|v|_{4.5}^2+|b|_4^2+|g|_4^2\Big].$$

By the identity $n^j\bar\partial^{10}\eta^j_{,\gamma\delta}=\bar\partial^{10}b_{\gamma\delta}-\sum_{k=1}^{10}C^{10}_k\bar\partial^{10-k}\eta_{,\gamma\delta}\bar\partial^k n^j$, we find that

$$\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta})_t n^j\bar\partial^{10}\eta^j_{,\gamma\delta}dS=\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta})_t\Big[\bar\partial^{10}b_{\gamma\delta}-\sum_{k=1}^{10}C^{10}_k\bar\partial^{10-k}\eta_{,\gamma\delta}\bar\partial^k n^j\Big]dS$$
$$=-\int_\Gamma\frac{1}{\sqrt a}\bar\partial^5[\sqrt a a^{\alpha\beta\gamma\delta}(b_{\alpha\beta})_t]\bar\partial^5 b_{\gamma\delta}dS+\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta})_t\sum_{k=1}^{10}C^{10}_k\bar\partial^{10-k}\eta^j_{,\gamma\delta}\bar\partial^k n^j dS$$
$$\le-\frac12\frac{d}{dt}\int_\Gamma a^{\alpha\beta\gamma\delta}\bar\partial^5 b_{\alpha\beta}\bar\partial^5 b_{\gamma\delta}dS+C_{\bar\delta}+\bar\delta\Big[|v|_{6.5}^2+|\eta|_7^2\Big]+C\Big[|b|_5^2+|g|_5^2\Big].$$

As a consequence, by (7.16), for κ small enough,

$$\kappa\sup_{t\in[0,T]}\varepsilon^3|b(t)|_5^2+\varepsilon^3\int_0^T|b(t)|_5^2dt\le\frac{C_{\bar\delta}}{\varepsilon}\Big[1+|a|_5^2+|\mathfrak b|_5^2+\varepsilon^2(|u_0|_4^2+|\Gamma|_6^2)\Big]$$
$$+C\varepsilon^3\int_0^T|g|_5^2dt+C\varepsilon\int_0^T|v|_5^2dt+\bar\delta\kappa\varepsilon^3\int_0^T\Big[|v|_{6.5}^2+|\eta|_7^2\Big]dt \tag{7.21}$$
$$+\frac{C}{\varepsilon^3}\int_0^T|h|_3^2dt+\bar\sigma\varepsilon^3\Big[C_{\bar\delta_1}|v|_{4.5}^2+\bar\delta_1(|b|_{4.5}^2+|g|_{4.5}^2)\Big](T).$$

By interpolation, (7.20) and (7.21) imply that

$$\varepsilon^3\int_0^T|b(t)|_{4.5}^2dt\le\mathcal M_{\bar\delta}(\varepsilon,\|u_0\|_4,|\Gamma|_{6.5},\|f\|_{\mathcal F^4(T)})+\Big[C\varepsilon+\frac{CT}{\varepsilon^3}\Big]\mathcal E_{\bar\sigma}(T) \tag{7.22a}$$
$$+\bar\sigma\Big[C_{\bar\delta_1}\varepsilon^2+\bar\delta_1+(C+\bar\delta_1)\varepsilon^2+\bar\delta\Big]\mathcal E_{\bar\sigma}(T),$$

$$\varepsilon\int_0^T|g|_{4.5}^2dt\le\mathcal M_{\bar\delta}(\varepsilon,\|u_0\|_4,|\Gamma|_{6.5},\|f\|_{\mathcal F^4(T)})+\left(\frac{C}{\varepsilon}+C+\bar\delta\right)\mathcal E_{\bar\sigma}(T). \tag{7.22b}$$

Later on we will denote $\mathcal M(\varepsilon,\|u_0\|_4,|\Gamma|_{6.5},\|f\|_{\mathcal F^4(T)})$ by $\mathcal M(\varepsilon,u_0,f)$ as well for simplicity.

8. κ-independent energy estimates. In order to obtain the energy estimate at the $L^2(0,T;H^5(\Omega))$ level, we test (2.5b) against $\zeta\bar\partial^8(\zeta v)$; time differentiate (2.5b) and test the resulting equations against $\zeta\bar\partial^4(\zeta v_t)$; and twice time differentiate (2.5b) and test the resulting equations against v_{tt}. The estimates due to the viscosity and the pressure terms in the interior are exactly the same as what we have in [5]. The only difference is the estimates over the boundary. Due to the membrane energy, we need to estimate

$$\mathrm I_1=\varepsilon\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta}-\mathfrak g_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^8 v^i_{,\delta}dSdt,$$
$$\mathrm I_2=\varepsilon\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(g_{\alpha\beta}-\mathfrak g_{\alpha\beta})\eta^i_{,\gamma}\Big]_t\bar\partial^4 v^i_{t,\delta}dSdt,$$
$$\mathrm I_3=\varepsilon\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(g_{\alpha\beta}-\mathfrak g_{\alpha\beta})\eta^i_{,\gamma}\Big]_{tt}v^i_{tt,\delta}dSdt,$$

while due to the bending energy it is required to estimate

$$\mathrm{II}_1 = \varepsilon^3 \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\Big[n^i \bar\partial^8 v^i_{,\gamma\delta} - \Gamma^\tau_{\gamma\delta} n^i \bar\partial^8 v^i_{,\tau}\Big] dSdt\,,$$

$$\mathrm{II}_2 = \varepsilon^3 \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}\Big[\Big((b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})n^i\Big)_t \bar\partial^4 v^i_{t,\gamma\delta} - \Big((b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\Gamma^\tau_{\gamma\delta} n^i\Big)_t \bar\partial^4 v^i_{t,\tau}\Big] dSdt\,,$$

$$\mathrm{II}_3 = \varepsilon^3 \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}\Big[\Big((b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})n^i\Big)_{tt} v^i_{tt,\gamma\delta} - \Big((b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\Gamma^\tau_{\gamma\delta} n^i\Big)_{tt} v^i_{tt,\tau}\Big] dSdt\,.$$

Furthermore, we also need to estimate the term due to the inertia:

$$\mathrm{III}_1 = \varepsilon \int_0^T \int_\Gamma v_t \cdot \bar\partial^8 v dS\,,\ \mathrm{III}_2 = \varepsilon \int_0^T \int_\Gamma v_{tt} \cdot \bar\partial^4 v_t dS\,,\ \mathrm{III}_3 = \varepsilon \int_0^T \int_\Gamma v_{ttt} \cdot v_{tt} dS\,.$$

Nevertheless, it is easy to see that

$$\mathrm{III}_1 = \varepsilon\Big[|v(T)|^2_4 - |u_0|^2_4\Big]\,,\ \mathrm{III}_2 = \varepsilon\Big[|v_t(T)|^2_2 - |\mathrm{w}_1|^2_4\Big]\,,\ \mathrm{III}_3 = \varepsilon\Big[|v_{tt}(T)|^2_0 - |\mathrm{w}_2|^2_0\Big]\,.$$

8.1. The estimate for $\mathbf{I_1}$. By $\partial_t g_{\gamma\delta} = \eta^i_{,\gamma} v^i_{,\delta} + v^i_{,\delta}\eta^i_{,\delta}$ and the symmetry and positivity of $a^{\alpha\beta\gamma\delta}$, we find that

$$\begin{aligned}\lambda_0|g(T) - \mathfrak{g}|^2_4 \le & \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^8 v^i_{,\delta} dSdt \\ & - \int_0^T \int_\Gamma \frac{1}{\sqrt{a}} \sum_{k=0}^3 C^4_k \bar\partial^{4-k}(\sqrt{a} a^{\alpha\beta\gamma\delta})\bar\partial^k(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\bar\partial^4(\eta^i_{,\gamma} v^i_{,\delta}) dSdt \\ & + \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta}) \sum_{k=1}^8 C^8_k \bar\partial^{8-k} v^i_{,\delta} \bar\partial^k \eta^i_{,\gamma} dSdt\end{aligned} \tag{8.1}$$

for some positive constant λ_0. The worst term in the second term of the right-hand side is

$$\int_0^T \int_\Gamma \frac{1}{\sqrt{a}} \sum_{k=0}^3 C^4_k \bar\partial^{4-k}(\sqrt{a} a^{\alpha\beta\gamma\delta})\bar\partial^k(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^4 v^i_{,\delta} dSdt,$$

which can be estimated by $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality. Therefore,

$$\begin{aligned}&\Big|\int_0^T \int_\Gamma \frac{1}{\sqrt{a}} \sum_{k=0}^3 C^4_k \bar\partial^{4-k}(\sqrt{a} a^{\alpha\beta\gamma\delta})\bar\partial^k(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta})\bar\partial^4(\eta^i_{,\gamma} v^i_{,\delta}) dSdt\Big| \\ &\le C\int_0^T |g - \mathfrak{g}|_{3.5}|v|_{4.5} dt \le CT^{3/2} + C\int_0^T \|v\|^2_5 dt\,.\end{aligned}$$

Integrating by parts for the last term in (8.1), by either $H^{1.5}(\Gamma)$-$H^{-1.5}(\Gamma)$ or $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality, from (7.18b) we find that

$$\begin{aligned}&\varepsilon\Big|\int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta} - \mathfrak{g}_{\alpha\beta}) \sum_{k=1}^8 C^8_k \bar\partial^{8-k} v^i_{,\delta}\bar\partial^k \eta^i_{,\gamma} dS\Big| \\ &\le C\varepsilon \int_0^T |g - \mathfrak{g}|_{3.5}|v|_{4.5}|\eta|_{5.5} dt \le C\varepsilon T^{1/4}\int_0^T (1 + |b|_{3.5} + |g|_{4.5})\|v\|_5 dt \\ &\le C + C\int_0^T \Big[\varepsilon^4|b|^2_{4.5} + \varepsilon^3|g|^2_{4.5}\Big] dt + \frac{\sqrt{T}}{\varepsilon}\int_0^T \|v\|^2_5 dt\,.\end{aligned}$$

Combining all the estimates above,

$$\lambda_0\varepsilon|g(T)|_2^2 \le \varepsilon\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}(g_{\alpha\beta}-\mathfrak{g}_{\alpha\beta})\eta^i_{,\gamma}\bar\partial^4 v^i_{,\delta}dSdt + \mathcal{M}_{\bar\delta}(\varepsilon,u_0,f)+\left[C\varepsilon^2+\frac{CT}{\varepsilon^2}+(C+\bar\sigma+\bar\delta)\varepsilon\right]\mathcal{E}_{\bar\sigma}(T)\,. \tag{8.2}$$

8.2. The estimate for II_1. We first estimate the first piece of II_1:

$$\varepsilon^3\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i\bar\partial^8 v^i_{,\gamma\delta}dSdt\,.$$

Since $n\cdot\bar\partial^8 v_{,\gamma\delta}=\bar\partial^8(b_{\gamma\delta}-\mathfrak{b}_{\gamma\delta})_t-\bar\partial^8(n^i_t\eta^i_{,\gamma\delta})-\sum_{k=1}^8 C^8_k\bar\partial^{8-k}v^i_{,\gamma\delta}\bar\partial^k n^i$, integrating by parts we find that

$$\begin{aligned}\lambda_0|b(T)-\mathfrak{b}|_4^2 \le &\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i\bar\partial^8 v^i_{,\gamma\delta}dSdt\\ &+\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\left[\bar\partial^8(n^i_t\eta^i_{,\gamma\delta})+\sum_{k=1}^8 C^8_k\bar\partial^{8-k}v^i_{,\gamma\delta}\bar\partial^k n^i\right]dSdt\\ &+\int_0^T\int_\Gamma\frac{1}{\sqrt a}\sum_{k=0}^3\bar\partial^{4-k}(\sqrt a a^{\alpha\beta\gamma\delta})\bar\partial^k(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\bar\partial^4\left[v^i_{,\gamma\delta}n^i+\eta^i_{,\gamma\delta}n^i_t\right]dSdt\,.\end{aligned}\tag{8.3}$$

For the second term on the right-hand side of (8.3), after integrating by parts, the worst term is of the form

$$\int_0^T\int_\Gamma a\left[\bar\partial^4(b-\mathfrak{b})(\bar\partial^5 v\bar\partial^2\eta+\bar\partial v\bar\partial^6\eta+\bar\partial^5 v\bar\partial n)+(b-\mathfrak{b})\bar\partial^5 v\bar\partial^5 n\right]dSdt\,.$$

By $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality,

$$\begin{aligned}&\varepsilon^3\left|\int_0^T\int_\Gamma a\left[\bar\partial^4(b-\mathfrak{b})(\bar\partial^5 v\bar\partial^2\eta+\bar\partial v\bar\partial^6\eta+\bar\partial^5 v\bar\partial n)+(b-\mathfrak{b})\bar\partial^5 v\bar\partial^5 n\right]dSdt\right|\\ &\le C\varepsilon^3\int_0^T\left[|b-\mathfrak{b}|_{4.5}(|v|_{4.5}+|\eta|_{5.5})+|b-\mathfrak{b}|_{1.5}|v|_{4.5}|n|_{5.5}\right]dt\\ &\le \mathcal{M}_{\bar\delta}(\varepsilon,u_0,f)+C\left[(1+\bar\delta)\varepsilon^{0.5}+\varepsilon^{1.5}+(1+\bar\sigma)\varepsilon^{2.5}+\frac{CT}{\varepsilon^{2.5}}\right]\mathcal{E}_{\bar\sigma}(T)\end{aligned}$$

(here $\bar\delta_1$ in (7.18a) is chosen to be ε^2 so that the inequality above can be derived), and as a result,

$$\begin{aligned}&\varepsilon^3\left|\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\left[\bar\partial^8(n^i_t\eta^i_{,\gamma\delta})+\sum_{k=1}^8 C^8_k\bar\partial^{8-k}v^i_{,\gamma\delta}\bar\partial^k n^i\right]dSdt\right|\\ &\le \mathcal{M}_{\bar\delta}(\varepsilon,u_0,f)+C\left[(1+\bar\delta)\varepsilon^{0.5}+\varepsilon^{1.5}+(1+\bar\sigma)\varepsilon^{2.5}+\frac{CT}{\varepsilon^{2.5}}\right]\mathcal{E}_{\bar\sigma}(T)\,.\end{aligned}$$

For the last term in (8.3), by either $H^{1.5}(\Gamma)$-$H^{-1.5}(\Gamma)$ or $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality, from (7.18a) we find that

$$\varepsilon^3 \left| \int_0^T \frac{1}{\sqrt{a}} \sum_{k=0}^{3} \bar\partial^{4-k}(\sqrt{a} a^{\alpha\beta\gamma\delta}) \bar\partial^k (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) \bar\partial^4 \Big[v^i_{,\gamma\delta} n^i + \eta^i_{,\gamma\delta} n^i_t \Big] dS \right|$$
$$\le \varepsilon^3 C(|\Gamma|_{5.5}) \int_0^T |b - \mathfrak{b}|_{4.5} \Big[|v|_{4.5} + |\eta|_{5.5} \Big] dt$$
$$\le \mathcal{M}_{\bar\delta}(\varepsilon, u_0, f) + C \left[(1+\bar\delta)\varepsilon^{0.5} + \varepsilon^{1.5} + (1+\bar\sigma)\varepsilon^{2.5} + \frac{CT}{\varepsilon^{2.5}} \right] \mathcal{E}_{\bar\sigma}(T)\,.$$

Combining all the estimates above,

$$\lambda_0 \varepsilon^3 |b(T) - \mathfrak{b}|_4^2 \le \varepsilon^3 \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) n^i \bar\partial^8 v^i_{,\gamma\delta} dS dt$$
$$+ \mathcal{M}_{\bar\delta}(\varepsilon, u_0, f) + C \left[(1+\bar\delta)\varepsilon^{0.5} + \varepsilon^{1.5} + (1+\bar\sigma)\varepsilon^{2.5} + \frac{CT}{\varepsilon^{2.5}} \right] \mathcal{E}_{\bar\sigma}(T)\,. \tag{8.4}$$

As for the second piece of II_1, we first note that

$$\int_\Gamma a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) \Gamma^\tau_{\gamma\delta} n^i \bar\partial^8 v^i_{,\tau} dS$$
$$= \int_\Gamma a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) \Gamma^\tau_{\gamma\delta} \left[\bar\partial^7 (b_{\cdot\tau})_t - \bar\partial^7 (\bar\partial \eta^i_{,\tau} n^i_t) - \sum_{k=1}^{7} C^7_k \bar\partial^{8-k} v^i_{,\tau} \bar\partial^k n^i \right] dS\,. \tag{8.5}$$

For the first term in the bracket, we study the time integral and integrate by parts in time and space to obtain

$$\int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) \Gamma^\tau_{\gamma\delta} \bar\partial^7 (b_{\cdot\tau})_t dS dt = \int_0^T \int_\Gamma \bar\partial^3 \Big[a^{\alpha\beta\gamma\delta} (b_{\alpha\beta})_t \Gamma^\tau_{\gamma\delta} \Big] \bar\partial^4 b_{\cdot\tau} dS dt$$
$$+ \int_0^T \int_\Gamma \bar\partial^3 \Big[a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) (\Gamma^\tau_{\gamma\delta})_t \Big] \bar\partial^4 b_{\cdot\tau} dS dt\,.$$

The worst term on the right-hand side is from the first term, when all the tangential derivatives hit $(b_{\alpha\beta})_t$. This particular term can be estimated by $H^{-0.5}(\Gamma)$-$H^{0.5}(\Gamma)$ duality, and we find that

$$\left| \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta} \bar\partial^3 (b_{\alpha\beta})_t \Gamma^\tau_{\gamma\delta} \bar\partial^4 b_{\cdot\tau} dS dt \right| \le C \int_0^T |v|_{4.5} |\Gamma^{\cdot}_{\cdot\cdot}|_{1.5} |b|_{4.5} dt\,. \tag{8.6}$$

All the other terms can be estimated directly by Hölder's inequality, hence

$$\left| \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) \Gamma^\tau_{\gamma\delta} \bar\partial^7 (b_{\cdot\tau})_t dS dt \right| \le C \int_0^T |v|_{4.5} |b|_{4.5} dt\,.$$

For the remaining terms in (8.5), integrating by parts (in space) and $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality imply that

$$\left| \int_\Gamma a^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta}) \Gamma^\tau_{\gamma\delta} \left[\bar\partial^7 (\bar\partial \eta^i_{,\tau} n^i_t) + \sum_{k=1}^{7} C^7_k \bar\partial^{8-k} v^i_{,\tau} \bar\partial^k n^i \right] dS \right|$$
$$\le C |b - \mathfrak{b}|_{3.5} \Big[|\eta|_{5.5} + |n|_{5.5} \Big] |v|_{4.5}\,.$$

Therefore, by Young's inequality and (7.22a),

$$\varepsilon^3 \left| \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\Gamma^\tau_{\gamma\delta} n^i \bar\partial^8 v^i_{,\tau} dS dt \right| \le \mathcal{M}_{\bar\delta}(\varepsilon, u_0, f) + \Big[C\varepsilon^2 + \frac{CT}{\varepsilon^2} + (\bar\sigma + \bar\delta)\varepsilon + T^{1/6} \Big] \mathcal{E}_{\bar\sigma}(T) \,. \tag{8.7}$$

Combining (8.2), (8.4), and (8.7), we find that

$$\sup_{t\in[0,T]} \Big[\|\bar\partial^4(\zeta v)\|_0^2 + \bar\sigma\varepsilon|\bar\partial^4 v|_0^2 + \varepsilon|g|_4^2 + \varepsilon^3|b|_4^2 \Big](t) + \int_0^T \|\nabla\bar\partial^4(\zeta v)(t)\|_0^2 dt \le \mathcal{M}_{\bar\delta}(\varepsilon, u_0, f) + C\Big[(1+\bar\delta)\varepsilon^{0.5} + \frac{T}{\varepsilon^{2.5}} + T^{1/6} \Big] \mathcal{E}_{\bar\sigma}(T) \,. \tag{8.8}$$

Remark 18. In the case of the 3D fluid, estimation of the error term

$$\int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta} - \mathfrak{b}_{\alpha\beta})\Gamma^\tau_{\gamma\delta} n^i \bar\partial^8 v^i_{,\tau} dS dt$$

creates a so-called Sobolev embedding obstruction, which necessitates the use of the high-regularity solution space $L^2(0,T;H^5(\Omega))$.

To see why this obstruction prevents us from using the space $L^2(0,T;H^3(\Omega))$ to close the energy estimates, suppose for the sake of contradiction that we attempt to use this lower-regularity space. We would then instead test (2.5b) against $\zeta\bar\partial^4(\zeta v)$ and find that we need to estimate an error term which has a cubic nonlinearity in the integrand which has the derivative count of the following integral:

$$\int_0^T \int_\Gamma \bar\partial^2\eta\, \bar\partial^2\eta\, \bar\partial^5 v \cdot n\, dS dt \,.$$

Integrating by parts in time and in space, the most problematic error terms have the derivative count of the following integral:

$$\int_0^T \int_\Gamma \bar\partial^3 v\, \bar\partial^2\eta\, \bar\partial^4\eta \cdot n\, dS dt \,.$$

Using the trilinear estimate

$$\langle h_1 h_2, h_3\rangle_{H^{0.5}(\Gamma)} \le C_s |h_1|_{1.5} |h_2|_{0.5} |h_3|_{-0.5} \,, \tag{8.9}$$

and letting $h_1 = \bar\partial^2\eta$, $\bar\partial^4\eta \cdot n$, and $h_3 = \bar\partial^3 v$, we find that

$$\left| \int_0^T \int_\Gamma \bar\partial^3 v\, \bar\partial^2\eta\, \bar\partial^4\eta\, dS dt \right| \le C_s \int_0^T |v|_{2.5} |\eta|_{3.5} |\bar\partial^4\eta \cdot n|_{0.5} dt \,. \tag{8.10}$$

Using the testing procedure described above, energy estimates would yield regularity for

$$v \in L^2(0,T;H^3(\Omega))$$

and (since $\bar\partial^4\eta \cdot n$ scales like $\bar\partial^2 b$) for

$$\bar\partial^4\eta \cdot n \in L^2(0,T;H^{0.5}(\Gamma)) \,.$$

It would then follow from Hölder's inequality that a bound for the time integral on the right-hand side of (8.10) would follow if

$$\eta \text{ is bounded in } L^\infty(0,T;H^{3.5}(\Gamma)).$$

Unfortunately, in three dimensions this is not the case; in particular, with $v \in L^2(0,T;H^3(\Omega))$, elliptic estimates on the boundary would show that η is only in $L^2(0,T;H^{3.5}(\Gamma))$.

On the other hand, by raising the regularity of the initial data and using the energy space $L^2(0,T;H^5(\Omega))$ for v, these type of error terms can easily be controlled.

8.3. The estimates for I_2 and I_3. The term I_2 can be treated as a lower order term. By $H^{2.5}$-$H^{-2.5}$ duality,

$$\mathrm{I}_2 \le C\varepsilon \int_0^T |v|_{3.5}|v_t|_{2.5}dt \le C\varepsilon\mathcal{E}_{\bar\sigma}(T). \tag{8.11}$$

Similarly, I_3 can also be treated as a lower order term. By $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality,

$$\mathrm{I}_3 \le C\varepsilon \int_0^T \Big[|v_t|_{2.5} + |v|_{2.5}\Big]|v_{tt}|_{0.5}dt \le C\varepsilon\mathcal{E}_{\bar\sigma}(T). \tag{8.12}$$

Remark 19. We can also perform energy estimates as in section 8.1 and two more energy contributions $\sup_{t\in[0,T]}|\bar\partial^3 v(t)\cdot\bar\partial\eta(t)|_0^2$ and $\sup_{t\in[0,T]}|\bar\partial v_t(t)\cdot\bar\partial\eta(t)|_0^2$ can be found; however, due to the viscosity we have already known that $v \in L^\infty(0,T;H^{3.5}(\Gamma))$ and $v_t \in L^\infty(0,T;H^{1.5}(\Gamma))$. These extra energy contributions are then useless, so we can treat these two terms as lower order terms.

8.4. The estimate for II_2. Since

$$\begin{aligned} n^i\bar\partial^4 v^i_{t,\gamma\delta} &= \bar\partial^4(v^i_{t,\gamma\delta}n^i) - \sum_{k=1}^4 C^4_k\bar\partial^{4-k}v^i_{t,\gamma\delta}\bar\partial^k n^i, \\ (b_{\alpha\beta})_t &= v^j_{,\alpha\beta}n^j + \eta^j_{,\alpha\beta}n^j_t, \end{aligned}$$

we find that

$$\begin{aligned} \int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i\Big]_t\bar\partial^4 v^i_{t,\gamma\delta}dS &= \int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i_t\bar\partial^4 v^i_{t,\gamma\delta}dS \\ &+ \int_\Gamma a^{\alpha\beta\gamma\delta}\Big[v^j_{,\alpha\beta}n^j + \eta^j_{,\alpha\beta}n^j_t\Big]\Big[\bar\partial^4(v^i_{t,\gamma\delta}n^i) - \sum_{k=1}^4 C^4_k\bar\partial^{4-k}v^i_{t,\gamma\delta}\bar\partial^k n^i\Big]dS. \end{aligned} \tag{8.13}$$

For the first term on the right-hand side of (8.13), we apply $H^{3.5}(\Gamma)$-$H^{-3.5}(\Gamma)$ duality to obtain

$$\left|\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i_t\bar\partial^4 v^i_{t,\gamma\delta}dS\right| \le C|b-\mathfrak{b}|_{3.5}|v|_{4.5}|v_t|_{2.5},$$

which implies that

$$\varepsilon^3\left|\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i_t\bar\partial^4 v^i_{t,\gamma\delta}dSdt\right| \le CT^{1/6}\mathcal{E}_{\bar\sigma}(T).$$

For the second term on the right-hand side of (8.13), we first note that by $H^{2.5}(\Gamma)$-$H^{-2.5}(\Gamma)$ duality and Hölder's inequality

$$\left|\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[v^j_{,\alpha\beta}n^j+\eta^j_{,\alpha\beta}n^j_t\Big]\sum_{k=1}^4 C^4_k\bar\partial^{4-k}v^i_{t,\gamma\delta}\bar\partial^k n^i dS\right|\le C\Big[|v|_{4.5}+|n|_{4.5}\Big]|v_t|_{2.5},$$

and $H^{3.5}(\Gamma)$-$H^{-3.5}(\Gamma)$ duality implies that

$$\left|\int_\Gamma a^{\alpha\beta\gamma\delta}\eta^j_{,\alpha\beta}n^j_t\bar\partial^4(v^i_{t,\gamma\delta}n^i)dS\right|\le C\Big[|\eta|_{5.5}+|v|_{4.5}\Big]|v_t|_{2.5}\,.$$

As for the rest of (8.13), we integrate by parts and find that

$$\begin{aligned}\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta}n^i\bar\partial^4(v^j_{t,\gamma\delta}n^j)dS&=\frac12\frac{d}{dt}\int_\Gamma a^{\alpha\beta\gamma\delta}\bar\partial^2(v^i_{,\alpha\beta}n^i)\bar\partial^2(v^j_{,\gamma\delta}n^j)dS\\&+\int_\Gamma\frac{1}{\sqrt a}\sum_{k=0}^1 C^2_k\bar\partial^{2-k}(a^{\alpha\beta\gamma\delta})\bar\partial^k(v^i_{,\alpha\beta}n^i)\bar\partial^2(v^j_{t,\gamma\delta}n^j)dS\\&-\int_\Gamma a^{\alpha\beta\gamma\delta}\bar\partial^2(v^i_{,\alpha\beta}n^i)\bar\partial^2(v^j_{,\gamma\delta}n^j_t)dS\,.\end{aligned}$$

The first term on the right-hand side gives the energy contribution, and the last term is bounded by $C|v|_4^2$. For the second term, we use $H^{1.5}(\Gamma)$-$H^{-1.5}(\Gamma)$ duality and obtain

$$\left|\int_\Gamma\frac{1}{\sqrt a}\sum_{k=0}^1 C^2_k\bar\partial^{2-k}(a^{\alpha\beta\gamma\delta})\bar\partial^k(v^i_{,\alpha\beta}n^i)\bar\partial^2(v^j_{t,\gamma\delta}n^j)dS\right|\le C|v|_{4.5}|v_t|_{2.5}\,.$$

Therefore, for some $\lambda_0>0$,

$$\begin{aligned}\lambda_0\varepsilon^3|\bar\partial^2v(T)\cdot n(T)|_2^2&\le\varepsilon^3\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}v^i_{,\alpha\beta}n^i\bar\partial^4(v^j_{t,\gamma\delta}n^j)dSdt\\&+\mathcal M(\varepsilon,u_0,f)+C\Big[\varepsilon+\varepsilon^2+(\bar\sigma+1)\varepsilon^3+T+T^{1/6}\Big]\mathcal E_{\bar\sigma}(T)\,.\end{aligned}\tag{8.14}$$

As for the second piece of II_2, we apply $H^{2.5}(\Gamma)$-$H^{-2.5}(\Gamma)$ duality and find that

$$\left|\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(b_{\alpha\beta}-\mathfrak b_{\alpha\beta})\Gamma^\tau_{\gamma\delta}n^i\Big]_t\bar\partial^4v^i_{t,\tau}dSdt\right|\le C\Big[|v|_{4.5}+|b-\mathfrak b|_{2.5}|v|_{3.5}\Big]|v_t|_{2.5}\,.$$

Combining (8.11), (8.14), and the inequality above, we find that

$$\begin{aligned}&\sup_{t\in[0,T]}\Big[\|\bar\partial^2(\zeta v_t)\|_0^2+\bar\sigma\varepsilon|v_t|_2^2+\varepsilon^3|\bar\partial^2v\cdot n|_2^2\Big](t)+\int_0^T\|\nabla[\bar\partial^2(\zeta v_t)]\|_0^2dt\\&\le\mathcal M(\varepsilon,u_0,f)+C\Big[\varepsilon+\varepsilon^2+(\bar\sigma+1)\varepsilon^3+T+T^{1/6}\Big]\mathcal E_{\bar\sigma}(T)\,.\end{aligned}\tag{8.15}$$

8.5. The estimate for II_3.

$$\mathrm{II}_3 = \varepsilon^3 \int_0^T \int_\Gamma a^{\alpha\beta\gamma\delta}\Big[\Big((b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i\Big)_{tt} v^i_{tt,\gamma\delta} - \Big((b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\Gamma^\tau_{\gamma\delta}n^i\Big)_{tt} v^i_{tt,\tau}\Big] dSdt\,.$$

By Leibniz's rule,

$$\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i\Big]_{tt} v^i_{tt,\gamma\delta}dS = \int_\Gamma a^{\alpha\beta\gamma\delta}\Big[v^j_{t,\alpha\beta}n^j + 2v^j_{,\alpha\beta}n^j_t + \eta^j_{,\alpha\beta}n^j_{tt}\Big]n^i v^i_{tt,\gamma\delta}dS$$
$$+ 2\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[v^j_{,\alpha\beta}n^j + \eta^j_{,\alpha\beta}n^j_t\Big]n^i_t v^i_{tt,\gamma\delta}dS + \int_\Gamma a^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i_{tt}v^i_{tt,\gamma\delta}dS\,.$$

Similarly to (8.14), all the terms above can be estimated by $H^{1.5}(\Gamma)$-$H^{-1.5}(\Gamma)$ duality except $\int_\Gamma a^{\alpha\beta\gamma\delta}(v^j_{t,\alpha\beta}n^j)(n^i v^i_{tt,\gamma\delta})dS$. For this term, we note that

$$\int_\Gamma a^{\alpha\beta\gamma\delta}(v^j_{t,\alpha\beta}n^j)(n^i v^i_{tt,\gamma\delta})dS = \frac{1}{2}\frac{d}{dt}\int_\Gamma a^{\alpha\beta\gamma\delta}(v^i_{t,\alpha\beta}n^i)(v^j_{t,\gamma\delta}n^j)dS$$
$$- \int_\Gamma a^{\alpha\beta\gamma\delta}(v^i_{t,\alpha\beta}n^i_t)(v^j_{t,\gamma\delta}n^i)dS\,.$$

Hence for some $\lambda_0 > 0$,

$$\lambda_0\varepsilon^3|\bar\partial^2 v_t(T)\cdot n(T)|_0^2 \le \varepsilon^3\int_0^T\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})n^i\Big]_{tt} v^i_{tt,\gamma\delta}dSdt + \mathcal{M}(\varepsilon,u_0,f) + C\varepsilon^3\mathcal{E}_{\bar\sigma}(T)\,. \tag{8.16}$$

For the rest term of II_3, by $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality we find that

$$\bigg|\int_\Gamma a^{\alpha\beta\gamma\delta}\Big[(b_{\alpha\beta}-\mathfrak{b}_{\alpha\beta})\Gamma^\tau_{\gamma\delta}n^i\Big]_{tt} v^i_{tt,\tau}dS\bigg| \le C|v_t|_{2.5}|v_{tt}|_{0.5}\,.$$

Therefore, combining (8.12), (8.16), and the inequality above,

$$\sup_{t\in[0,T]}\Big[\|v_{tt}\|_0^2 + \bar\sigma\varepsilon|v_{tt}|_0^2 + \varepsilon^3|\bar\partial^2 v_t\cdot n|_0^2\Big](t) + \int_0^T \|\nabla v_{tt}(t)\|_0^2 dt \le \mathcal{M}(\varepsilon,u_0,f) + C(\varepsilon+\varepsilon^3)\mathcal{E}_{\bar\sigma}(T)\,. \tag{8.17}$$

8.6. Energy inequalities. Combining (7.22a), (8.8), (8.15), and (8.17), by (6.4) we find that

$$\mathcal{E}_{\bar\sigma}(T) \le \mathcal{M}_{\bar\delta}(\varepsilon,u_0,f) + C\bigg[(1+\bar\delta)\varepsilon^{0.5} + \frac{CT}{\varepsilon^{2.5}} + T^{1/6} + \bar\delta\bigg]\mathcal{E}_{\bar\sigma}(T)\,.$$

By choosing $\bar\delta > 0$, $\varepsilon > 0$, and $T = T(\varepsilon) > 0$ small enough, we conclude that

$$\mathcal{E}_{\bar\sigma}(T) \le \mathcal{M}(\varepsilon, \|u_0\|_5, |\Gamma|_{6.5}, \|f\|_{\mathcal{F}^4(T)})\,. \tag{8.18}$$

This κ-independent estimate enables us to extend the time interval $[0,T]$ in which v_κ is defined for some $T = T(\varepsilon)$ by the continuation argument as stated in section 9 of [8], and a solution $\mathfrak{v}_{\tilde\kappa}$ to (2.5) (still defined in $\Omega_{\tilde\kappa_j}$ and on $\Gamma_{\tilde\kappa_j}$) can be obtained by taking the weak (or strong) limit of v_κ as $\kappa \to 0$.

8.7. The regularity of $\Gamma(t)$. Let h denote the height function measuring the sign distance from $\Gamma(t)$ to Γ. The regularity of h is then the same as the regularity of $\Gamma(t)$. By the regularity of $g, b \in L^2(0,T;H^{2\mathfrak{n}-1.5}(\Gamma))$, we find that the mean curvature H is of class $L^1(0,T;H^{2\mathfrak{n}-1.5}(\Gamma))$ since $H = -\frac{1}{2}g^{\alpha\beta}b_{\alpha\beta}$ if $\mathfrak{n}=3$ or $H = -g^{-1}b$ if $\mathfrak{n}=2$. As computed in [5], by defining $(G_z)_{\alpha\beta} = \mathfrak{g}_{\alpha\beta} - 2z\mathfrak{b}_{\alpha\beta} + z^2\mathfrak{g}^{\gamma\delta}\mathfrak{b}_{\alpha\gamma}\mathfrak{b}_{\beta\delta}$ and $J_h = (1 + h_{,\alpha}G_h^{\alpha\beta}h_{,\beta})^{1/2}$, then in terms of h,

$$H = -(J_h^{-1}G_h^{\gamma\delta}h_{,\gamma})_{,\delta} + J_h^{-1}(-G_h^{\gamma\delta}h_{,\gamma}\Gamma^j_{j\delta} + \Gamma^j_{j\mathfrak{n}})\,,$$

where Γ^k_{ij} denotes the Christoffel symbols with respect to the metric G and only the first derivative of h is involved in Γ^k_{ij}. By elliptic regularity, since $H \in H^{2\mathfrak{n}-1.5}(\Gamma)$ for almost all $t > 0$, $h \in H^{2\mathfrak{n}+0.5}(\Gamma)$, which concludes that $\Gamma(t)$ is of class $H^{2\mathfrak{n}+0.5}$.

8.8. The limit as $\tilde{\kappa} \to 0$. As mentioned in Remark 17, the estimates above depend only on $|\Gamma|_{2\mathfrak{n}+0.5}$. Therefore, we can pass $\tilde{\kappa}$ to zero and obtain a solution $\mathfrak{v}$ to (2.5). To be more precise, since $\Omega_{\tilde{\kappa}_1} \subseteq \Omega_{\tilde{\kappa}_2}$ if $\tilde{\kappa}_1 \ge \tilde{\kappa}_2$, $\mathfrak{v}_{\tilde{\kappa}_2}$ satisfies (2.5a)–(2.5f) in $\Omega_{\tilde{\kappa}_1}$ if $\tilde{\kappa}_1 \ge \tilde{\kappa}_2$. Passing $\tilde{\kappa}_2$ to zero first, we find that the limit $\mathfrak{v}$ of $\mathfrak{v}_{\tilde{\kappa}_2}$ satisfies (2.5a)–(2.5f) in $\Omega_{\tilde{\kappa}_1}$ for all $\tilde{\kappa}_1 > 0$, and the conclusion follows.

9. Uniqueness. The proof of uniqueness for the case $\mathfrak{n}=3$ is essentially the same as the case $\mathfrak{n}=2$, and hence we prove the uniqueness result for the case $\mathfrak{n}=2$ to shorten the length of the proof. We also omit the factor 1/3 in front of the bending traction for further simplification.

Let v and $\tilde{v}$ in $\mathcal{V}^5(T)$ be two solutions to (2.5) (with q and $\tilde{q} \in L^2(0,T;H^4(\Omega)) \cap L^\infty(0,T;H^3(\Omega))$, q_t and $\tilde{q}_t \in L^2(0,T;H^2(\Omega))$, g, $\tilde{g}$, b, $\tilde{b} \in L^2(0,T;H^{4.5}(\Gamma)) \cap L^\infty(0,T;H^4(\Gamma))$), and $w = v - \tilde{v}$, $r = q - \tilde{q}$, $\mathcal{E} = \eta - \tilde{\eta}$. Then w, r, $\mathcal{E}$ satisfy

$$w^i_t - \nu(A^j_\ell A^k_\ell w^i_{,k})_{,j} = -A^j_i r_{,j} + (\delta F)^i \quad \text{in } (0,T)\times\Omega\,, \tag{9.1a}$$

$$A^j_i w^i_{,j} = \delta D \quad \text{in } (0,T)\times\Omega\,, \tag{9.1b}$$

$$\Big[r\mathrm{Id}^j_i - \nu(A^k_i w^j_{,k} + A^k_j w^i_{,k})\Big]A^\ell_j N_\ell = \bar{\sigma}\varepsilon w_t + \tilde{\mathcal{L}}(\mathcal{E}) + \sum_{k=1}^5 \delta L_k \quad \text{on } (0,T)\times\Gamma\,, \tag{9.1c}$$

$$w(0,x) = 0 \quad \text{in } \Omega\,, \tag{9.1d}$$

where

$$\begin{aligned}
(\delta F)^i &= f^i\circ\eta - f^i\circ\tilde{\eta} + \nu[(A^k_\ell A^j_\ell - \tilde{A}^k_\ell\tilde{A}^j_\ell)\tilde{v}^i_{,j}]_{,k} + \nu[(A^k_\ell A^j_i - \tilde{A}^k_\ell\tilde{A}^j_i)\tilde{v}^\ell_{,j}]_{,k}\\
&\quad - (A^k_i - \tilde{A}^k_i)\tilde{q}_{,k}\,,\\
\tilde{\mathcal{L}}(\mathcal{E}) &= -\frac{\varepsilon}{\sqrt{a}}\Big[\sqrt{a}^{-3}[(\mathcal{E}'\cdot\eta')\eta' + (\mathcal{E}'\cdot\tilde{\eta}')\tilde{\eta}']\Big]' + \frac{\varepsilon^3}{\sqrt{a}}\Big[\sqrt{a}^{-3}(\mathcal{E}''\cdot n)n\Big]''\,,\\
\delta D &= -(A^j_i - \tilde{A}^j_i)\tilde{v}^i_{,j}\,,\\
\delta L_1 &= \nu\Big[(A^k_i - \tilde{A}^k_i)\tilde{v}^j_{,k} + (A^k_j - \tilde{A}^k_j)\tilde{v}^i_{,k}\Big]A^\ell_j N_\ell + \nu\Big[\tilde{A}^k_i\tilde{v}^j_{,k} + \tilde{A}^k_j\tilde{v}^i_{,k}\Big](A^\ell_j - \tilde{A}^\ell_j)N_\ell\\
&\quad + \tilde{q}(A^\ell_i - \tilde{A}^\ell_i)N_\ell\,,\\
\delta L_2 &= -\frac{\varepsilon}{\sqrt{a}}\Big[\sqrt{a}^{-3}[(\tilde{\eta}'\cdot\mathcal{E}')\mathcal{E}' + (\tilde{g} - \mathfrak{g})\mathcal{E}']\Big]' + \frac{\varepsilon^3}{\sqrt{a}}\Big[\sqrt{a}^{-3}(\tilde{b} - \mathfrak{b})(n - \tilde{n})\Big]''\,,\\
\delta L_3 &= \frac{\varepsilon^3}{\sqrt{a}}\Big[\sqrt{a}^{-3}[(g^{-1} - \tilde{g}^{-1})(b - \mathfrak{b})g'n + \tilde{g}^{-1}(\tilde{b} - \mathfrak{b})\tilde{g}'(n - \tilde{n})]\Big]'\,,
\end{aligned}$$

$$\delta L_4 = \frac{\varepsilon^3}{\sqrt{a}}\Big[\sqrt{a}^{-3}\tilde{\eta}''\cdot(n-\tilde{n})n\Big]'',$$
$$\delta L_5 = \frac{\varepsilon^3}{\sqrt{a}}\Big[\sqrt{a}^{-3}[\tilde{g}(\tilde{b}-\mathfrak{b})(g-\tilde{g})'n+\tilde{g}^{-1}(b-\tilde{b})g'n]\Big]'.$$

Moreover, the following inequalities from [5] hold:

$$\|\delta D\|_k^2 + \|\delta F\|_k^2 \le Ct\int_0^t \|w\|_{k+1}^2 ds \quad \text{for } k=0,\,1,\,2, \tag{9.2a}$$

$$\|(\delta D)_t\|_0^2 + \|(\delta F)_t\|_0^2 \le C\sqrt{t}\int_0^t \Big[\|w\|_3^2 + \|w_t\|_1^2\Big] ds\,. \tag{9.2b}$$

Furthermore, w, r, and $\mathcal{E}$ satisfy the following variational form: for all $\varphi \in H^{1;2}(\Omega;\Gamma)$,

$$\langle w_t,\varphi\rangle + \bar{\sigma}\varepsilon\langle w_t,\varphi\rangle_\Gamma + \frac{\nu}{2}\int_\Omega D_A w : D_A\varphi dx + \int_\Gamma \sqrt{a}^{-4}(\mathcal{E}'\cdot\eta')(\varphi'\cdot\eta')dS \tag{9.3}$$
$$+\int_\Gamma \sqrt{a}^{-4}(\mathcal{E}''\cdot n)(\varphi''\cdot n)dS - (r, A_i^j\varphi^i_{,j})_{L^2(\Omega)} = \langle\delta F,\varphi\rangle - \sum_{k=1}^4\int_\Gamma(\delta L_k)\cdot\varphi dS\,.$$

9.1. Some a priori estimates. Similar to (6.1), solving a Stokes problem (formed from (9.1a) and (9.1b)) gives us

$$\|w\|_2^2 + \|r\|_1^2 \le C\Big[\|\delta F\|_0^2 + \|w_t\|_0^2 + \|\delta a\|_1^2 + |w|_{1.5}^2\Big]$$
$$\le C\left[\|w_t\|_0^2 + |w|_{1.5}^2 + t\int_0^t \|w\|_2^2 ds\right], \tag{9.4a}$$

$$\|w\|_3^2 + \|r\|_2^2 \le C\left[\|w_t\|_1^2 + |w|_{2.5}^2 + t\int_0^t \|w\|_3^2 ds\right]. \tag{9.4b}$$

For T small enough, (9.4b) implies that

$$\int_0^t \Big[\|w\|_3^2 + \|r\|_2^2\Big] ds \le C\int_0^t \Big[\|w_t\|_1^2 + |w|_{2.5}^2\Big] ds\,. \tag{9.5}$$

Since $g-\tilde{g} = (\eta'+\tilde{\eta}')\mathcal{E}'$ and $b-\tilde{b} = \mathcal{E}''\cdot n + \tilde{\eta}''\cdot(n-\tilde{n})$, for $s > (\mathfrak{n}-1)/2$,

$$|g-\tilde{g}|_s^2 \le C\Big[|\eta|_{s+1}^2 + |\tilde{\eta}|_{s+1}^2\Big]|\mathcal{E}|_{s+1}^2\,,$$
$$|b-\tilde{b}|_s^2 \le C\Big[|\mathcal{E}''\cdot n|_s^2 + |\tilde{\eta}|_{s+2}^2|n-\tilde{n}|_s^2\Big].$$

For $(\mathfrak{n}-1)/2 < s \le 3$, by (7.2b),

$$|n-\tilde{n}|_{s+1}^2 \le C\big[|\mathcal{E}|_{s+1}^2 + |\mathcal{E}''\cdot n|_s^2 + |n-\tilde{n}|_s^2\big].$$

Therefore, for $(\mathfrak{n}-1)/2 < s \le 3$, by interpolation,

$$|n-\tilde{n}|_{s+1}^2 \le C\Big[|\mathcal{E}|_{s+1}^2 + |\mathcal{E}''\cdot n|_s^2\Big], \tag{9.6}$$

and as a result,

$$|g-\tilde{g}|_s^2 \le C|\mathcal{E}|_{s+1}^2\,, \quad |b-\tilde{b}|_s^2 \le C\Big[|\mathcal{E}''\cdot n|_s^2 + |\mathcal{E}|_s^2\Big]. \tag{9.7}$$

Next, we estimate $\mathcal{E}$ in terms of $\mathcal{E}''\cdot n$ and $\mathcal{E}'\cdot\eta'$. Since

$$\mathcal{E}'' = (\mathcal{E}''\cdot\eta')\eta' + (\mathcal{E}''\cdot n)n = (\mathcal{E}'\cdot\eta')'\eta' - (\mathcal{E}'\cdot\eta'')\eta' + (\mathcal{E}''\cdot n)n\,,$$

we find that for $s > (\mathfrak{n}-1)/2$,

$$|\mathcal{E}|_{s+2} \le C\Big[|\mathcal{E}'\cdot\eta'|_{s+1}|\eta|_{s+1} + |\mathcal{E}|_{s+1}|\eta|_{s+2}|\eta|_{s+1} + |\mathcal{E}''\cdot n|_s|n|_s\Big]; \tag{9.8}$$

hence for $(\mathfrak{n}-1)/2 < s \le 3$,

$$|\mathcal{E}|_{s+2} \le C\Big[|\mathcal{E}'\cdot\eta'|_{s+1} + |\mathcal{E}|_{s+1} + |\mathcal{E}''\cdot n|_s\Big]\,.$$

In particular, for $s = 1.5$,

$$\begin{aligned}|\mathcal{E}|^2_{3.5} &\le C\Big[|\mathcal{E}'\cdot\eta'|^2_{2.5} + |\mathcal{E}|^2_{2.5} + |\mathcal{E}''\cdot n|^2_{1.5}\Big]\\ &\le C\left[|\mathcal{E}'\cdot\eta'|^2_{2.5} + t\int_0^t \|w\|^2_3 ds + |\mathcal{E}''\cdot n|^2_{1.5}\right]\,.\end{aligned} \tag{9.9}$$

Next, we prove (9.6) for $s = 2.5$ through the study of the boundary condition (9.1c). We rewrite (9.1c) as

$$\tilde{\mathcal{L}}(\mathcal{E}) = -\Big[(D_A w)^j_i - r\mathrm{Id}^j_i\Big]A^\ell_j N_\ell - \bar\sigma\varepsilon w_t - \sum_{k=1}^5 \delta L_k\,. \tag{9.10}$$

Same as the argument in section 7, testing (9.10) against $n(\mathcal{E}^{(6)}\cdot n)$ and $\mathcal{E}^{(6)}$, by (9.9) and interpolation we find that

$$\begin{aligned}\varepsilon^3\int_0^t |\mathcal{E}''\cdot n|^2_{2.5}ds &\le C\bar\sigma\varepsilon\left[\|w\|^2_2 + |\mathcal{E}''\cdot n|^2_{1.5} + \int_0^t \|w\|^2_3 ds\right](t) + C_\varepsilon t^2\int_0^t \|w\|^2_3 ds\\ &\quad + C\varepsilon^3\int_0^t |\mathcal{E}'\cdot\eta'|^2_{2.5}ds + \frac{C}{\varepsilon^3}\int_0^t\left[|\delta F\cdot n|^2_{0.5} + \sum_{k=1}^6 |\delta L_k\cdot n|^2_{0.5}\right]ds\,,\\ \varepsilon\int_0^t\Big[|\mathcal{E}'\cdot\eta'|^2_{2.5} + |\mathcal{E}'\cdot\tilde\eta'|^2_{2.5}\Big]ds &\le C\bar\sigma\varepsilon\left[|w|^2_2 + |\mathcal{E}|^2_3 + \int_0^t \|w\|^2_3\right](t) + C_\varepsilon t^2\int_0^t \|w\|^2_3 ds\\ &\quad + \frac{C}{\varepsilon}\int_0^t\left[|\delta F|^2_{1.5} + \sum_{k=1}^3 |\delta L_k|^2_{1.5}\right]ds\,,\end{aligned}$$

where we use

$$\begin{aligned}&\left|\int_\Gamma \delta L_4\cdot\mathcal{E}^{(6)}dS\right| = \varepsilon^3\left|\int_\Gamma \sqrt{a}^{-4}\tilde\eta''\cdot(n-\tilde n)(\mathcal{E}^{(8)}\cdot n)dS\right|\\ &= \varepsilon^3\left|\int_\Gamma \sqrt{a}^{-4}\tilde\eta''\cdot(n-\tilde n)\left[(\mathcal{E}''\cdot n)^{(6)} - \sum_{k=1}^6 C^6_k\mathcal{E}^{8-k}\cdot n^{(k)}\right]dS\right|\\ &\le C_{\bar\delta}\varepsilon^3|n-\tilde n|^2_3 + \bar\delta\varepsilon^3\Big[|\mathcal{E}''\cdot n|^2_3 + |\mathcal{E}|^2_4\Big]\end{aligned}$$

and

$$\left|\int_\Gamma \delta L_5 \cdot \mathcal{E}^{(6)} dS\right| = \varepsilon^3 \left|\int_\Gamma \sqrt{a}^{-4}\Big[\tilde{g}(\tilde{b}-\mathfrak{b})(g-\tilde{g})' + \tilde{g}^{-1}(b-\tilde{b})g'\Big](\mathcal{E}^{(7)}\cdot n) dS\right|$$

$$= \varepsilon^3 \left|\int_\Gamma \sqrt{a}^{-4}\Big[\tilde{g}(\tilde{b}-\mathfrak{b})(g-\tilde{g})' + \tilde{g}^{-1}(b-\tilde{b})g'\Big]\Big[(\mathcal{E}''\cdot n)^{(5)} - \sum_{k=1}^{5} C^5_k \mathcal{E}^{7-k}\cdot n^{(k)}\Big] dS\right|$$

$$\leq C_{\bar{\delta}}\varepsilon^3\Big[t|g-\tilde{g}|^2_3 + |b-\tilde{b}|^2_2\Big] + \bar{\delta}\varepsilon^3\Big[|\mathcal{E}''\cdot n|^2_3 + |\mathcal{E}|^2_4\Big]$$

to obtain the estimate of $\varepsilon\int_0^t \big[|\mathcal{E}'\cdot\eta'|^2_{2.5} + |\mathcal{E}'\cdot\tilde{\eta}'|^2_{2.5}\big]ds$.

By (9.6), (9.7), and (9.9),

$$\sum_{k=1}^{5} |\delta L_k \cdot n|^2_{0.5} \leq C\varepsilon^2 t\int_0^t \|w\|^2_3 ds + C\Big[\varepsilon^2|\mathcal{E}''\cdot n|^2_{1.5} + t\varepsilon^6|\mathcal{E}'\cdot\eta'|^2_{2.5}\Big],$$

$$\sum_{k=1}^{3} |\delta L_k|^2_{1.5} \leq C_{\bar{\delta}}\ \varepsilon^2 t\int_0^t \|w\|^2_3 ds + C\Big[\varepsilon^2 t|\mathcal{E}'\cdot\eta'|^2_{2.5} + (\delta+t)\varepsilon^6|\mathcal{E}''\cdot n|^2_{2.5}\Big].$$

Choosing $\bar{\delta} > 0$ and $T > 0$ small enough, for $\varepsilon \ll 1$ we find that

$$\varepsilon^3\int_0^t |\mathcal{E}''\cdot n|^2_{2.5} ds \leq C\bar{\sigma}\varepsilon\Big[\|w\|^2_2 + |\mathcal{E}''\cdot n|^2_{1.5}\Big](t) + \frac{C}{\varepsilon^3}\int_0^t \Big[\|w_t\|^2_0 + |w|^2_{1.5}\Big]ds$$

$$+ C(\bar{\sigma}\varepsilon + t^2)\int_0^t \Big[\|w_t\|^2_1 + |w|^2_{2.5}\Big]ds, \tag{9.11a}$$

(9.11b)

$$\varepsilon\int_0^t \Big[|\mathcal{E}'\cdot\eta'|^2_{2.5} + |\mathcal{E}'\cdot\tilde{\eta}'|^2_{2.5}\Big]ds \leq C\bar{\sigma}\varepsilon\Big[|w|^2_2 + |\mathcal{E}|^2_3\Big](t) + \frac{C}{\varepsilon}\int_0^t \Big[\|w_t\|^2_1 + |w|^2_{2.5}\Big]ds.$$

9.2. Estimates for w_t. We study the time differentiated problem first. Time differentiating (9.3) and then testing the resulting equation against w_t, we find that

(9.12)

$$\frac{1}{2}\frac{d}{dt}\Big[\|w_t\|^2_0 + \bar{\sigma}\varepsilon|w_t|^2_0 + \varepsilon|\sqrt{a}^{-\frac{3}{2}}w'\cdot\eta'|^2_0 + \varepsilon|\sqrt{a}^{-\frac{3}{2}}w'\cdot\tilde{\eta}'|^2_0 + \varepsilon^3|\sqrt{a}^{-\frac{3}{2}}w''\cdot n|^2_0\Big]$$

$$+ \frac{\nu}{2}\|D_A w_t\|^2_0 = \langle(\delta F)_t, w_t\rangle - \langle r_t, A^j_i w^i_{t,j}\rangle - \langle r, (A^j_i)_t w^i_{t,j}\rangle - \sum_{k=1}^{5}\langle(\delta L_k)_t, w_t\rangle_\Gamma$$

$$- \frac{\nu}{2}\langle (A^k_i)_t w^j_{,k} + (A^k_j)_t w^j_{,k}, (D_A w_t)^j_i\rangle - \frac{\nu}{2}\langle (D_A w)^j_i, (A^k_i)_t w^i_{t,k} + (A^k_j)_t w^j_{t,k}\rangle$$

$$+ \varepsilon^3\int_\Gamma \sqrt{a}^{-3}\Big[(w'\cdot v')(w'\cdot\eta') + (w'\cdot\tilde{v}')(w'\cdot\tilde{\eta}')\Big]dS$$

$$+ \varepsilon^3\int_\Gamma \sqrt{a}^{-3}(w''\cdot n_t)(w''\cdot n)dS.$$

It is clear that the terms due to viscosity on the right-hand side are bounded by $C\|w\|_1\|w_t\|_1$, and by (5.13) and Young's inequality,

$$\Big|((A^k_i)_t w^j_{,k} + (A^k_j)_t w^i_{,k}, (D_A w_t)^i_j)_{L^2(\Omega)} + \langle (D_A w)^i_j, (A^k_i)_t w^j_{t,k} + (A^k_j)_t w^i_{t,k}\rangle\Big|$$
$$\le C_{\bar\delta}\int_0^t \|w_t\|^2_1 ds + \bar\delta\|w_t\|^2_1 .$$

For the term involving δF, by (9.2b) and (9.5) we find that

$$|\langle (\delta F)_t, w_t\rangle| \le C_{\bar\delta}\sqrt{t}\int_0^t |w|^2_{2.5} ds + (C_{\bar\delta} t + \bar\delta)\int_0^t \|w_t\|^2_1 ds .$$

For the terms involving r, integration by parts implies that

$$\left|\int_0^t \langle r_t, A^j_i w^i_{t,j}\rangle ds + \int_0^t \langle r, (A^j_i)_t w^i_{t,j} ds\rangle\right|$$
$$\le |\langle r, (\delta D)_{tt}\rangle(t)| + C_{\bar\delta}\int_0^t \|r\|^2_0 ds + \bar\delta\int_0^t \|\nabla w_t\|^2_0 ds + C\int_0^t \|\nabla w\|^2_0 ds.$$

The worst term in $\langle r, (\delta D)_{tt}\rangle$ is $\langle r, A\nabla w_t A : \nabla\tilde v\rangle$ (which comes from the fact that all the time derivatives hit $(A-\tilde A)$), and in this case, integrating by parts in space implies that

$$|\langle r, A\nabla w_t A : \nabla\tilde v\rangle| \le \left|\int_\Gamma r A^j_k w^k_t A^\ell_i N_\ell dS\right| + C\|r\|_1\|w_t\|_0 .$$

Therefore, by Young's inequality we conclude that

$$\left|\int_0^t \langle r_t, A^j_i w^i_{t,j}\rangle ds + \int_0^t \langle r, (A^j_i)_t w^i_{t,j} ds\rangle\right|$$
$$\le \bar\delta\|r\|^2_1 + C_{\bar\delta} t\int_0^t \|w\|^2_3 ds + C_{\bar\delta}\int_0^t \|r\|^2_0 ds + (Ct+\bar\delta)\int_0^t \|\nabla w_t\|^2_0 ds .$$

It is also clear that the last two terms of (9.12) are bounded by $C\|w\|^2_{2.5}$. To complete the computations, we only need to estimate $\sum_{k=1}^5 \langle (\delta L_k)_t, w_t\rangle_\Gamma$.

By interpolation and Young's inequality,

$$\left|\sum_{k=1}^5 \langle (\delta L_k)_t, w_t\rangle_\Gamma\right| \le Ct\int_0^t \|w\|^2_3 ds + C_{\bar\delta}\varepsilon^2|w|^2_{2.5} + \bar\delta\|w_t\|^2_1 ,$$

where $\sum_{k=2}^5 |(\delta L_k)_t|_{-0.5} \le C(\varepsilon+\varepsilon^3+\varepsilon^3 t)|w|_{2.5}$ and $H^{-0.5}(\Gamma)$-$H^{0.5}(\Gamma)$ duality are used to obtain the estimate for k from 2 to 5.

Time integrating (9.12), combining all the inequalities above, and choosing $\bar\delta > 0$ and $T > 0$ small enough, by (9.4b) we find that

$$Y(t) + \int_0^t \|w_t\|^2_1 ds \le C\int_0^t Y(t) ds + C(\varepsilon^3 + \sqrt{t})\int_0^t |w|^2_{2.5} ds, \tag{9.13}$$

where

$$Y(t) = \|w_t(t)\|^2_0 + \varepsilon|(w'\cdot\eta')(t)|^2_0 + \varepsilon|(w'\cdot\tilde\eta')(t)|^2_0 + \varepsilon^3|(w''\cdot n)(t)|^2_0 .$$

9.3. Estimates for w''. Let $\varphi = (\zeta^2 w'')''$ in (9.3). Then

(9.14)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|\zeta w''\|_0^2 + \bar\sigma\varepsilon|w''|_0^2 + \varepsilon|\sqrt{a}^{-\frac{3}{2}}(\mathcal{E}'\cdot\eta')''|_0^2 + \varepsilon|\sqrt{a}^{-\frac{3}{2}}(\mathcal{E}'\cdot\tilde\eta')''|_0^2 \\
&\quad + \varepsilon^3|\sqrt{a}^{-\frac{3}{2}}(\mathcal{E}''\cdot n)''|_0^2\Big] + \frac{\nu}{2}\|\zeta D_A w''\|_0^2 = -\langle r, A_i^j(\zeta^2 w^{i\prime\prime})''_{,j}\rangle + \langle \delta F, (\zeta^2 w'')''\rangle \\
&- \sum_{k=1}^{5}\langle \delta L_k, w^{(4)}\rangle_\Gamma - \nu\int_\Omega \zeta\Big[2(A_i^{k\prime} w^{j\prime}_{,k} + A_j^{k\prime} w^{i\prime}_{,k}) + (A_i^{k\prime\prime} w^j_{,k} + a_j^{k\prime\prime} w^i_{,k})\Big] A_i^k \zeta w^{j\prime\prime}_{,k}\, dx \\
&- 2\nu\int_\Omega \zeta\Big[(A_i^{k\prime} w^j_{,k} + a_j^{k\prime} w^i_{,k}) + (A_i^k w^{j\prime}_{,k} + A_j^k w^{i\prime}_{,k})\Big] A_i^{k\prime}\zeta w^{j\prime\prime}_{,k}\, dx \\
&+ \varepsilon\sum_{k=1}^{4}\int_\Gamma \sqrt{a}^{-3} C_k^4\Big[(\mathcal{E}'\cdot\eta')w^{(5-k)}\cdot\eta^{(1+k)} + (\mathcal{E}'\cdot\tilde\eta')w^{(5-k)}\cdot\tilde\eta^{(1+k)}\Big]dS \quad (\equiv A_1) \\
&+ \varepsilon^3\sum_{k=1}^{4}\int_\Gamma \Big[\sqrt{a}^{-3} C_k^4(\mathcal{E}''\cdot n)w^{(6-k)}\cdot n^{(k)}\Big]dS \qquad (\equiv A_2) \\
&- \varepsilon\sum_{k=0}^{1}\int_\Gamma \frac{1}{\sqrt{a}} C_k^2\Big[(\mathcal{E}'\cdot\eta')^{(k)}(\mathcal{E}'\cdot\eta')'' + (\mathcal{E}'\cdot\tilde\eta')^{(k)}(\mathcal{E}'\cdot\tilde\eta')''\Big](\sqrt{a}^{-3})^{(2-k)}dS \\
&+ \int_\Gamma \sqrt{a}^{-3}\Big[\varepsilon(\mathcal{E}'\cdot v')''(\mathcal{E}'\cdot\eta')'' + \varepsilon(\mathcal{E}'\cdot\tilde v')''(\mathcal{E}'\cdot\tilde\eta')'' + \varepsilon^3(\mathcal{E}''\cdot n_t)''(\mathcal{E}''\cdot n)''\Big]dS \\
&+ \varepsilon^3\int_\Gamma \frac{1}{\sqrt{a}}\Big[(\mathcal{E}''\cdot n)(\sqrt{a}^{-3})'' + 2(\mathcal{E}''\cdot n)'(\sqrt{a}^{-3})'\Big](\mathcal{E}''\cdot n)''dS\,.
\end{aligned}$$

For the terms involving r, integrating by parts if necessary, by $A_i^j w^i_{,j} = \delta D$ we find that

$$\begin{aligned}
|\langle r, A_i^j(\zeta^2 w^{i\prime\prime})''_{,j}\rangle| &\le C_{\bar\delta}\|w\|_2^2 + C_{\bar\delta}\|\delta D\|_2^2 + \bar\delta\|r\|_2^2 \\
&\le C_{\bar\delta}\|w\|_0^2 + C_{\bar\delta} t\int_0^t \|w\|_3^2 ds + \bar\delta\Big[\|w\|_3^2 + \|r\|_2^2\Big]\,.
\end{aligned}$$

By (9.2a) and interpolation,

$$|\langle \delta F, (\zeta^2 w'')''\rangle| + |\langle \delta L_1, w^{(4)}\rangle_\Gamma| \le C_{\bar\delta} t\int_0^t \|w\|_3^2 ds + \bar\delta\|w\|_3^2\,.$$

The terms due to viscosity can be bounded by $C\|w\|_{2.5}\|w\|_3$. Therefore, by interpolation and Young's inequality,

$$\begin{aligned}
&\left|\int_\Omega \zeta\Big[2(A_i^{k\prime} w^{j\prime}_{,k} + A_j^{k\prime} w^{i\prime}_{,k}) + (A_i^{k\prime\prime} w^j_{,k} + A_j^{k\prime\prime} w^i_{,k})\Big] a_i^k \zeta w^{j\prime\prime}_{,k}\, dx\right| \\
&+ \left|\int_\Omega \zeta\Big[(A_i^{k\prime} w^j_{,k} + a_j^{k\prime} w^i_{,k}) + (A_i^k w^{j\prime}_{,k} + A_j^k w^{i\prime}_{,k})\Big] A_i^{k\prime}\zeta w^{j\prime\prime}_{,k}\, dx\right| \le C_{\bar\delta}\|w\|_2^2 + \bar\delta\|w\|_3^2\,.
\end{aligned}$$

For terms A_1 and A_2, by $H^{1.5}(\Gamma)$-$H^{-1.5}(\Gamma)$ or $H^{2.5}(\Gamma)$-$H^{-2.5}(\Gamma)$ duality, we find that

$$A_1 + A_2 \le C_{\bar\delta}\Big[\varepsilon^2|\mathcal{E}'\cdot\eta'|_{1.5}^2 + \varepsilon^2|\mathcal{E}'\cdot\tilde\eta'|_{1.5}^2 + \varepsilon^6|\mathcal{E}''\cdot n|_{2.5}^2\Big] + \bar\delta\|w\|_3^2\,.$$

Therefore, time integrating (9.14), by (9.11) we find that

$$\Big[\|\zeta w''\|_0^2 + \bar\sigma\varepsilon|w|_2^2 + \varepsilon|\mathcal{E}'\cdot\eta'|_2^2 + \varepsilon|\mathcal{E}'\cdot\tilde\eta'|_2^2 + \varepsilon^3|\mathcal{E}''\cdot n|_2^2\Big](t) + \int_0^t |w|_{2.5}^2 ds$$
$$\le C_{\bar\delta}\int_0^t \Big[\|w_t\|_0^2 + |\mathcal{E}'\cdot\eta'|_2^2 + |\mathcal{E}'\cdot\tilde\eta'|_2^2\Big]ds + (C_{\bar\delta}t + \bar\delta + C\bar\sigma\varepsilon^4)\int_0^t \Big[\|w_t\|_1^2 + |w|_{2.5}^2\Big]ds$$
$$+ \left|\sum_{k=2}^5 \int_0^t \langle \delta L_k, w^{(4)}\rangle_\Gamma ds\right|.$$

For the last term of the inequality above, we first note that by $H^{1.5}(\Gamma)$-$H^{-1.5}(\Gamma)$ duality,

$$\langle \delta L_2, w^{(4)}\rangle_\Gamma + \langle \delta L_3, w^{(4)}\rangle_\Gamma \le C_{\bar\delta}t\Big[\varepsilon^2|\mathcal{E}|_{3.5}^2 + \varepsilon^6|n-\tilde n|_{3.5}^2\Big] + \bar\delta\|w\|_3^2\,.$$

For $\langle \delta L_4, w^{(4)}\rangle_\Gamma$, we study the time integral and integrate by parts to obtain

$$\int_0^t \langle \delta L_4, w^{(4)}\rangle_\Gamma ds = -\int_0^t \langle (\delta L_4)_t, \mathcal{E}^{(4)}\rangle_\Gamma ds$$
$$= -\varepsilon^3\int_0^t\int_\Gamma \sqrt{a}^{-3}\Big[\tilde\eta''\cdot(n-\tilde n)n_t + \tilde v''\cdot(n-\tilde n)n + \tilde\eta''\cdot(n-\tilde n)_t n\Big]\mathcal{E}^{(6)} dS ds,$$

and by $H^s(\Gamma)$-$H^{-s}(\Gamma)$ duality,

$$\left|\int_0^t \langle \delta L_4, w^{(4)}\rangle_\Gamma ds\right| \le C_{\bar\delta}\varepsilon^3\int_0^t \Big[|n-\tilde n|_{2.5}^2 + |(n-\tilde n)_t|_{1.5}^2\Big]ds$$
$$+ \bar\delta\varepsilon^3\int_0^t \Big[|\mathcal{E}|_{3.5}^2 + |\mathcal{E}''\cdot n|_{2.5}^2\Big]ds\,,$$

where we use $\mathcal{E}^{(6)}\cdot n = (\mathcal{E}''\cdot n)^{(4)} - \sum_{k=1}^4 C_k^4 \mathcal{E}^{6-k}\cdot n^{(k)}$ to estimate the last term of the integral. The term $\langle \delta L_5, w^{(4)}\rangle_\Gamma$ can be estimated in a similar fashion, and we find that

$$\left|\int_0^t \langle \delta L_5, w^{(4)}\rangle_\Gamma ds\right| \le C_{\bar\delta}\varepsilon^3\int_0^t \Big[t|(g-\tilde g)_t|_{1.5}^2 + |(b-\tilde b)_t|_{0.5}^2\Big]ds$$
$$+ \bar\delta\varepsilon^3\int_0^t \Big[|\mathcal{E}|_{3.5}^2 + |\mathcal{E}''\cdot n|_{2.5}^2\Big]ds\,.$$

Combining all the estimates above, we conclude that

(9.15)
$$Z(t) + \int_0^t |w|_{2.5}^2 ds \le C_{\bar\delta}\int_0^t \Big[\|w_t\|_1^2 + Y(s)\Big]ds + C(\varepsilon,\bar\delta,t)\int_0^t \Big[\|w_t\|_1^2 + |w|_{2.5}^2\Big]ds,$$

where

$$Z(t) = \varepsilon|(\mathcal{E}'\cdot\eta')(t)|_2^2 + \varepsilon|(\mathcal{E}'\cdot\tilde\eta')(t)|_2^2 + \varepsilon^3|(\mathcal{E}''\cdot n)(t)|_2^2$$

and $C(\varepsilon,\bar\delta,t)$ can be made small for $\varepsilon \ll 1$ and proper choice of $\bar\delta$ and T. With $\varepsilon \ll 1$, (9.13) and (9.15) together imply that $Y + Z = 0$ by choosing $\bar\delta > 0$, $T > 0$ small enough. Thus, we conclude that $w = 0$, and hence the solution to (2.5) is unique.

10. List of notation.

$\mathfrak{n}$	The dimension of the space
u	The Eulerian velocity
p	The Eulerian pressure
u_0	The initial condition of the velocity field
$g_{\alpha\beta}$	The metric tensor of the moving surface
$b_{\alpha\beta}$	The curvature tensor of the moving surface
$\mathfrak{g}_{\alpha\beta}$	The metric tensor of the initial equilibrium state
$\mathfrak{b}_{\alpha\beta}$	The curvature tensor of the initial equilibrium state
$a^{\alpha\beta\gamma\delta}$	The elasticity tensor
$\bar{\sigma}$	$\bar{\sigma} = 1$ or 0 indicates the case with or without inertia is considered
$\mathcal{V}^k(T)$	The collection of $v \in L^2(0,T;L^2(\Omega))$ so that $\partial_t^\ell v \in L^2(0,T;H^{k-2\ell}(\Omega))$
$\mathcal{V}_v$	The collection of $w \in H^{1;2}(\Omega;\Gamma)$ so that $A_i^j w^i_{,j} = 0$
$\mathcal{V}_v(T)$	The collection of $w \in L^2(0,T;H^{1;2}(\Omega;\Gamma))$ so that $A_i^j w^i_{,j} = 0$
$\\| \cdot \\|_s$	The $H^s(\Omega)$-norm of the object
$\lvert \cdot \rvert_s$	The $H^s(\Gamma)$-norm of the object
$\bar{\partial}$	The tangential derivative for the case $\mathfrak{n} = 3$
$'$	The tangential derivative for the case $\mathfrak{n} = 2$
$\mathbf{e}$	The identity map on $\mathbb{R}^\mathfrak{n}$ satisfying $\mathbf{e}(x) = x$
$\langle \cdot , \cdot \rangle_X$	The duality pairing between space X and its dual X'
$\langle \cdot , \cdot \rangle$	The duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$
$\langle \cdot , \cdot \rangle_\Gamma$	The duality pairing between $H^2(\Gamma)$ and $H^{-2}(\Gamma)$
$\mathcal{F}^k(T)$	The collection of $f \in L^2(0,T;L^2(\mathbb{R}^\mathfrak{n}))$ so that $\partial_t^\ell f \in L^2(0,T;H^{k-2\ell}(\mathbb{R}^\mathfrak{n}))$
$\sqrt{a}$	The square root of the determinant of the initial metric
η	The flow map of the fluid velocity
A_i^j	The cofactor matrix of $\nabla\eta$
v	The Lagrangian velocity field, $v = u \circ \eta$
q	The Lagrangian pressure, $q = p \circ \eta$
q_0	The initial value of q
w_1	The initial value of v_t
w_2	The initial value of v_{tt}
q_1	The initial value of q_t
$\mathbf{P}_{\tan}$	The orthogonal projection onto the tangent plane of Γ
$H^{1;2}(\Omega;\Gamma)$	The space of $H^1(\Omega)$-functions with H^2 traces on the boundary
$\tilde{u}_0$, $\tilde{\mathrm{w}}_k$, $\tilde{q}_k$	The regularized initial data
v_θ	The solution to the penalized problem
q_θ	The penalized pressure defined as $\tilde{q}_0 + t\tilde{q}_1 - \frac{1}{\theta} A_i^j v^i_{\theta,j}$
v_κ	The solution to the linearized and regularized equation (5.5)

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