Arch. Rational Mech. Anal. 221 (2016) 987–1033
Digital Object Identifier (DOI) 10.1007/s00205-016-0977-z

On the Impossibility of Finite-Time Splash Singularities for Vortex Sheets

Daniel Coutand & Steve Shkoller

Communicated by P. Constantin

Abstract

In fluid dynamics, an interface *splash* singularity occurs when a locally smooth interface self-intersects in finite time. By means of elementary arguments, we prove that such a singularity cannot occur in finite time for vortex sheet evolution, that is for the two-phase incompressible Euler equations. We prove this by contradiction; we assume that a splash singularity does indeed occur in finite time. Based on this assumption, we find precise blow-up rates for the components of the velocity gradient which, in turn, allow us to characterize the geometry of the evolving interface just prior to self-intersection. The constraints on the geometry then lead to an impossible outcome, showing that our assumption of a finite-time splash singularity was false.

Contents

1. Introduction

1.1. The Interface Splash Singularity

The fluid interface *splash singularity* was introduced by Castro et al. in [11]. A *splash singularity* occurs when a fluid interface remains locally smooth but self-intersects in finite time. For the two-dimensional water waves problem, Castro et al. [11] showed that a splash singularity occurs in finite time using methods from complex analysis together with a clever transformation of the equations. In Coutand and Shkoller [18], we showed the existence of a finite-time splash singularity for the water waves equations in two or three-dimensions (and, more generally, for the one-phase Euler equations), using a very different approach, founded upon an approximation of the self-intersecting fluid domain by a sequence of smooth fluid domains, each with non self-intersecting boundary.

1.2. The Two-Fluid Incompressible Euler Equations

A natural question, then, is whether a splash singularity can occur for vortex sheet evolution, in which two phases of the fluid are present. Consider the two-phase incompressible Euler equations: let $\mathcal{D} \subseteq \mathbb{R}^2$ denote an open, bounded set, which comprises the volume occupied by two incompressible and inviscid fluids with different densities. At the initial time $t = 0$, we let Ω^+ denote the volume occupied by the *lower* fluid with density ρ^+ and we let Ω^- denote the volume occupied by the *upper* fluid with density ρ^-. Mathematically, the sets Ω^+ and Ω^- denote two disjoint open bounded subsets of $\mathcal{D}$ such that $\overline{\mathcal{D}} = \overline{\Omega^+} \cup \overline{\Omega^-}$ and $\Omega^+ \cap \Omega^- = \emptyset$. The *material interface* at time $t = 0$ is given by $\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-}$, and $\partial\mathcal{D} = \partial(\Omega^- \cup \Omega^+)/\,\Gamma$. (We can also consider the case that $\Omega^+ = \mathbb{T} \times (-1, 0)$, $\Omega^- = \mathbb{T} \times (0, 1)$, and $\Gamma = \mathbb{T} \times \{0\}$).

For time $t \in [0, T]$ for some $T > 0$ fixed, $\Omega^+(t)$ and $\Omega^-(t)$ denote the time-dependent volumes of the two fluids, respectively, separated by the moving material interface $\Gamma(t)$. Let $u^\pm$ and $p^\pm$ denote the velocity field and pressure function, respectively, in $\Omega^\pm(t)$. A planar vortex sheet $\Gamma(t)$ evolves according to the incompressible and irrotational Euler equations:

$$\rho^\pm(u_t^\pm + u^\pm \cdot Du^\pm) + Dp^\pm = -\rho^\pm g e_2 \quad \text{in} \quad \Omega^\pm(t), \tag{1.1a}$$

$$\operatorname{curl} u^\pm = 0, \quad \operatorname{div} u^\pm = 0 \quad \text{in} \quad \Omega^\pm(t), \tag{1.1b}$$

$$p^+ - p^- = \sigma H \quad \text{on} \quad \Gamma(t), \tag{1.1c}$$

$$(u^+ - u^-) \cdot \mathcal{N} = 0 \quad \text{on} \quad \Gamma(t), \tag{1.1d}$$

$$u^- \cdot N = 0 \quad \text{on} \quad \partial\mathcal{D}, \tag{1.1e}$$

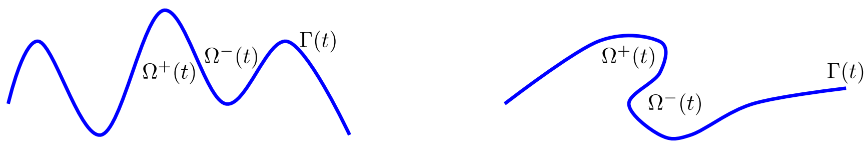

Fig. 1. Two examples of the evolution of a vortex sheet $\Gamma(t)$ by the Euler equations. The two fluid regions are denoted by $\Omega^+(t)$ and $\Omega^-(t)$

$$u(0) = u_0 \quad \text{on} \quad \{t = 0\} \times \mathcal{D}, \tag{1.1f}$$

$$\mathcal{V}(\Gamma(t)) = u^+(t) \cdot \mathcal{N}(t), \tag{1.1g}$$

where $\mathcal{V}(\Gamma(t))$ denotes the speed of the moving interface $\Gamma(t)$ in the normal direction, and $\mathcal{N}(\cdot, t)$ denotes the outward-pointing unit normal to $\Gamma(t)$ (pointing into $\Omega^-(t)$), N denotes the outward-pointing unit normal to the fixed boundary $\partial\mathcal{D}$, g denotes gravity, and e_2 is the vertical unit vector $(0, 1)$. Equation (1.1g) indicates that $\Gamma(t)$ moves with the normal component of the fluid velocity. The variables $0 < \rho^\pm$ denote the densities of the two fluids occupying $\Omega^\pm(t)$, respectively, $H(t)$ is twice the mean curvature of $\Gamma(t)$, and $\sigma > 0$ is the surface tension parameter which we will henceforth set to one. For notational simplicity, we will also set $\rho^+ = 1$ and $\rho^- = 1$ (Fig. 1).

Via an elementary proof by contradiction, we prove that a finite-time splash singularity cannot occur for vortex sheets governed by (1.1). We rule-out a single splash singularity in which one self-intersection occurs, as well as the case that many (finite or infinite) simultaneous self-intersections occur. We also rule-out a *splat* singularity, wherein the interface $\Gamma(t)$ self-intersects along a curve (see [11] and [18] for a precise definition).

1.3. Outline of the Paper

In Section 2, we introduce Lagrangian coordinates (using the flow of u^-) for the purpose of fixing the domain and the material interface. Rather than using an arbitrary parameterization of the evolving interface $\Gamma(t)$, we specifically use the Lagrangian parameterization which has some important features for our analysis that general parameterizations do not. With this parameterization defined, we state the main theorem of the paper in Section 3 which states that a finite-time splash singularity cannot occur in this setting. In Section 4, we derive the evolution equations for the vorticity along the interface as well as the evolution equation for the tangential derivative of the vorticity; the latter plays a fundamental role in our analysis. In particular, under the assumption that the tangential derivative of vorticity blows-up in finite time, we find the precise blow-up rates for the components of $\nabla u^-(\cdot, t)$. Letting $\eta(\cdot, t) : \Gamma \to \Gamma(t)$ denote the Lagrangian parameterization of the vortex sheet, and supposing that the two reference points x_0 and x_1 in Γ evolve toward one another so that $|\eta(x_0, t) - \eta(x_1, t)| \to 0$ as $t \to T$, in Section 6, we find the evolution equation for the distance $\delta\eta(t) = \eta(x_0, t) - \eta(x_1, t)$ between the two contact points. We can determine that the two portions of the curve $\Gamma(t)$ converge towards self-intersection in an essentially horizontal approach.

Finally, using the evolution equation for $\boldsymbol{\delta}\eta(t)$, we prove our main theorem in Section 7; in particular, we show that our assumption of a finite-time self-intersection of the curve $\Gamma(t)$ as $t \to T$ leads to the following contradiction: we first show that $u_1^-(\eta(x_0, T), T) - u_1^-(\eta(x_1, T), T) = 0$, where $u_1^- = u^- \cdot \mathrm{e}_1$ and e_1 is the tangent vector at $\eta(x_0, T)$, and then we proceed to show that $u_1^-(\eta(x_0, T), T) - u_1^-(\eta(x_1, T), T) \neq 0$. We first arrive at this contradiction for a single splash singularity, meaning that one self-intersection point exists for $\Gamma(T)$; then, we proceed to prove that a finite (or even infinite) number of self-intersections also cannot occur. We conclude by showing that a *splat* singularity, wherein $\Gamma(T)$ self-intersects along a curve rather than a point, also cannot occur.

1.4. A Brief History of Prior Results

1.4.1. Local-in-Time Well-Posedness We begin with a short history of the local-in-time existence theory for the free-boundary incompressible Euler equations. For the *irrotational* case of the water waves problem, and for two dimensional fluids (and hence one dimensional interfaces), the earliest local existence results were obtained by NALIMOV [31], YOSIHARA [41], and CRAIG [12] for initial data near equilibrium. BEALE et al. [8] proved that the linearization of the two dimensional water wave problem is well-posed if the Rayleigh–Taylor sign condition $\frac{\partial p}{\partial n} < 0$ on $\Gamma \times \{t = 0\}$ is satisfied by the initial data (see [33] and [36]). WU [37] established local well-posedness for the two dimensional water waves problem and showed that, due to irrotationality, the Taylor sign condition is satisfied. Later AMBROSE and MASMOUDI [5], proved local well-posedness of the two dimensional water waves problem as the limit of zero surface tension. DISCONZI and EBIN [20,21] have considered the limit of surface tension tending to infinity. For three dimensional fluids (and two dimensional interfaces), WU [38] used Clifford analysis to prove local existence of the three dimensional water waves problem with *infinite depth*, again showing that the Rayleigh–Taylor sign condition is always satisfied in the irrotational case by virtue of the maximum principle holding for the potential flow. LANNES [29] provided a proof for the *finite depth case with varying bottom*. Recently, ALAZARD et al. [1] have established low regularity solutions (below the Sobolev embedding) for the water waves equations. See also [6,7].

The first local well-posedness result for the three dimensional incompressible Euler equations without the irrotationality assumption was obtained by LINDBLAD [30] in the case that the domain is diffeomorphic to the unit ball using a Nash–Moser iteration, following the a prior estimates of [15]. COUTAND and SHKOLLER [16] proved local well-posedness for arbitrary initial geometries that have at least H^3-class boundaries without derivative loss; see also [17]. SHATAH and ZENG [34] established a priori estimates for this problem using an infinite-dimensional geometric formulation, and ZHANG and ZHANG [42] proved well-posedness by extending the complex-analytic method of WU [38] to allow for vorticity. Again, in the latter case the domain was with infinite depth.

1.4.2. Long-Time Existence It is of great interest to understand if solutions to the Euler equations can be extended for all time when the data is sufficiently smooth and small, or if a finite-time singularity can be predicted for other types of initial conditions.

Because of irrotationality, the water waves problem does not suffer from vorticity concentration; therefore, singularity formation involves only the loss of regularity of the interface or interface collision. In the case that the irrotational fluid is infinite in the horizontal directions, certain dispersive-type properties can be made use of. For sufficiently smooth and small data, Alvarez-Samaniego and Lannes [4] proved existence of solutions to the water waves problem on large time-intervals (larger than predicted by energy estimates), and provided a rigorous justification for a variety of asymptotic regimes. By constructing a transformation to remove the quadratic nonlinearity, combined with decay estimates for the linearized problem (on the infinite half-space domain), Wu [39] established an almost global existence result (existence on time intervals which are exponential in the size of the data) for the two dimensional water waves problem with sufficiently small data. In a different framework, Alazard et al. [1] have also proven this result. Using position-velocity potential holomorphic coordinates, Hunter et al. [25] have also proved almost global existence of the two dimensional water waves problem.

Wu [40] proved global existence in three dimensional for small data. Using the method of spacetime resonances, Germain et al. [24] also established global existence for the three dimensional irrotational problem for sufficiently small data. More recently, global existence for the two dimensional water waves problem with small data was established by Ionescu and Pusateri [28], Alazard and Delort [2,3], and Ifrim and Tataru [26,27].

1.4.3. The Finite-Time Splash and Splat Singularity The finite-time *splash* and *splat singularities* were introduced by Castro et al. [11]; therein, using methods from complex analysis, they proved that a locally smooth interface can self-intersect in finite time for the two dimensional water waves equations and hence established the existence of finite-time splash and splat singularities (see also [9] and [10]). In Coutand and Shkoller [18], we established the existence of finite-time splash and splat singularities for the two dimensional and three dimensional water waves and Euler equations (with vorticity) using an approximation of the self-intersecting domain by a sequence of standard Sobolev-class domains, each with non self-intersecting boundary. Our approach can be applied to many one-phase hyperbolic free-boundary problems, and shows that splash singularities can occur with surface tension, with compressibility, with magnetic fields, and for many one-phase hyperbolic free-boundary problems.

Recently, Fefferman et al. [22] have proven that a splash singularity cannot occur for planar vortex sheets (or two-fluid interfaces) with surface tension. Their proof relies on a sophisticated harmonic analysis of the integral kernel of the Birkhoff–Rott equation. Other than vortex sheet evolution for the two-phase Euler equations, it is of interest to determine the possibility of finite-time splash singularities for other fluids models. In this regard, Gancedo and Strain [23] have recently shown that a finite-time splash singularity cannot occur for the three-phase Muskat equations. In addition to the study of other fluids models, it is also of great interest to determine a mechanism for the loss of regularity of the evolving interface, which, in turn, could allow for finite-time self-intersection.

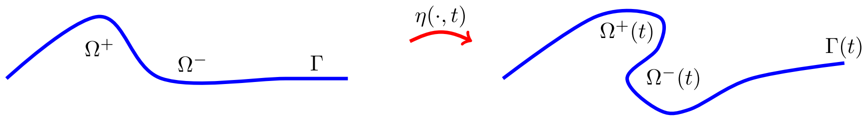

Fig. 2. The mapping $\eta(\cdot, t)$ fixes the two fluid domains and the interface. The moving interface $\Gamma(t)$ is the image of Γ by $\eta(\cdot, t)$

2. Fixing the Fluid Domains Using the Lagrangian Flow of $u-$

Let $\tilde{\eta}$ denote the Lagrangian flow map of u^- in Ω^- so that $\tilde{\eta}_t(x,t) = u^-(\tilde{\eta}(x,t),t)$ for $x \in \Omega^-$ and $t \in (0,T)$, with initial condition $\tilde{\eta}(x,0) = x$ (Fig. 2). Since $\operatorname{div} u^- = 0$, it follows that $\det \nabla\tilde{\eta} = 1$. By a theorem of [19], we define $\Psi : \Omega^+ \to \Omega^+(t)$ as incompressible extension of $\tilde{\eta}$, satisfying $\det \nabla\Psi = 1$ and $\|\Psi\|_{H^s(\Omega^+)} \leqq C\|\eta^-|_\Gamma\|_{H^{s-1/2}(\Gamma)}$ for $s > 2$. We then set

$$\eta(x,t) = \begin{cases} \tilde{\eta}(x,t), & x \in \overline{\Omega^-} \\ \Psi(x,t), & x \in \Omega^+ \end{cases} .$$

We define the following quantities set on the fixed domains and boundary:

$$\begin{aligned} v^\pm &= u^\pm \circ \eta, && \text{in } \ \Omega^\pm \times [0,T], \\ q^\pm &= p^\pm \circ \eta, && \text{in } \ \Omega^\pm \times [0,T], \\ A &= [\nabla\eta]^{-1}, && \text{in } \ \mathcal{D} \times [0,T], \\ \mathcal{H} &= H \circ \eta, && \text{on } \ \Gamma \times [0,T], \\ \delta v &= v^+ - v^-, && \text{on } \ \Gamma \times [0,T]. \end{aligned}$$

The momentum equations (1.1a) can then be written on the fixed domains $\Omega^\pm$ as

$$v_t^+ + \nabla v^+ A\,(v^+ - \Psi_t) + A^T \nabla q^+ = -g e_2 \quad \text{in} \quad \Omega^+ \times [0,T], \tag{2.1a}$$

$$v_t^- + A^T \nabla q^- = -g e_2 \quad \text{in} \quad \Omega^- \times [0,T], \tag{2.1b}$$

and the pressure jump condition (1.1c) is $\delta q = \mathcal{H}$ on $\Gamma \times [0,T]$, where $\delta q = q^+ - q^-$.

Using the Einstein summation convention, $[\nabla v^+ A\,(v^+ - \Psi_t)]^i = v^{+i},_r A^r_j (v^+_j - \partial_t \Psi_j)$. This is the advection term; when Ψ is the identity map, we recover the Eulerian description, while if Ψ is the Lagrangian flow map, then we recover the Lagrangian description. The form (2.1.a) is called the Arbitrary Lagrangian Eulerian (ALE) description of the fluid flow in Ω^+

3. The Main Result

In [13,14], we proved that if at time $t = 0$, $u_0^\pm \in H^k(\Omega^\pm)$ and Γ of class H^{k+1} for integers $k \geqq 3$, then there exists a solution $(u^\pm(\cdot,t), \Gamma(t))$ of the system (1.1) satisfying $u^\pm \in L^\infty(0,T_0; H^k(\Omega^\pm(t)))$ with $\Gamma(t)$ being of class H^{k+1}, for all $t \in [0,T_0]$, for some $T_0 > 0$. (See also [35] and [32]).

Theorem 3.1. (No finite-time splash singularity). *Let $\mathcal{D}$ be a bounded domain of class H^4. We assume the existence of a closed curve $\Gamma \subset \mathcal{D}$ of class $W^{4,\infty}$ which does not self-intersect and such that $\mathcal{D} = \Omega^+ \cup \Gamma \cup \Omega^-$, where the open sets Ω^+ and Ω^- are connected and disjoint and do not intersect Γ. Our assumption of non self-intersection means that Ω^+ and Ω^- are both (locally) on one side of Γ.*

Let $u^\pm$ be a solution to (1.1) on $[0, T)$ such that $u^\pm \in H^3(\Omega^\pm(t))$ and $\Gamma(t)$ is of class $W^{4,\infty}$ for each $t \in [0, T)$. Suppose that

(1) *$\Omega^+(t)$ and $\Omega^-(t)$ are both (locally) on one side of $\Gamma(t)$ for all $t \in [0, T)$;*
(2) *there exists a constant $0 < \mathcal{M} < \infty$, such that*

$$\text{for all } \; t \in [0, T), \quad \operatorname{dist}(\Gamma(t), \partial\mathcal{D}) > \frac{1}{\mathcal{M}}.$$

and

$$\sup_{t\in[0,T)} \left(\left\| u^+(\cdot, t) \right\|_{W^{2,\infty}(\Gamma(t))} + \| H(\cdot, t) \|_{W^{2,\infty}(\Gamma(t))} \right) < \mathcal{M}. \tag{3.1}$$

Then $\Gamma(t)$ cannot self-intersect at time $t = T$; that is, there does not exist a finite-time splash singularity.

Note, that we give a precise definition for the $W^{k,\infty}(\Gamma(t))$-norm below in Definition 4.1.

Remark 1. The condition (1) in Theorem 3.1, requiring $\Omega^+(t)$ and $\Omega^-(t)$ to both (locally) be on one side of $\Gamma(t)$ for all $t \in [0, T)$, is equivalent to requiring the chord-arc function to be strictly positive for all t in $[0, T)$ (without specifying a lower bound as $t \to T$, other than 0).

Remark 2. In Theorem 3.1, we have assumed that $\mathcal{D} = \Omega^+(t) \cup \Gamma(t) \cup \Omega^-(t)$ is a bounded domain simply because the local well-posedness theorem for the two-phase Euler equations given in [13] used such a geometry; however, as our proof by contradiction relies on a local analysis in a spacetime region near an assumed point (or points) of self-intersection of the curve $\Gamma(t)$, we can also treat the case that our two fluids occupy all of $\mathbb{R}^2$ or occupy a channel geometry with periodic boundary conditions in the horizontal direction.

As part of condition (2) in Theorem 3.1 for the case that $\mathcal{D}$ is bounded, we assume that $\operatorname{dist}(\Gamma(t), \partial\mathcal{D}) > \frac{1}{\mathcal{M}}$ so that the moving interface $\Gamma(t)$ stays away from the fixed domain boundary $\partial\mathcal{D}$.

4. Evolution Equations on Γ for the Vorticity and Its Tangential Derivative

4.1. Geometric Quantities Defined on Γ and $\Gamma(t)$

We set

$$\mathcal{N}(x, t) = \text{ unit normal vector field on } \Gamma(t), \quad n = \mathcal{N} \circ \eta$$
$$\mathcal{T}(x, t) = \text{ unit tangent vector field on } \Gamma(t), \quad \tau = \mathcal{T} \circ \eta.$$

We choose the unit-normal $\mathcal{N}(\cdot, t)$ to point into $\Omega^-(t)$. In a sufficiently small neighborhood $\mathcal{U}$ of the material interface Γ at $t = 0$, we choose a local chart $\theta : B(0, 1) \to \mathcal{U}$. The unit ball $B(0, 1)$ has coordinates (x_1, x_2), and $\theta : \{(x_1, x_2) : x_2 = 0\} \to \mathcal{U} \cap \Gamma$, $\theta\{(x_1, x_2) : x_2 > 0\} \to \mathcal{U} \cap \Omega^-$, and $\theta\{(x_1, x_2) : x_2 < 0\} \to \mathcal{U} \cap \Omega^+$. In order to define a tangent vector, we also assume that the length $|\theta'(x_1, 0)|$ of the vector $\theta'(x_1, 0)$ is bounded away from 0 by some constant $C > 0$. For notational convenience in our computations, we shall write $\eta \circ \theta$ simply as η. We define

$$G(x, t) = |\eta'(x, t)|^{-1}, \quad \text{where } (\cdot)' = \partial(\cdot)/\partial x_1.$$

Hence,

$$\tau(x, t) = G\eta'(x, t),\ n(x, t) = G\eta'^{\perp}(x, t),\ x^{\perp} = (-x_2, x_1). \tag{4.1}$$

On $\Gamma(t)$, we let $\nabla_{\mathcal{T}}$ denote the tangential derivative, that is, the derivative in the direction of the unit tangent vector $\mathcal{T}$. Let f denote any Eulerian quantity. Then, by the chain-rule,

$$(\nabla_{\mathcal{T}} f) \circ \eta = G(f \circ \eta)'. \tag{4.2}$$

Definition 4.1. ($W^{k,\infty}(\Gamma(t))$-norm). For a function $f(\cdot, t) : \Gamma(t) \to \mathbb{R}$ and integers $k \geqq 0$, we define

$$\|f(\cdot, t)\|_{W^{k,\infty}(\Gamma(t))} = \sum_{i=0}^{k} \|\nabla^i_{\mathcal{T}} f(\cdot, t)\|_{L^\infty(\Gamma(t))}.$$

Remark 3. From our assumed bounds (3.1) we have that $|\nabla_{\mathcal{T}} u^+| \leqq \mathcal{M}$. Since $\operatorname{div} u^+ = 0$, we have that $|\nabla_{\mathcal{N}} u^+ \cdot \mathcal{N}| = |\nabla_{\mathcal{T}} u^+ \cdot \mathcal{T}| \leqq \mathcal{M}$, and since $\operatorname{curl} u^+ = 0$, $|\nabla_{\mathcal{N}} u^+ \cdot \mathcal{T}| = |-\nabla_{\mathcal{T}} u^+ \cdot \mathcal{N}| \leqq \mathcal{M}$, which shows that

$$\|\nabla u^+\|_{L^\infty(\Gamma(t))} \leqq \mathcal{M} \tag{4.3}$$

(where the norm of a matrix is chosen to be the maximum of the absolute value of all four components).

Remark 4. We now define ϕ to be the flow map of u^+ in $\overline{\Omega^+}$. With the chart θ introduced above, and with $x = (x_1, 0)$, we then infer from $\phi_t(\theta(x), t) = u^+(\phi(\theta(x), t), t)$ that

$$[\phi_t(\theta(x), t)]' = \nabla u^+(\phi(\theta(x), t), t)\ [\phi(\theta(x), t)]'.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|[\phi(\theta(x), t)]'\right|^2 &= 2[\phi(\theta(x), t)]' \cdot \left(\nabla u^+(\phi(\theta(x), t), t)\ [\phi(\theta(x), t)]'\right)\\ &\geqq -4\mathcal{M}|[\phi(\theta(x), t)]'|^2,\end{aligned}$$

where the inequality follows from (4.3). Thus

$$|[\phi(\theta(x), t)]'|^2 \geqq \mathrm{e}^{-4\mathcal{M}t}|\theta'(x)|^2 \geqq \mathrm{e}^{-4\mathcal{M}t} C^2 > 0. \tag{4.4}$$

Therefore, the unit tangent vector to $\Gamma(t)$ can be defined simply as

$$\mathcal{T}(\phi(\theta(x), t)) = \frac{[\phi(\theta(x), t)]'}{|[\phi(\theta(x), t)]'|},$$

or with our notational convention of writing $\phi \circ \theta$ simply as ϕ,

$$\mathcal{T}(\phi) = \frac{\phi'}{|\phi'|}.$$

Remark 5. Using the same argument as in Remark 4, if $\|\nabla u^-(\cdot, t)\|_{L^\infty(\Omega(t))}$ is bounded from above (which is the case for $t < T$ for a solution $u^- \in L^\infty(0, T; H^3(\Omega^-(t)))$ so long as there is no self-intersection of $\Gamma(t)$), then the flow map η of u^- satisfies an identity similar to (4.4), ensuring that the definition of $G(x, t)$ is well-defined for all $t \in [0, T)$.

4.2. Evolution Equation for the Vorticity on Γ

Equation (2.1a) is $v_t^+ + \nabla v^+ \, A \, (v^+ - \Psi_t) + A^T \nabla q^+ = -g e_2$. By definition, on Γ, $\Psi_t = v^-$, so that $v^+ - \Psi_t = \delta v$. Since $\delta v \cdot n = 0$ on Γ, we see that $\delta v = (\delta v \cdot \tau)\tau$. Hence, the advection term can be written (using the Einstein summation convention) as $\frac{\partial v^+}{\partial x_r} A^r_j \tau_j (\delta v \cdot \tau)$. From (4.1), $\tau_j = G \eta'_j$ which in our local coordinate system is the same as $G \frac{\partial \eta_j}{\partial x_1}$. Since $A = [\nabla \eta]^{-1}$, we see that $A^r_j \frac{\partial \eta_j}{\partial x_1} = \delta^r_1$, where δ^r_1 denotes the Kronecker delta.

It follows that on Γ, (2.1a) takes the form

$$v_t^+ + G v^{+'} \delta v \cdot \tau + A^T \nabla q^+ = -g e_2. \tag{4.5}$$

Equation (2.1b) does not have the advection term, and remains the same on Γ. Subtracting (2.1b) from (4.5a), taking the scalar product of this difference with τ, and using that $\delta q = \mathcal{H}$, yields

$$\delta v_t \cdot \tau + G v^{+'} \cdot \tau (\delta v \cdot \tau) + G \mathcal{H}' = 0,$$

from which it follows that

$$(\delta v \cdot \tau)_t + G v^{+'} \cdot \tau (\delta v \cdot \tau) + G \mathcal{H}' = 0 \quad \text{on } \Gamma \times [0, T), \tag{4.6}$$

where we have used the fact that $\tau_t = G(v' \cdot n) n$ and $\delta v \cdot n = 0$. Using (4.2), we write (4.6) as

$$(\delta v \cdot \tau)_t + [\nabla_{\mathcal{T}} u^+ \cdot \mathcal{T} \circ \eta](\delta v \cdot \tau) + \nabla_{\mathcal{T}} H \circ \eta = 0 \quad \text{on } \Gamma \times [0, T). \tag{4.7}$$

4.3. Evolution Equation for Derivative of Vorticity $\nabla_{\mathcal{T}} \delta u \cdot \mathcal{T}$

On Γ, we denote the tangential derivative by $\nabla_{\mathcal{T}}$. The chain-rule (4.2) shows that the tangential derivative of vorticity along particle trajectories can be written as

$$[\nabla_{\mathcal{T}}\delta u \cdot \mathcal{T}] \circ \eta = G\delta v' \cdot \tau. \tag{4.8}$$

Our analysis will rely on the evolution equation for $G\delta v' \cdot \tau$. By differentiating (4.7), we find that

$$(\delta v' \cdot \tau)_t + [Gv^{+'} \cdot \tau](\delta v' \cdot \tau) + (\delta v \cdot \tau)[Gv^{+'} \cdot \tau]' + (G\mathcal{H}')' = 0. \tag{4.9}$$

Defining our "forcing function" $\mathcal{A}$ to be

$$\begin{aligned}\mathcal{A} &= (\delta v \cdot \tau)G[Gv^{+'} \cdot \tau]' + G(G\mathcal{H}')' \\ &= (\delta v \cdot \tau)\nabla_{\mathcal{T}}(\nabla_{\mathcal{T}}u^+ \cdot \mathcal{T}) \circ \eta + \nabla_{\mathcal{T}}(\nabla_{\mathcal{T}}H) \circ \eta,\end{aligned} \tag{4.10}$$

we see that Equation (4.9) is simply

$$(\delta v' \cdot \tau)_t + Gv^{+'} \cdot \tau(\delta v' \cdot \tau) + G^{-1}\mathcal{A} = 0. \tag{4.11}$$

Multiplying (4.11) by G and commuting G with the time-derivative shows that

$$(G\delta v' \cdot \tau)_t + G(v^{-'} \cdot \tau + v^{+'} \cdot \tau)(G\delta v' \cdot \tau) + \mathcal{A} = 0.$$

Writing $v^{-'} \cdot \tau = -\delta v' \cdot \tau + v^{+'} \cdot \tau$, we arrive at the desired evolution equation

$$(G\delta v' \cdot \tau)_t - (G\delta v' \cdot \tau)^2 + 2Gv^{+'} \cdot \tau(G\delta v' \cdot \tau) + \mathcal{A} = 0. \tag{4.12}$$

Notice that the coefficient $2Gv^{+'} \cdot \tau = 2\nabla_{\mathcal{T}}u^+ \cdot \mathcal{T} \circ \eta$, as well as the forcing function $\mathcal{A}$, are both bounded as a consequence of our assumed bounds (3.1) on u^+ and the parameterization of $z(\cdot, t)$ of $\Gamma(t)$.

Remark 6. In [22], FEFFERMAN et al. use the notation $z(\alpha, t)$ to denote a smooth parameterization of $\Gamma(t)$. In our analysis, we will make use of the Lagrangian parameterization $\eta(x, t)$ of $\Gamma(t)$ for points x in the reference curve Γ. Our notation η' corresponds to $\partial_\alpha z$ in [22]. Furthermore, our $\delta v \cdot \tau$ is the same as $\frac{\omega}{|\partial_\alpha z|}$ in [22]. The tangential derivative of vorticity $[\nabla_{\mathcal{T}}\delta u \cdot \mathcal{T}] \circ \eta$ corresponds to $\partial_\alpha\left(\frac{\omega}{|\partial_\alpha z|}\right)/|\partial_\alpha z|$ in [22].

5. Bounds for ∇u^- and the Rate of Blow-up

Lemma 5.1. *Assuming* (3.1),

$$\sup_{t\in[0,T]} \|v^-(\cdot, t)\|_{W^{1,\infty}(\Gamma)} \lesssim \mathcal{M}. \tag{5.1}$$

Proof. With $\tau_0 = \tau(x, 0)$, solving (4.7) using an integrating factor, we find that

$$\begin{aligned}\delta v \cdot \tau = \delta u_0 \cdot \tau_0 \ \exp\left(-\int_0^t Gv^{+'} \cdot \tau\right) - \exp\left(-\int_0^t Gv^{+'} \cdot \tau\right) \\ \int_0^t \nabla_{\mathcal{T}}H \circ \eta \ \exp\left(\int_0^s Gv^{+'} \cdot \tau\right) \mathrm{d}s.\end{aligned} \tag{5.2}$$

We set $\mathcal{I}(t) = \exp\left(\int_0^t \|Gv^{+'} \cdot \tau\|_{L^\infty(\Gamma)}\right)$. Since $Gv^{+'} \cdot \tau = [\nabla_{\mathcal{T}} u^+ \cdot \mathcal{T}] \circ \eta$, by (3.1), $\mathcal{I}(t)$ is bounded. It follows from (5.2) that

$$\|\delta v \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)} \leqq \mathcal{I}(t)\|\delta u_0\|_{L^\infty(\Gamma)} + \mathcal{I}(t) \int_0^t \|\nabla_{\mathcal{T}} H \circ \eta\|_{L^\infty(\Gamma)}.$$

Again from (3.1), the tangential derivative of the mean curvature $\nabla_{\mathcal{T}} H \in W^{1,\infty}(\Gamma)$ so we see that $\|\delta v \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)}$ is bounded.

Next, as $\delta v \cdot n = 0$, and $v^+ \cdot n$ is bounded according to (3.1), we find that $\|v^-(\cdot, t)\|_{L^\infty(\Gamma)} \lesssim \mathcal{M}$ for all $t \in [0, T]$. Then, from (4.11),

$$\begin{aligned}\delta v' \cdot \tau = \delta u_0' \cdot \tau_0 \ \exp\left(-\int_0^t Gv^{+'} \cdot \tau\right) - \exp\left(-\int_0^t Gv^{+'} \cdot \tau\right) \\ \int_0^t G^{-1}\mathcal{A} \exp\left(\int_0^s Gv^{+'} \cdot \tau\right) \mathrm{d}s,\end{aligned}$$

so that with $G^{-1} = |\eta'|$,

$$\|\delta v' \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)} \leqq \mathcal{I}(t)\|\delta u_0' \cdot \tau_0\|_{L^\infty(\Gamma)} + \mathcal{I}(t) \int_0^t \||\eta'(\cdot, s)|\mathcal{A}(\cdot, s)\|_{L^\infty(\Gamma)} \mathrm{d}s. \tag{5.3}$$

From the fundamental theorem of calculus,

$$\begin{aligned}|\eta'(\cdot, s)| &\leqq |\eta'(\cdot, 0)| + \int_0^s |v^{-'}(\cdot, r)|\mathrm{d}r \\ &\leqq |\eta'(\cdot, 0)| + \int_0^s |v^{+'}(\cdot, r)|\mathrm{d}r + \int_0^s |\delta v'(\cdot, r)|\mathrm{d}r \\ &\leqq \mathcal{M} + \int_0^s |\delta v'(\cdot, r) \cdot n(\cdot, r)|\mathrm{d}r + \int_0^s |\delta v'(\cdot, r) \cdot \tau(\cdot, r)|\mathrm{d}r,\end{aligned}$$

where we have used our assumed bounds (3.1) for the last inequality. Next, since $\delta v \cdot n = 0$ on Γ, we see that $\delta v' \cdot n = -\delta v \cdot n'$; as $n' = (H \circ \eta)\tau$, and as $\|H \circ \eta(\cdot, t)\|_{L^\infty(\Gamma)}$ and $\|\delta v \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)}$ are bounded, we see from (5.3) that

$$\|\delta v' \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)} \lesssim \mathcal{M} + T\mathcal{M} \int_0^t \|\delta v' \cdot \tau(\cdot, s)\|_{L^\infty(\Gamma)} \mathrm{d}s.$$

By taking the convention that $\lesssim$ incorporates T (which we view in this paper as a given constant, namely the eventual finite-time of self-intersection), this shows that

$$\|\delta v' \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)} \lesssim \mathcal{M} + \mathcal{M} \int_0^t \|\delta v' \cdot \tau(\cdot, s)\|_{L^\infty(\Gamma)} \mathrm{d}s.$$

Hence, by Gronwall's inequality, $\sup_{t\in[0,T]} \|\delta v' \cdot \tau(\cdot, t)\|_{L^\infty(\Gamma)}$ is bounded. We have already shown that $\sup_{t\in[0,T]} \|\delta v' \cdot n(\cdot, t)\|_{L^\infty(\Gamma)}$ is bounded; thus, $\sup_{t\in[0,T]} \|\delta v'\|_{L^\infty(\Gamma)}$ is bounded, from which we may conclude that $\|v^{-'}(\cdot, t)\|_{L^\infty(\Gamma)} \lesssim \mathcal{M}$ for all $t \in [0, T]$. □

Remark 7. Note that u^- is Lipschitz continuous, uniformly on any time interval $[0, t]$ with $t < T$. This, in turn, allows us to define the Lagrangian flow map η in a classical sense for any time interval $[0, t]$ for $t < T$. We then extend this definition of η to the time interval $[0, T]$ by $\eta(x, T) = x + \int_0^T v^-(x, s)ds$ by the bounds in Lemma 5.1.

Lemma 5.2. *Assuming* (3.1),

$$\sup_{t\in[0,T]} \|\nabla u^-(\cdot, t)\|_{L^\infty(\eta(\Omega^-,t))} \lesssim \frac{\mathcal{M}}{\min_\Gamma |\eta'(\cdot, t)|}. \tag{5.4}$$

Proof. From (4.8) and Lemma 5.1, $\|[\nabla_{\mathcal{T}}\delta u \cdot \mathcal{T}] \circ \eta\|_{L^\infty(\Gamma)} \lesssim \mathcal{M}/\min_\Gamma |\eta'(\cdot, t)|$. Then, we see that $\max_{y\in\eta(\Gamma,t)} |\nabla_{\mathcal{T}}\delta u \cdot \mathcal{T}| \lesssim \mathcal{M}/\min_\Gamma |\eta'(\cdot, t)|$. Hence, with our assumed bounds (3.1),

$$\max_{y\in\eta(\Gamma,t)} \left|\nabla_{\mathcal{T}} u^- \cdot \mathcal{T}\right| \lesssim \frac{\mathcal{M}}{\min_\Gamma |\eta'(\cdot, t)|}. \tag{5.5}$$

Next, as $\delta u \cdot \mathcal{N} = 0$ (where recall that $\delta u = u^+ - u^-$ on $\Gamma(t)$), we have the identity $0 = \nabla_{\mathcal{T}}(\delta u \cdot \mathcal{N}) = (\nabla_{\mathcal{T}}\delta u) \cdot \mathcal{N} + \delta u \cdot \nabla_{\mathcal{T}}\mathcal{N}$; hence, we see that

$$\nabla_{\mathcal{T}} u^- \cdot \mathcal{N} = \nabla_{\mathcal{T}} u^+ \cdot \mathcal{N} + \delta u \cdot \nabla_{\mathcal{T}}\mathcal{N}.$$

Lemma 5.1 provides us with $L^\infty(\Gamma)$ control of u^-; hence, with (3.1), it follows that

$$\max_{y\in\eta(\Gamma,t)} \left|[\nabla_{\mathcal{T}} u^- \cdot \mathcal{N}](y)\right| \lesssim \mathcal{M}. \tag{5.6}$$

The inequalities (5.5) and (5.6) together with the fact that $\operatorname{div} u^- = \operatorname{curl} u^- = 0$ in $\eta(\Omega^-, t)$ implies that for any $t < T$,

$$\|\nabla u^-(\cdot, t)\|_{L^\infty(\eta(\Gamma,t))} \lesssim \frac{\mathcal{M}}{\min_\Gamma |\eta'(\cdot, t)|}. \tag{5.7}$$

As $\Delta\nabla u^- = 0$ in $\eta(\Omega^-, t)$, the maximum and minimum principle applied to each component of ∇u^-, together with (5.7), provide the inequality (5.4). □

Remark 8. As a consequence of Lemma 5.2, we see that $\sup_{y\in\Gamma(t)} \|\nabla u^-(y, t)\|_{L^\infty(\eta(\Omega^-,t))} \to \infty$ as $t \to T$ if and only if $\lim_{t\to T} |\eta'(x, t)| \to 0$ for some $x \in \Gamma$. If we assume that there are distinct points $x_0, x_1 \in \Gamma$ which come into contact, such that $\eta(x_0, T) = \eta(x_1, T)$ and that such an intersection point is unique at time $t = T$, then $|\nabla u^-(\cdot, t)|$ can only blow-up at the contact point $\eta(x_0, T)$.

The explanation is as follows: since u^- is harmonic, by using a smooth cut-off function φ whose support does not intersect $\eta(x_0, T)$, and proceeding as in the proof of (5.40) (just after (5.30)), elliptic estimates show that $|\nabla u^-(x, t)|$ must be bounded for $x \in \operatorname{spt}(\varphi)$, namely away from x_0.

Next, suppose that $|\nabla u^-(\eta(x_0, t), t)|$ remains bounded as $t \to T$; then, by employing a similar argument as we used to establish (4.4) (considering now the flow η of u^-), we obtain that $|\eta'(x_0, t)| \geqq \lambda > 0$ as $t \to T$ for some constant λ. By continuity of η', this means that $|\eta'(x, t)| > 0$ in a small neighborhood of $\eta(x_0, t)$ which means that, by Lemma 5.2, $|\nabla u^-(x, t)|$ cannot blow-up as $t \to T$ for x close to x_0.

Theorem 5.1. *With the assumed bounds* (3.1)*, if there is a sequence* $t_n \to T$ *such that*

$$\max_{x\in\Gamma} |[\nabla_{\mathcal{T}}\delta u \cdot \mathcal{T}](\eta(x,t_n),t_n)| \to \infty, \tag{5.8}$$

then for $0<\varepsilon \ll 1$*, there exists* $t_0(\varepsilon)$ *such that* $T - t_0(\varepsilon) < \varepsilon$ *and*

$$\max_{y\in\eta(\overline{\Omega^-},t)} |\nabla u^-(y,t)| \leqq \frac{1+\varepsilon}{T-t} \quad \forall t \in [t_0(\varepsilon),T). \tag{5.9}$$

Furthermore, if there exists a unique point of $\Gamma(T)$ *such that there are two distinct points* $x_0, x_1 \in \Gamma$ *with* $\eta(x_0,T)=\eta(x_1,T)$ *with tangent vector to* $\Gamma(T)$ *at* $\eta(x_0,T)$ *given by* e_1*, then*

$$\max_{y\in\eta(\overline{\Omega^-},t)} \left|\frac{\partial u_2^-}{\partial x_1}(y,t)\right| \leqq \frac{\varepsilon}{T-t} \quad \forall t \in [t_0(\varepsilon),T). \tag{5.10}$$

Remark 9. We note that $0<\varepsilon\ll 1$ is a *fixed* positive constant which only depends on the initial data and the bound $\mathcal{M}$ in (3.1). Note also that $t_0(\varepsilon)$ depends on ε, and will be chosen closer and closer to T in the course of the proof, and is eventually fixed as a function of ε.

Proof. *Step 1. Blow-up rate for the derivative of vorticity* $[\nabla_{\mathcal{T}}\delta u\cdot\mathcal{T}](\eta(x_0,t),t)$ *as* $t\to T$. We first suppose that for some $x_0\in\Gamma$, $|[\nabla_{\mathcal{T}}\delta u\cdot\mathcal{T}](\eta(x_0,t_n),t_n)|\to\infty$, and establish that $[\nabla_{\mathcal{T}}\delta u\cdot\mathcal{T}](\eta(x_0,t),t)$ (which, recall, equals $G\delta v'\cdot\tau(x_0,t)$) has a precise blow-up rate.

We set

$$\mathfrak{X}(x_0,t) = G\delta v'\cdot\tau(x_0,t),$$

and define the coefficient function

$$\mathfrak{A}(x_0,t) = 2Gv^{+'}\cdot\tau(x_0,t).$$

Then, (4.12) reads

$$\mathfrak{X}_t(x_0,t) - \mathfrak{X}^2(x_0,t) + \mathfrak{A}(x_0,t)\,\mathfrak{X}(x_0,t) = -\mathcal{A}(x_0,t), \tag{5.11}$$

where $\mathcal{A}(x,t)$ is defined in (4.10). This equation can be written as

$$\begin{aligned}&\left[\exp\int_0^t \mathfrak{A}(x_0,s)\mathrm{d}s\ \mathfrak{X}(x_0,t)\right]_t - \exp\int_0^t \mathfrak{A}(x_0,s)\mathrm{d}s\ \mathfrak{X}^2(x_0,t)\\ &\quad = -\exp\int_0^t \mathfrak{A}(x_0,s)\mathrm{d}s\ \mathcal{A}(x_0,t)\end{aligned}$$

so that

$$\begin{aligned}\int_0^t \exp\left(\int_0^s \mathfrak{A}(x_0,r)\mathrm{d}r\right)\mathfrak{X}^2(x_0,s)\mathrm{d}s &= \exp\left(\int_0^t \mathfrak{A}(x_0,s)\mathrm{d}s\right)\mathfrak{X}(x_0,t) - \mathfrak{X}(x_0,0)\\ &\quad + \int_0^t \exp\left(\int_0^s \mathfrak{A}(x_0,r)\mathrm{d}r\right)\mathcal{A}(x_0,s)\mathrm{d}s.\end{aligned} \tag{5.12}$$

Thanks to (3.1), $\mathfrak{A}(x_0, t)$ has a minimum and maximum on $[0, T]$. Hence, there are positive constants c_1, c_2, c_3 such that for any $t \in [0, T)$,

$$c_1 \int_0^t \mathcal{X}^2(x_0, s)\mathrm{d}s - c_3 \leqq \mathcal{X}(x_0, t) \leqq c_2 \int_0^t \mathcal{X}^2(x_0, s)\mathrm{d}s + c_3,$$

and by (5.8), the limit as $t \to T$ is well-defined and

$$\lim_{t\to T} \mathcal{X}(x_0, t) = \infty. \tag{5.13}$$

For $t > \bar{t}_0$ sufficiently close to T, we can then divide (5.11) by $\mathcal{X}^2$, and integrate from $\bar{t}_0$ to t, to find that

$$-\frac{1}{\mathcal{X}(x_0, t)} + \frac{1}{\mathcal{X}(x_0, \bar{t}_0)} - t + \bar{t}_0 + \int_{t_0}^t \left(\frac{\mathfrak{A}(x_0, s)}{\mathcal{X}(x_0, s)} + \frac{\mathcal{A}(x_0, s)}{\mathcal{X}^2(x_0, s)}\right) \mathrm{d}s = 0.$$

Using the limit in (5.13),

$$\frac{1}{\mathcal{X}(x_0, \bar{t}_0)} - T + \bar{t}_0 + \int_{t_0}^T \left(\frac{\mathfrak{A}(x_0, s)}{\mathcal{X}(x_0, s)} + \frac{\mathcal{A}(x_0, s)}{\mathcal{X}^2(x_0, s)}\right) \mathrm{d}s = 0, \tag{5.14}$$

from which we obtain the following identity: for $t \in [t_0, T)$,

$$\mathcal{X}(x_0, t) = \left[T - t - \int_t^T \left(\frac{\mathfrak{A}(x_0, s)}{\mathcal{X}(x_0, s)} + \frac{\mathcal{A}(x_0, s)}{\mathcal{X}^2(x_0, s)}\right) \mathrm{d}s\right]^{-1}, \tag{5.15}$$

since we can replace t_0 with t in (5.14).

From (5.13), this formula implies that the integrand is small as t is close to T, and then provides the rate of blow-up:

$$\lim_{t\to T} \mathcal{X}(x_0, t)(T - t) = 1.$$

Using (3.1), we see that

$$\lim_{t\to T} [\nabla_{\mathcal{T}} u^- \cdot \mathcal{T}](\eta(x_0, t), t)\ (T - t) = -1. \tag{5.16}$$

Step 2. Maximum of vorticity derivative blows-up on $\Gamma(t)$. Having established the blow-up rate for $[\nabla_{\mathcal{T}} \delta u \cdot \mathcal{T}](\eta(x_0, t), t)$, we shall next prove that for any $t \in [0, T)$, the quantity $\max_{x\in\Gamma} [\nabla_{\mathcal{T}} \delta u \cdot \mathcal{T}](\eta(x, t), t)$ (which equals $\max_{x\in\Gamma} G\delta v' \cdot \tau(x, t)$) has the same blow-up rate. For each $x \in \Gamma$ and $t \in [0, T)$, we set

$$\mathfrak{A}(x, t) = 2Gv^{+'} \cdot \tau(x, t) \quad \text{and} \quad \mathcal{X}(x, t) = G\delta v' \cdot \tau(x, t). \tag{5.17}$$

Following (5.12), we see that

$$\begin{aligned} \mathcal{X}(x, t) \geqq \exp\left(-\int_0^t \mathfrak{A}(x, s)\mathrm{d}s\right) \mathcal{X}(x, 0) - \exp\left(-\int_0^t \mathfrak{A}(x, s)\mathrm{d}s\right) \\ \times \int_0^t \exp\left(\int_0^s \mathfrak{A}(x, r)\mathrm{d}r\right) \mathcal{A}(x, s)\mathrm{d}s\,; \end{aligned} \tag{5.18}$$

hence, there exists a positive constant c_4 such that $\mathcal{X}(x,t) > -c_4$. Since $\mathcal{X}_t = \mathcal{X}^2 - \mathfrak{A}\,\mathcal{X} - \mathcal{A}$, there is a positive constant c_5,

$$\mathcal{X}_t > \mathcal{X}^2/2 - c_5.$$

It follows that if $\mathcal{X}(x,t_0) \geqq \sqrt{2c_5}$, then $\mathcal{X}(x,\cdot)$ is increasing on $[t_0,T)$. For $x \in \Gamma$ we choose $t_0(\varepsilon) < T$ sufficiently close to T so that for $0 < \varepsilon \ll 1$ fixed,

$$\mathcal{X}(x,t_0(\varepsilon)) > \sqrt{2c_5} + 1 + \frac{8c_6}{\varepsilon}, \quad c_6 = \sup_{(t,x)\in[0,T]\times\Gamma} \left(|\mathfrak{A}(x,t)| + \mathcal{A}(x,t)|\right), \tag{5.19}$$

with c_6 denoting a bounded constant thanks to (3.1). Since $\mathcal{X}(x,\cdot)$ is increasing for such an x, for $t \in [t_0(\varepsilon),T)$, the limit of $\mathcal{X}(x,t)$ as $t \to T$ is well-defined in the interval $(1 + \sqrt{2c_5} + 8c_6/\varepsilon, \infty]$, and thus so is the limit of $\frac{1}{\mathcal{X}(x,t)}$. Analogous to (5.15), we obtain that

$$\mathcal{X}(x,t) = \left[\frac{1}{\lim_{t\to T}\mathcal{X}(x,t)} + T - t + \int_T^t \left(\frac{\mathfrak{A}(x,s)}{\mathcal{X}(x,s)} + \frac{\mathcal{A}(x,s)}{\mathcal{X}^2(x,s)}\right) \mathrm{d}s\right]^{-1}.$$

From (5.19), we then have that for all $t \in [t_0(\varepsilon),T)$,

$$\mathcal{X}(x,t) \leqq \left[\frac{1}{\lim_{t\to T}\mathcal{X}(x,t)} + (T-t)(1-\varepsilon)\right]^{-1}$$

and since $\lim_{t\to T}\mathcal{X}(x,t) \geqq 0$, then for all $t < T$,

$$\mathcal{X}(x,t) \leqq \frac{1}{(T-t)(1-\varepsilon)}. \tag{5.20}$$

Step 3. Blow-up rate for ∇u^- in $\overline{\Omega^-(t)}$ as $t \to T$. From (5.20), for any $t \in [t_0(\varepsilon),T)$,

$$\max_{y\in\eta(\Gamma,t)} |[\nabla_{\mathcal{T}}\delta u \cdot \mathcal{T}](y,t)| \leqq \frac{1+2\varepsilon}{(T-t)}. \tag{5.21}$$

The inequalities (5.6) and (5.21), together with the fact that $\operatorname{div} u^- = \operatorname{curl} u^- = 0$ in $\eta(\Omega^-,t)$, show that

$$\max_{y\in\eta(\Gamma,t)} |\nabla u^-(y,t)| \leqq \frac{1+2\varepsilon}{T-t}, \tag{5.22}$$

where $\max_{y\in\eta(\Gamma,t)} |\nabla u^-(y,t)|$ denotes the maximum over all of the components of the matrix ∇u^-. Now, for any fixed $t \in [0,T)$, since each component of ∇u^- is harmonic in the domain $\eta(\Omega^-,t)$, the maximum and minimum principles together with the boundary estimate (5.22) shows that (5.9) holds.

Step 4. Asymptotic estimates for the components of ∇u^- as $t \to T$ in an ε-neighborhood of the splash. Since

$$\frac{\partial u^-}{\partial x_1} := \nabla_{\mathrm{e}_1} u^- = (\mathcal{T}\cdot \mathrm{e}_1)\nabla_{\mathcal{T}} u^- + (\mathcal{N}\cdot \mathrm{e}_1)\nabla_{\mathcal{N}} u^-,$$

we have that

$$\begin{aligned}\frac{\partial u_2^-}{\partial x_1} &= (\mathcal{T}\cdot \mathrm{e}_1)\nabla_{\mathcal{T}} u^- \cdot (\mathcal{T}\cdot \mathrm{e}_2\, \mathcal{T} + \mathcal{N}\cdot \mathrm{e}_2\, \mathcal{N}) \\ &\quad + (\mathcal{N}\cdot \mathrm{e}_1)\nabla_{\mathcal{N}} u^- \cdot (\mathcal{T}\cdot \mathrm{e}_2\, \mathcal{T} + \mathcal{N}\cdot \mathrm{e}_2\, \mathcal{N}) \\ &= (\mathcal{T}\cdot \mathrm{e}_1)(\mathcal{T}\cdot \mathrm{e}_2)\nabla_{\mathcal{T}} u^- \cdot \mathcal{T} + (\mathcal{T}\cdot \mathrm{e}_1)(\mathcal{N}\cdot \mathrm{e}_2)\nabla_{\mathcal{T}} u^- \cdot \mathcal{N} \\ &\quad + (\mathcal{T}\cdot \mathrm{e}_2)(\mathcal{N}\cdot \mathrm{e}_1)\nabla_{\mathcal{N}} u^- \cdot \mathcal{T} + (\mathcal{N}\cdot \mathrm{e}_1)(\mathcal{N}\cdot \mathrm{e}_2)\nabla_{\mathcal{N}} u^- \cdot \mathcal{N}. \end{aligned} \tag{5.23}$$

By rotating our coordinate system, if necessary, we suppose that the tangent and normal directions to $\Gamma(T)$ at $\eta(x_0, T)$ are given by the standard basis vectors $\mathrm{e}_1 = (1, 0)$ and $\mathrm{e}_2 = (0, 1)$, respectively (which we refer to as the horizontal and vertical directions, respectively).

Next, choose a point $\eta(x, t) \in \Gamma(t)$ in a small neighborhood of $\eta(x_0, t)$, and let the curve $\mathcal{S}(t)$ denote that portion of $\Gamma(t)$ that connects $\eta(x_0, t)$ to $\eta(x, t)$. Let $\vec{l}(t) : [0, 1] \to \mathcal{S}(t)$ denote a unit-speed parameterization such that $\vec{l}(t)(1) = \eta(x, t)$ and $\vec{l}(t)(0) = \eta(x_0, t)$. Then,

$$\begin{aligned}\mathcal{N}(\eta(x,t),t)\cdot \mathrm{e}_1 - \mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1 &= \int_{\mathcal{S}(t)} \nabla(\mathcal{N}\cdot \mathrm{e}_1)\cdot \mathrm{d}\vec{l} \\ \mathcal{T}(\eta(x,t),t)\cdot \mathrm{e}_2 - \mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2 &= \int_{\mathcal{S}(t)} \nabla(\mathcal{T}\cdot \mathrm{e}_2)\cdot \mathrm{d}\vec{l}. \end{aligned} \tag{5.24}$$

From our assumed bounds (3.1), there is a constant $c_7 > 0$ such that for $t \leqq T$

$$\begin{aligned}&|\mathcal{N}(\eta(x,t),t)\cdot \mathrm{e}_1 - \mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1| + |\mathcal{T}(\eta(x,t),t)\cdot \mathrm{e}_2 - \mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2| \\ &\quad \leqq c_7|\eta(x,t) - \eta(x_0,t)|. \end{aligned} \tag{5.25}$$

Next, with $G = |\eta'|^{-1}$, we compute that

$$\begin{aligned}\tau_t &= (G v^{-'}\cdot n)\, n = -(G\delta v'\cdot n)\, n + (G v^{+'}\cdot n)\, n \\ &= (G\, n'\cdot \delta v)\, n + (G v^{+'}\cdot n)\, n = \big[(\nabla_{\mathcal{T}}\mathcal{N}\cdot \delta u)\, \mathcal{N} + (\nabla_{\mathcal{T}} u^+\cdot \mathcal{N})\, \mathcal{N}\big]\circ \eta, \end{aligned}$$

where we have used (4.2) in the last equality. There is a similar formula for $n_t = -(G v^{-'}\cdot n)\, \tau$. It follows from Lemma 5.1 and our assumed bounds (3.1) that

$$\sup_{t\in[0,T]} \big(\|\tau_t(\cdot,t)\|_{L^\infty(\Gamma)} + \|n_t(\cdot,t)\|_{L^\infty(\Gamma)}\big) \lesssim \mathcal{M}. \tag{5.26}$$

Then, using the fundamental theorem of calculus, we see that

$$\begin{aligned}\mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1 &= \mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1 - \mathcal{N}(\eta(x_0,T),T)\cdot \mathrm{e}_1 = \int_T^t \partial_t n(x_0,s)\cdot \mathrm{e}_1 \mathrm{d}s \\ \mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2 &= \mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2 - \mathcal{T}(\eta(x_0,T),T)\cdot \mathrm{e}_2 = \int_T^t \partial_t \tau(x_0,s)\cdot \mathrm{e}_2 \mathrm{d}s, \end{aligned}$$

so that (by readjusting the constant c_7 if necessary), we have that

$$|\mathcal{N}(\eta(x_0,t),t)\cdot \mathrm{e}_1| + |\mathcal{T}(\eta(x_0,t),t)\cdot \mathrm{e}_2| \leqq c_7(T - t). \tag{5.27}$$

Next, we choose $t_0(\varepsilon) \in [0, T)$ and a sufficiently small neighborhood $\gamma_0(\varepsilon) \subset \Gamma$ of x_0 s.t.

$$\left\{ \begin{array}{l} (T-t) < \min\left(\frac{\varepsilon}{100c_7(1+\mathcal{M})}, \varepsilon\right) \quad \text{and} \quad |\eta(x,t) - \eta(x_0,t)| < \frac{\varepsilon}{2c_7} \\ |\mathcal{N}(\eta(x,t),t)\cdot \mathrm{e}_1| + |\mathcal{T}(\eta(x,t),t)\cdot \mathrm{e}_2| < \varepsilon \end{array} \right\} \quad \forall\, x \in \gamma_0(\varepsilon),\ t \in [t_0(\varepsilon), T), \tag{5.28}$$

where the constant c_7 was defined in (5.25) Consequently, from (5.6), (5.22) and (5.23), we see that

$$\left| \frac{\partial u_2^-}{\partial x_1}(\eta(x,t),t) \right| \leqq \frac{3\varepsilon}{T-t} + |\nabla_{\mathcal{T}} u^- \cdot \mathcal{N}|(\eta(x,t),t) + 2\varepsilon |\nabla_{\mathcal{N}} u^- \cdot \mathcal{T}|(\eta(x,t),t),$$

which thanks to (5.6) and the fact that $\operatorname{curl} u^- = \nabla_{\mathcal{T}} u^- \cdot \mathcal{N} - \nabla_{\mathcal{N}} u^- \cdot \mathcal{T} = 0$, provides us with

$$\left| \frac{\partial u_2^-}{\partial x_1}(\eta(x,t),t) \right| \leqq \frac{3\varepsilon}{T-t} + c_8 \mathcal{M} \ \ \forall\, x \in \gamma_0(\varepsilon),\ t \in [t_0(\varepsilon), T),$$

for a constant $c_8 > 0$. Thus , by choosing $t_0(\varepsilon)$ closer to T if necessary, we have that

$$\left| \frac{\partial u_2^-}{\partial x_1}(\eta(x,t),t) \right| \leqq \frac{3\varepsilon}{T-t} \ \ \forall\, x \in \gamma_0(\varepsilon),\ t \in [t_0(\varepsilon), T). \tag{5.29}$$

In a similar fashion, we choose $t_0(\varepsilon) \in [0, T)$ and a sufficiently small neighborhood $\gamma_1(\varepsilon) \subset \Gamma$ of x_1 s.t.

$$\left\{ \begin{array}{l} (T-t) < \min\left(\frac{\varepsilon}{100c_7(1+\mathcal{M})}, \varepsilon\right) \quad \text{and} \quad |\eta(x,t) - \eta(x_1,t)| < \frac{\varepsilon}{2c_7} \\ |\mathcal{N}(\eta(x,t),t)\cdot \mathrm{e}_1| + |\mathcal{T}(\eta(x,t),t)\cdot \mathrm{e}_2| < \varepsilon \end{array} \right\} \quad \forall\, x \in \gamma_1(\varepsilon),\ t \in [t_0(\varepsilon), T) \tag{5.30}$$

and such that the inequality (5.28) holds. Now, we choose $x \in \Gamma$ but in the complement of $\gamma_0(\varepsilon) \cup \gamma_1(\varepsilon)$. For such an x, we have that $|\nabla u^-(\eta(x,t),t)| \leqq \mathcal{M}_\varepsilon < \infty$. This bound is obtained as follows.

For each $t \in [t_0(\varepsilon), T)$, we let $\mathcal{B}_{\varepsilon,t} \subset \mathbb{R}^2$ denote a small closed ball containing $\eta(\gamma_0(\varepsilon),t) \cup \eta(\gamma_1(\varepsilon),t)$. The ball $\mathcal{B}_{\varepsilon,t}$ can be taken with a fixed radius independent of $t \in [t_0(\varepsilon), T)$ (for $T - t_0(\varepsilon)$ sufficiently small), with a center which is simply translated as t varies. This is possible as we assume at that there is a single point of self-intersection for the curve $\Gamma(T)$, and so the width of the domain $\Omega^-(t)$ cannot shrink to zero in other locations as $t \to T$. With the unit tangent vector field $\mathcal{T}$ defined on $\Gamma(t)$, we define a smooth extension of $\mathcal{T}$ to the set $\Omega^-(t) \cap \mathcal{B}^c_{\varepsilon,t}$, which is possible since the interface $\Gamma(t) \cap \mathcal{B}^c_{\varepsilon,t}$ remains $W^{4,\infty}$ for all $t \in [0, T]$; we continue to denote this extension by $\mathcal{T}$, and we note that the extension of $\mathcal{T}$ does not necessarily have modulus 1. Since $\Gamma(t) \cap \mathcal{B}^c_{\varepsilon,t}$ does not self-intersect for all $t \in [0, T]$ by the hypothesis (1) of Theorem 3.1, there exists a minimum positive radius $r_\varepsilon > 0$ such that for all $x \in \Gamma(t) \cap \mathcal{B}^c_{\varepsilon,t}$ and all $t \in [t_0(\varepsilon), T]$, there exists a translated open ball $\mathfrak{B}_{\varepsilon,t,x}(r_\varepsilon) \subset \Omega^-(t)$ of radius r_ε with $x \in \partial \mathfrak{B}_{\varepsilon,t,x}(r_\varepsilon)$. In other words, for each $x \in \Gamma(t)$ *away from the region of self-intersection*, there exists an open ball of smallest radius r_ε that is contained in the set $\Omega^-(t)$ and such that x is on the sphere of smallest radius.

We note that the radius $r_\varepsilon \to 0$ as $\varepsilon \to 0$; hence, on the domain $\Omega^-(t) \cap \mathcal{B}^c_{\varepsilon,t}$, we have an estimate of the type

$$\|\mathcal{T}\|_{H^3(\Omega^-(t)\cap\mathcal{B}^c_{\varepsilon,t})} \lesssim C(\mathcal{M}, \varepsilon), \tag{5.31}$$

where $C(\mathcal{M}, \varepsilon) > 0$ denotes a constant depending on $\mathcal{M}$ and ε (with $C(\mathcal{M}, \varepsilon) \to \infty$ as $\varepsilon \to 0$).

We now introduce the stream function ψ^- such that $u^- = \nabla^\perp\psi^-$; then,

$$\mathcal{T} \cdot \nabla\psi^- = u^- \cdot \mathcal{N} = u^+ \cdot \mathcal{N} \quad \text{on } \Gamma(t),$$

which then shows, using our bounds in (3.1), that

$$\|\psi^-\|_{H^3(\Gamma(t))} \lesssim C(\mathcal{M}, \varepsilon). \tag{5.32}$$

Furthermore, due to the conservation law

$$\frac{1}{2}\|u^-(t)\|^2_{L^2(\Omega^-(t))} + \text{ length of } \Gamma(t) = \frac{1}{2}\|u_0^-\|^2_{L^2(\Omega^-)} + \text{ length of } \Gamma, \tag{5.33}$$

we have that

$$\|\psi^-\|_{H^1(\Omega^-(t))} \lesssim C(\mathcal{M}, \varepsilon), \tag{5.34}$$

where we have used that $\|\psi^-\|_{H^1(\Omega^-(t))} \leqq C(\|\nabla\psi^-\|_{L^2(\Omega^-(t))} + \|\psi^-\|_{H^3(\Gamma(t))})$ and (5.32).

Next, we fix $t \in [t_0(\varepsilon), T]$, and choose a smooth cut-off function $0 \leqq \varphi(\cdot, t) \leqq 1$ whose support is contained in the complement of $\mathcal{B}_{\varepsilon,t}$. Since $\Gamma(t)\cap\mathcal{B}^c_{\varepsilon,t}$ is assumed to be of class $W^{4,\infty}$ for each $t \in [0, T]$, we consider the following elliptic problem:

$$\begin{aligned} \Delta(\varphi\psi^-) &= 2\nabla\varphi \cdot \nabla\psi^- + \Delta\varphi\ \psi^- && \text{in } \Omega^-_\varepsilon(t), \\ \varphi\psi^- &= \varphi\psi^- && \text{on } \partial\Omega^-_\varepsilon(t), \end{aligned}$$

where $\Omega^-_\varepsilon(t)$ is a smooth open subset of $\Omega^-(t)$ containing $\Omega^-(t)\cap\mathcal{B}^c_{\varepsilon,t}$, and where we have used the fact that ψ^- is harmonic, since $\operatorname{curl} u^- = 0$. From (5.34), (5.32), we have by elliptic regularity that

$$\|\varphi\psi^-\|_{H^2(\Omega^-_\varepsilon(t))} \lesssim C(\mathcal{M}, \varepsilon). \tag{5.36}$$

We next consider the elliptic problem:

$$\begin{aligned} \Delta(\varphi\mathcal{T} \cdot \nabla(\varphi\psi^-)) &= 2\nabla(\varphi\mathcal{T}_i) \cdot \nabla\frac{\partial(\varphi\psi^-)}{\partial x_i} + \Delta(\varphi\mathcal{T}_i)\,\frac{\partial(\varphi\psi^-)}{\partial x_i} \\ &\quad + \varphi\mathcal{T} \cdot \nabla\left[2\nabla\varphi \cdot \nabla\psi^- + \Delta\varphi\ \psi^-\right] && \text{in } \Omega^\varepsilon, \\ \varphi\mathcal{T} \cdot \nabla(\varphi\psi^-) &= \varphi\mathcal{T} \cdot \nabla(\varphi\psi^-) && \text{on } \partial\Omega\varepsilon. \end{aligned}$$

Due to (5.36), (5.32) and (5.31), we have by elliptic regularity:

$$\|\varphi\mathcal{T} \cdot \nabla(\varphi\psi)\|_{H^2(\Omega^\varepsilon)} \lesssim C(\mathcal{M}, \varepsilon). \tag{5.38}$$

In the same manner as we obtained (5.38) from (5.36), we can also obtain that

$$\|\varphi\mathcal{T}\cdot\nabla(\varphi\mathcal{T}\cdot\nabla(\varphi\psi))\|_{H^2(\Omega^\varepsilon)} \lesssim C(\mathcal{M},\varepsilon). \tag{5.39}$$

By the trace theorem and the Sobolev embedding theorem, we infer from (5.39) that

$$\|\varphi^3\nabla u^-\cdot\mathcal{N}(\cdot,t)\|_{L^\infty(\Gamma(t))} \lesssim C(\mathcal{M},\varepsilon).$$

Since u^- is divergence and curl free this immediately ensures by the algebraic expression of the divergence and curl that

$$\|\varphi^3\nabla u^-(\cdot,t)\|_{L^\infty(\Gamma(t))} \lesssim C(\mathcal{M},\varepsilon), \tag{5.40}$$

showing that $\nabla u^-\cdot\mathcal{N}(\eta(x,t),t)$ is bounded for $\eta(x,t)$ outside of $\mathcal{B}_{\varepsilon,t}$. Therefore, our previous estimates obtained for x in $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ ensure that for all $x\in\Gamma$, $|\nabla u^-(\eta(x,t),t)| < \frac{\varepsilon}{T-t}$ for t sufficiently close to T; thus, for $T-t_0(\varepsilon)$ sufficiently small (which means that once again, we have taken $t_0(\varepsilon)$ even closer to T if necessary),

$$\max_{y\in\eta(\Gamma,t)}\left|\frac{\partial u_2^-}{\partial x_1}(y,t)\right| \leqq \frac{3\varepsilon}{T-t} \quad \forall t\in[t_0(\varepsilon),T),$$

which, thanks to the maximum and minimum principles applied to the harmonic function $\frac{\partial u_2^-}{\partial x_1}$, provides us with (5.10). Since $0<\varepsilon\ll 1$, we replace 3ε by ε, and replace $1+2\varepsilon$ by $1+\varepsilon$. This completes the proof. □

Corollary 5.1. *With* (3.1) *and* (5.8) *holding, and for* $0<\varepsilon\ll 1$, *there exists* $t_0(\varepsilon)\in[T-\varepsilon,T)$ *such that*

$$\|\nabla_{\mathcal{T}}u^-\cdot\mathcal{T}(\cdot,t)\|_{L^\infty(\Gamma(t))} \leqq \frac{1+(1+2c_6)(T-t)}{T-t} \quad \forall t\in[t_0(\varepsilon),T), \tag{5.41}$$

where the constant c_6 *is defined in* (5.19).

Proof. Using the notation from the proof of Theorem 5.1,

$$\mathcal{X}(x,t) = \nabla_{\mathcal{T}}\delta u\cdot\mathcal{T}(\eta(x,t),t).$$

and we recall that $\mathcal{X}(x_0,t)=\chi(t)$ and that $\mathcal{X}(x,t)$ satisfies

$$\mathcal{X}_t(x,t) - \mathcal{X}^2(x,t) + \mathfrak{A}(x,t)\,\mathcal{X}(x,t) = -\mathcal{A}(x,t). \tag{5.42}$$

We let $\delta t = T-t$, and fix $0<\varepsilon\ll 1$. Since $\lim_{t\to T}\mathcal{X}(x_0,t)(T-t)=1$, for δt sufficiently small, we have that

$$(1-\varepsilon)\delta t^{-1} \leqq \mathcal{X}(x_0,t) \leqq (1+\varepsilon)\delta t^{-1}.$$

Substituting this inequality into (5.15), we see that

$$\mathcal{X}(x_0,t) \leqq \frac{1}{(1-c_6\delta t)\delta t} \leqq \frac{1+2c_6\delta t}{\delta t}. \tag{5.43}$$

If we replace x_0 with x_1, then (5.43) continues to hold.

Now, for the sake of contradiction, we will assume that there exists a sequence of points (x^*, t^*), with $x^* \in \Gamma$ and t^* converging to T, such that

$$\mathcal{X}(x^*, t^*) > \frac{1 + (1 + 2c_6)(T - t^*)}{T - t^*}. \tag{5.44}$$

We will later prove that set of possible contact points $x_i \in \Gamma$, such that $\eta(x_0, T) = \eta(x_1, T) = \eta(x_i, T)$, is finite. Then, from this set of all possible reference points which can self-intersect at time $t = T$, we relabel x_0 so that x_0 is the limit of a subsequence of points x^* converging toward it along Γ. Henceforth, we restrict the sequence of points (x^*, t^*) to the subsequence which converges to the point x_0.

By Remark 7, if there exists $C > 0$ such that $|\nabla u^-(\eta(x_0, t), t)| \leqq C$ for any $t \in [t_0(\varepsilon), T)$, we would also have the existence of a neighborhood of x_0 on Γ such that for any x in this neighborhood, $|\nabla u^-(\eta(x, t), t)| \leqq 2C$ for any $t \in [t_0(\varepsilon), T)$. This would then make (5.44) impossible. Therefore, we have $\mathcal{X}(x_0, T) \to \infty$ as $t \to T$.

We assume that this point is x_0 (for otherwise we can reverse the labels on the two points x_0 and x_1). Notice that since $x^* \to x_0$ as $t^* \to T$, then for $T - t^*$ sufficiently small,

$$|x^* - x_0| < \varepsilon. \tag{5.45}$$

We define

$$\mathcal{Y}(t) = \mathcal{X}(x^*, t) - \mathcal{X}(x_0, t) \quad \text{and} \quad \mathcal{Z}(t) = \mathcal{X}(x^*, t) + \mathcal{X}(x_0, t),$$
$$\delta\mathfrak{A}(t) = \mathfrak{A}(x^*, t) - \mathfrak{A}(x_0, t) \quad \text{and} \quad \delta\mathcal{A}(t) = \mathcal{A}(x^*, t) - \mathcal{A}(x_0, t).$$

Then, setting $\mathcal{P}(t) = \mathcal{Z}(t) - \mathfrak{A}(x^*, t)$, from (5.42), $\mathcal{Y}(t)$ satisfies

$$\mathcal{Y}_t(t) - \mathcal{P}(t)\mathcal{Y}(t) = -\delta\mathfrak{A}(t)\mathcal{X}(x_0, t) - \delta\mathcal{A}(t),$$

and hence

$$\left[\mathrm{e}^{-\int_{t^*}^t \mathcal{P}(s)\mathrm{d}s}\mathcal{Y}(t)\right]_t = -\mathrm{e}^{-\int_{t^*}^t \mathcal{P}(s)\mathrm{d}s}\left[\delta\mathfrak{A}(t)\mathcal{X}(x_0, t) + \delta\mathcal{A}(t)\right].$$

Integrating from t^* to t, we see that

$$\mathcal{Y}(t) = \mathrm{e}^{\int_{t^*}^t \mathcal{P}(s)\mathrm{d}s}\left(\mathcal{Y}(t^*) - \int_{t^*}^t \mathrm{e}^{-\int_{t^*}^s \mathcal{P}(r)\mathrm{d}r}\left[\delta\mathfrak{A}(s)\mathcal{X}(x_0, s) + \delta\mathcal{A}(s)\right]\mathrm{d}s\right). \tag{5.46}$$

Our goal is to show that $\mathcal{Y}(t) \geqq 0$, for all $t \geqq t^*$. By (5.43) and (5.44), we see that

$$\mathcal{Y}(t^*) > 1, \tag{5.47}$$

so all we need to prove is that the second term on the right-hand side of (5.46),

$$\kappa(t^*, t) = -\int_{t^*}^t \mathrm{e}^{-\int_{t^*}^s \mathcal{P}(r)\mathrm{d}r}\left[\delta\mathfrak{A}(s)\mathcal{X}(x_0, s) + \delta\mathcal{A}(s)\right]\mathrm{d}s, \tag{5.48}$$

is very small for t^* and t close to T.

We first consider $-\int_{t^*}^{s} \mathcal{P}(r)\mathrm{d}r$ which is equal to $-\int_{t^*}^{s} \mathcal{Z}(r)\mathrm{d}r + \int_{t^*}^{s} \mathfrak{A}(x^*, r)\mathrm{d}r$. Since $\mathcal{X}(x^*, t)$ is positive, we see that $\mathcal{Z}(t) > \mathcal{X}(x_0, t)$ and so $-\mathcal{Z}(t) < -\mathcal{X}(x_0, t)$, and as we noted above, $\mathcal{X}(x_0, t) > (1-\varepsilon)\delta t^{-1}$. Hence $-\int_{t^*}^{s} \mathcal{Z}(r)\mathrm{d}r < -\int_{t^*}^{s} \mathcal{X}(x_0, r)\mathrm{d}r$, so that

$$\mathrm{e}^{-\int_{t^*}^{s} \mathcal{Z}(r)\mathrm{d}r} < \mathrm{e}^{-\int_{t^*}^{s} \mathcal{X}(x_0,r)\mathrm{d}r} \leqq \mathrm{e}^{-\int_{t^*}^{s} \frac{1-\varepsilon}{T-r}\mathrm{d}r} = \left[\frac{T-s}{T-t^*}\right]^{1-\varepsilon}$$

and since $\mathrm{e}^{\int_{t^*}^{s} \mathfrak{A}(x^*,r)\mathrm{d}r} \lesssim \mathcal{M}$, then

$$\mathrm{e}^{-\int_{t^*}^{s} \mathcal{P}(r)\mathrm{d}r} \lesssim \mathcal{M}\left[\frac{T-s}{T-t^*}\right]^{1-\varepsilon}.$$

From (5.48), we see that

$$\begin{aligned}|\kappa(t^*, t)| &\lesssim \mathcal{M}\int_{t^*}^{t}\left[\frac{(T-s)}{T-t^*}\right]^{1-\varepsilon}\left(\frac{1+\varepsilon}{T-s}\delta\mathfrak{A}(s) + \delta\mathcal{A}(s)\right)\mathrm{d}s \\ &\lesssim \frac{\mathcal{M}(1+\varepsilon)}{(T-t^*)^{1-\varepsilon}}\int_{t^*}^{t}(T-s)^{-\varepsilon}\delta\mathfrak{A}(s)\mathrm{d}s + \frac{\mathcal{M}}{(T-t^*)^{1-\varepsilon}} \\ &\quad \int_{t^*}^{t}(T-s)^{1-\varepsilon}\delta\mathcal{A}(s)\mathrm{d}s.\end{aligned}$$

Let $\vec{r}$ denote a unit-speed parameterization of the path $\gamma \subset \Gamma$ starting at x_0 and ending at x^*. From (5.17), $\mathfrak{A}(x, t) = 2Gv^{+'} \cdot \tau(x, t)$, so that thanks to our assumed bounds (3.1), we see that

$$\delta\mathfrak{A}(t) = \int_{\gamma} \nabla\mathfrak{A} \cdot \mathrm{d}\vec{r} \lesssim \mathcal{M}|x^* - x_0| \lesssim \mathcal{M}\varepsilon,$$

the last inequality following from (5.45). It follows that

$$\begin{aligned}|\kappa(t^*, t)| &\lesssim \frac{\varepsilon\mathcal{M}(T-t)^{1-\varepsilon}}{(T-t^*)^{1-\varepsilon}} + \varepsilon\mathcal{M} + \frac{\mathcal{M}(T-t)^{2-\varepsilon}}{(T-t^*)^{1-\varepsilon}} + \mathcal{M}(T-t^*) \\ &\lesssim \mathcal{M}\left[\varepsilon + (T-t^*)\right] \quad \forall t \in [t^*, T).\end{aligned}$$

Hence, for $T - t^*$ sufficiently small, and $t \in [t^*, T)$, we have $|\kappa(t^*, t)| < 1$. Thanks to (5.47), this implies that for such any such t^*, and for all $t \in [t^*, T)$, $\mathcal{Y}(t) \geqq 0$, which by the definition of $\mathcal{Y}(t)$, implies that

$$\mathcal{X}(x^*, t) \geqq \mathcal{X}(x_0, t),$$

and thus $\lim_{t\to T} \mathcal{X}(x^*, t) = \infty$. Now, from our assumption of a single splash contact in this section, this implies that either $x^* = x_0$ or $x^* = x_1$ or $x^* = x_i$. Since x^* is sequence in Γ converging to x_0, we then have $x^* = x_0$. Thus, by (5.43) and (5.44), we then have

$$1 < 0,$$

which is the contradiction needed to establish that our assumption (5.44) was wrong.

By definition of $\mathcal{X}(x, t)$, this then shows that $\sup_{y\in\Gamma(t)} |\nabla_{\mathcal{T}}\delta u \cdot \mathcal{T}(\cdot, t)| \leqq \frac{1+(1+2c_6)(T-t)}{T-t}$ for all $t \in [t_0(\varepsilon), T)$ with $T - t_0(\varepsilon)$ taken sufficiently small. Together with our assumed bounds (3.1) on u^+, this completes the proof. □

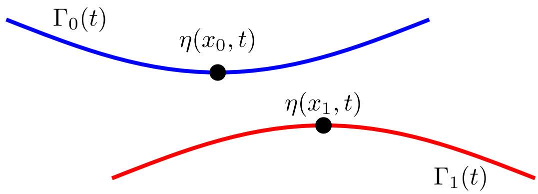

Fig. 3. For t sufficiently close to T, the interface $\Gamma(t)$ has a local neighborhood of $\eta(x_0, t)$ called $\Gamma_0(t) := \eta(\gamma_0(\varepsilon), t)$ and a local neighborhood of $\eta(x_1, t)$ called $\Gamma_1(t) := \eta(\gamma_1(\varepsilon), t)$

6. The Interface Geometry Near the Assumed Blow-up

For the sake of contradiction, we assume the existence of two points x_0 and x_1 in the reference interface Γ at time $t = 0$ which evolve towards a splash singularity at time $t = T$; namely $\eta(x_0, T) = \eta(x_1, T)$. In this section, we assume that this is the only point of self-intersection at time $t = T$, and that no self-intersection of $\Gamma(t)$ occurs for any $t < T$. There may indeed also exist additional points $x_i \in \Gamma$, such that $\eta(x_i, T) = \eta(x_0, T) = \eta(x_1, T)$, but in the course of our analysis, we will prove that there can only be a finite number of such points. In the case that these additional points $x_i \in \Gamma$ exist, we moreover show that we can relabel the point x_1 so that for time t sufficiently close to T, the points $\eta(x_1, t)$ and $\eta(x_0, t)$ are such that the vertical open segment joining $\eta(x_0, t)$ to a small neighborhood of $\eta(x_1, t)$ on $\Gamma(t)$ is contained in $\Omega^-(t)$, as we depict in Fig. 3. We will then prove that our assumption of a finite-time splash singularity leads to a contradiction, and is hence impossible.

If a splash singularity occurs at time T, then of course $\lim_{t\to T} |\eta(x_0, t) - \eta(x_1, t)| = 0$. In this section, we find the evolution equation for the distance between the two points $\eta(x_0, t)$ and $\eta(x_1, t)$.

Recall that the tangent and normal directions to $\Gamma(T)$ at $\eta(x_0, T) = \eta(x_1, T)$ are given by the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, respectively. In what follows, we will consider $0 < \varepsilon \ll 1$ fixed and sufficiently small.

With Γ denoting the initial interface at time $t = 0$ and $0 < \varepsilon \ll 1$, recall the definition of the two small neighborhoods $\gamma_0(\varepsilon) \subset \Gamma$ and $\gamma_1(\varepsilon) \subset \Gamma$ given in (5.28) and (5.30), respectively. According to these definitions, we may fix $\varepsilon > 0$ sufficiently small so that for each $x \in \gamma_0(\varepsilon) \cup \gamma_1(\varepsilon)$ and for all $t \in [t_0(\varepsilon), T]$, $\mathcal{N}(\eta(x_i, t), t)$ and $\mathcal{T}(\eta(x_i, t), t)$ $(i = 0, 1)$ are almost parallel with e_2 and e_1, respectively; in particular, the inequalities given in (5.28) and (5.30) *provide a quantitative estimate* for the term "almost parallel." Hence, by the definition of (5.28) and (5.30),

$$\Gamma_0(t) := \eta(\gamma_0(\varepsilon), t) \quad \text{and} \quad \Gamma_1(t) := \eta(\gamma_1(\varepsilon), t)$$

are almost flat neighborhoods of $\eta(x_0, t)$ and $\eta(x_1, t)$ for all $t \in [t_0(\varepsilon), T]$.

Next, we define

$$\boldsymbol{\delta}\eta(t) = \eta(x_0, t) - \eta(x_1, t) \quad \text{and} \quad \boldsymbol{\delta}u^-(t) = u^-(\eta(x_0, t), t) - u^-(\eta(x_1, t), t), \tag{6.1}$$

and

$$\delta\eta_1 = \delta\eta \cdot \mathrm{e}_1,\ \delta\eta_2 = \delta\eta \cdot \mathrm{e}_2 \quad \text{and} \quad \delta u_1^- = \delta u^- \cdot \mathrm{e}_1,\ \delta u_2^- = \delta u^- \cdot \mathrm{e}_2.$$

Since η is the flow of the velocity u^-, we see that for any $t \in [t_0(\varepsilon), T)$,

$$\partial_t \delta\eta = u^-(\eta(x_0, t), t) - u^-(\eta(x_1, t), t). \tag{6.2}$$

Definition 6.1. (*Distance function on* $\Gamma(t)$). We denote by $d_{\Gamma(t)}(X, Y)$ the distance along $\Gamma(t)$ between two points X and Y of $\Gamma(t)$. Let $\gamma_{X,Y}(t) \subset \Gamma(t)$ denote that portion of $\Gamma(t)$ connecting the points X and Y.

In order to establish our main result, we need the following lemmas.

Lemma 6.1. *Let X and Y denote two points in $\Gamma(t)$. Then,*

$$\left|\mathcal{T}(X, t) - \mathcal{T}(Y, t)\right| \leqq \mathcal{M} d_{\Gamma(t)}(X, Y),$$

and, if $X_1 \geqq Y_1$ and $\mathcal{T}_1 \geqq 0$ on $\gamma_{X,Y}(t)$,

$$X_1 - Y_1 \geqq \min_{Z \in \gamma_{X,Y}(t)} \mathcal{T}_1(Z, t)\, d_{\Gamma(t)}(X, Y).$$

Proof. Let $\theta : [0, 1] \to \Gamma$ denote a $W^{4,\infty}$-class parameterization of the reference interface Γ. There exists $\alpha(t), \beta(t) \in [0, 1]$ such that $X = \eta(\theta(\alpha(t)), t)$ and $Y = \eta(\theta(\beta(t)), t)$.

We set $\tilde{\eta} = \eta \circ \theta$. Then for $\alpha(t) \leqq s \leqq \beta(s)$, we have that

$$\mathcal{T}(\tilde{\eta}(\alpha(t), t), t) - \mathcal{T}(\tilde{\eta}(\beta(t), t), t) = \int_{\beta(t)}^{\alpha(t)} \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{T}(\tilde{\eta}(s, t), t)\mathrm{d}s.$$

We write $\mathcal{T}(\tilde{\eta}(s, t), t)$ as $\mathcal{T}(\tilde{\eta})(s, t)$ and employ the chain-rule to find that

$$\begin{aligned}
\mathcal{T}(\tilde{\eta}(\alpha(t), t), t) - \mathcal{T}(\tilde{\eta}(\beta(t), t), t) &= \int_{\beta(t)}^{\alpha(t)} \partial_i \mathcal{T}(\tilde{\eta})(s, t)\ \tilde{\eta}_i'(s, t)\ \mathrm{d}s \\
&= \int_{\beta(t)}^{\alpha(t)} \nabla_{\mathcal{T}} \mathcal{T}(\tilde{\eta})(s, t)\ \left|\tilde{\eta}'(s, t)\right| \mathrm{d}s \\
&= \int_{\beta(t)}^{\alpha(t)} H\mathcal{N}(\tilde{\eta})(s, t)\ \left|\tilde{\eta}'(s, t)\right| \mathrm{d}s,
\end{aligned}$$

where from (4.2), $\nabla_{\mathcal{T}} \mathcal{T}(\tilde{\eta}) = G(G\tilde{\eta}')'$ which is equal to $H\mathcal{N}(\tilde{\eta})$. Therefore, from (3.1),

$$\left|\mathcal{T}(X, t) - \mathcal{T}(Y, t)\right| \leqq \mathcal{M} \int_{\beta(t)}^{\alpha(t)} |[\eta \circ \theta]'(s, t)| \mathrm{d}s \leqq \mathcal{M} d_{\Gamma(t)}(X, Y).$$

Next, we have that

$$\begin{aligned}
X_1 - Y_1 = \int_{\beta(t)}^{\alpha(t)} (\eta \circ \theta)_1'(s, t)\ \mathrm{d}s &= \int_{\beta(t)}^{\alpha(t)} \mathcal{T}_1((\eta \circ \theta)(s, t), t) |(\eta \circ \theta)'(s, t)|\ \mathrm{d}s \\
&\geqq \min_{Z \in \gamma_{X,Y}(t)} \mathcal{T}_1(Z, t)\, d_{\Gamma(t)}(X, Y),
\end{aligned} \tag{6.3}$$

if $X_1 \geqq Y_1$ and $\mathcal{T}_1 \geqq 0$ on $\gamma_{X,Y}(t)$. □

Lemma 6.2. *For* $0 < \varepsilon \ll 1$ *fixed, let* $\gamma_1(\varepsilon)$ *denote the curve defined in* (5.30)*. Then, for all* $t \in [t_0(\varepsilon), T]$*, there exist points* $X^l(t)$ *and* $X^r(t)$ *in the curve* $\eta(\gamma_1(\varepsilon), t)$ *such that*

$$\begin{aligned}\eta_1(x_1, t) - \frac{\varepsilon}{2c_7} \leqq X_1^l(t) \leqq \eta_1(x_1, t) - \frac{\varepsilon}{4c_7} < \eta_1(x_1, t) + \frac{\varepsilon}{4c_7} \\ \leqq X_1^r(t) \leqq \eta_1(x_1, t) + \frac{\varepsilon}{2c_7},\end{aligned}$$

where the constant c_7 *is defined in* (5.25)*,* $\eta_1 = \eta\cdot e_1$*,* $X_1^l = X^l\cdot e_1$*, and* $X_1^r = X^r\cdot e_1$*.*

Proof. According to our definition (5.30) of $\gamma_1(\varepsilon)$,

$$|\mathcal{T}_1(\eta(x, t), t)| > 1 - \varepsilon \;\; \forall \;\; x \in \gamma_1(\varepsilon), \;\; t \in [t_0(\varepsilon), T).$$

Let us assume we are in the case

$$\mathcal{T}_1(\eta(x, t), t) > 1 - \varepsilon \;\; \forall \;\; x \in \gamma_1(\varepsilon), \;\; t \in [t_0(\varepsilon), T), \tag{6.4}$$

the other case

$$\mathcal{T}_1(\eta(x, t), t) < -1 + \varepsilon \;\; \forall \;\; x \in \gamma_1(\varepsilon), \;\; t \in [t_0(\varepsilon), T),$$

being treated in a way similar as what follows. Next, let X denote a point $\eta(\gamma_1(\varepsilon), t)$ such that $X_1 < \eta_1(x_1, t)$, and (by fixing ε even smaller if necessary) satisfying

$$d_{\Gamma(t)}(X, \eta(x_1, t)) = \frac{\varepsilon}{4c_7(1 - \varepsilon)}. \tag{6.5}$$

By (6.4), (6.5) and Lemma 6.1, for all $t \in [t_0(\varepsilon), T)$,

$$\eta_1(x_1, t) - X_1 \geqq (1 - \varepsilon) d_{\Gamma(t)}(X, \eta(x_1, t)) \geqq \frac{\varepsilon}{4c_7}.$$

On the other hand, by (6.3), we also have that

$$\eta_1(x_1, t) - X_1 \leqq d_{\Gamma(t)}(X, \eta(x_1, t)) = \frac{\varepsilon}{4c_7(1 - \varepsilon)} \leqq \frac{\varepsilon}{2c_7},$$

for $\varepsilon > 0$ small enough. We then set $X^l(t) = X$.

The same argument also provides the point $X^r(t)$ which is on the right of $\eta(x_1, t)$. □

Our next result establishes the evolution equation for $\boldsymbol{\delta}\eta(t)$.

Theorem 6.1. (Evolution equation for $\boldsymbol{\delta}\eta(t)$). *With the assumed bounds* (3.1)*, and for* $x_0, x_1 \in \Gamma$ *such that* $|\eta(x_0, t) - \eta(x_1, t)| \to 0$ *as* $t \to T$*, if* $|[\nabla_{\mathcal{T}} \delta u \cdot \mathcal{T}](\eta(x_0, t), t)| \to \infty$ *as* $t \to T$*, then for* $0 < \varepsilon \ll 1$ *taken sufficiently small and fixed, and* $t_0(\varepsilon) \in [T - \varepsilon, T)$*, we have that for all* $t \in [t_0(\varepsilon), T)$*,*

$$\partial_t \boldsymbol{\delta}\eta(t) = \mathcal{M}(t)\boldsymbol{\delta}\eta(t) \quad \textit{where } \mathcal{M}(t) = \frac{1}{T - t}\begin{bmatrix} -\beta_1(t) & \varepsilon_1(t) \\ \mathcal{E}_2(t) & \alpha_2(t)\end{bmatrix}, \tag{6.6}$$

where the matrix coefficients

$$\beta_1(t), \alpha_2(t) \in [-2\varepsilon, 1 + 2c_9(T - t)] \quad \textit{and} \quad \varepsilon_1(t), \mathcal{E}_2(t) \in [-2\varepsilon, 2\varepsilon],$$

and where $c_9 = 1 + 2c_6$*, where* c_6 *is defined* (5.19)*.*

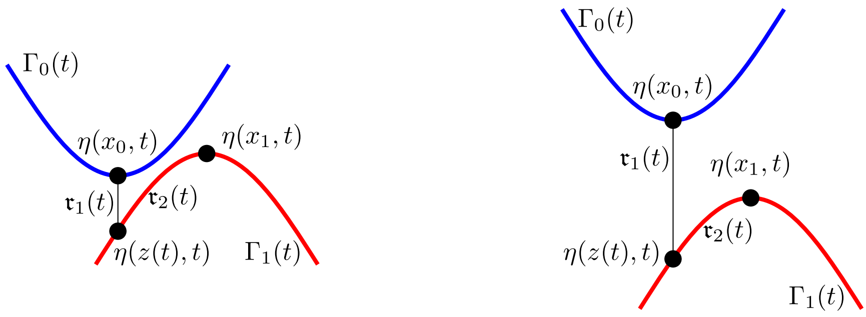

Fig. 4. *Left* $\eta_2(x_0, t) \leqq \eta_2(x_1, t)$ *Right* $\eta_2(x_0, t) > \eta_2(x_1, t)$

Proof. *Step 1. The geometric set-up.* Fig. 4 shows the geometry of the two approaching curves at some instant of time $t \in [t_0(\varepsilon), T)$: the left side of the figure shows the case that $\eta_2(x_0, t) \leqq \eta_2(x_1, t)$ and the right side of the figure shows the case that $\eta_2(x_0, t) > \eta_2(x_1, t)$.[1] Our idea is to connect $\eta(x_0, t)$ with $\eta(x_1, t)$ using a specially chosen path.

We remind the reader of two facts that we shall make use of: (1) for $t \in [t_0(\varepsilon), T)$ sufficiently small, the two approaching curves $\Gamma_0(t)$ and $\Gamma_1(t)$ are nearly flat, as described in (5.28) and (5.30); (2) there are two small neighborhoods $\gamma_0(\varepsilon) \subset \Gamma$ and $\gamma_1(\varepsilon) \subset \Gamma$ that are defined in (5.28) and (5.30), respectively.

We now explain why for $\varepsilon > 0$ chosen sufficiently small, the vertical projection of $\eta(x_0, t)$ must intersect $\eta(\gamma_1(\varepsilon), t)$ at *one unique* point, for any $t \in [t_0(\varepsilon), T]$. Due to Lemma 6.2, for $\varepsilon > 0$ small enough, there exists a point $x \in \gamma_1(\varepsilon)$ and another point $y \in \gamma_1(\varepsilon)$ such that for all $t \in [t_0(\varepsilon), T)$,

$$\eta_1(x, t) + \frac{\varepsilon}{4c_7} \leqq \eta_1(x_1, t) \leqq \eta_1(y, t) - \frac{\varepsilon}{4c_7}.$$

Now, by the fundamental theorem of calculus,

$$|\eta(x_1, t) - \eta(x_1, T)| \leqq \left| \int_t^T v^-(x_1, s)\mathrm{d}s \right| \leqq \mathcal{M}(T - t),$$

where we have used Lemma 5.1 to bound v^-. From (5.30), $T - t_0(\varepsilon) \leqq \frac{\varepsilon}{100c_7\mathcal{M}}$; it follows that

$$|\eta(x_1, t) - \eta(x_1, T)| \leqq \frac{\varepsilon}{100c_7}.$$

Similarly, $|\eta(x_0, t) - \eta(x_0, T)| \leqq \frac{\varepsilon}{100c_7}$ and using that $\eta(x_0, T) = \eta(x_1, T)$, we see that (by taking ε even smaller if necessary)

$$\eta_1(x, t) \leqq \eta_1(x, t) + \frac{\varepsilon}{5c_7} \leqq \eta_1(x_0, t) \leqq \eta_1(y, t) - \frac{\varepsilon}{5c_7} \leqq \eta_1(y, t).$$

[1] The actual curves $\Gamma_0(t)$ and $\Gamma_1(t)$ are almost flat near the assumed splash point, but we have made the slopes large to clearly demonstrate the paths $\mathfrak{r}_1(t)$ and $\mathfrak{r}_2(t)$; moreover, both $\Gamma_0(t)$ and $\Gamma_1(t)$ can have very small oscillations near the contact points and do not have to be parabolas. On the other hand, any potential small oscillations along the curves do not effect the qualitative picture in any way.

By the intermediate value theorem, this shows that there exists $\eta(z(t), t) \in \eta(\gamma_1(\varepsilon), t)$ such that $\eta_1(z(t), t) = \eta_1(x_0, t)$, and hence

$$\eta_1(x, t) + \frac{\varepsilon}{5c_7} \leqq \eta_1(z(t), t) \leqq \eta_1(y, t) - \frac{\varepsilon}{5c_7}. \tag{6.7}$$

This proves the existence of a point $\eta(z(t), t)$ in the curve $\Gamma_1(t) := \eta(\gamma_1(\varepsilon), t)$ which has the same horizontal component as the point $\eta(x_0, t)$ for every $t \in [t_0(\varepsilon), T)$.

Let us now show that there cannot be a second point in this intersection. We proceed by contradiction, and assume the existence of a different point $Z(t) \in \gamma_1(\varepsilon)$ such that $Z(t) \neq z(t)$ and satisfies (6.7). Since $\eta_1(z(t), t) = \eta_1(Z(t), t)$, by Rolle's theorem, there exists $c(t) \in \gamma_1(\varepsilon)$ such that

$$\eta_1'(c(t), t) = 0. \tag{6.8}$$

Since for any $t < T$, $\det \nabla\eta = 1$, we then have that for $t < T$, $|\eta'| \neq 0$, and (6.8) provides

$$0 = \frac{\eta_1'(c(t), t)}{|\eta'(c(t), t), t)|} = \mathcal{T}(\eta(c(t), t), t) \cdot \mathrm{e}_1. \tag{6.9}$$

Therefore, $\mathcal{T}(\eta(c(t), t), t) = (\mathcal{T}(\eta(c(t), t), t) \cdot \mathrm{e}_2)\, \mathrm{e}_2$, which with (5.28) provides

$$1 = |\tau(\eta(c(t), t), t) \cdot \mathrm{e}_2| \leqq \varepsilon,$$

which is a contradiction as $\varepsilon < 1$.

As shown in Fig. 4, we define $\mathfrak{r}_1(t)$ to be the vertical line segment connecting $\eta(x_0, t) \in \Gamma_0(t)$ to $\Gamma_1(t)$. Let us now explain why the path $\mathfrak{r}_1(t)$ can always be assumed to be contained in the closure of $\Omega^-(t)$.

We assume that the path $\mathfrak{r}_1(t)$ is not contained in the closure of $\Omega^-(t)$. Then, since $\Omega^+(t)$ is an open and connected set, $\Omega^+(t) \cap \mathfrak{r}_1(t)$ is a union of segments $\mathcal{S}_i :=]\mathcal{X}_i(t), \mathcal{Y}_i(t)[$, with $\mathcal{X}_i(t) > \mathcal{Y}_i(t)$ for each i, and each segment $\mathcal{S}_i$ lies strictly above the next segment S_{i+1}.

We now show that there can only be a finite number of such segments $\mathcal{S}_i$. Let $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ be two such consecutive segments. Let $\mathfrak{c}_i(t) \subset \Gamma(t)$ denote the portion of $\Gamma(t)$ connecting the point $\mathcal{Y}_i \in \mathcal{S}_i$ to $\mathcal{X}_{i+1} \in \mathcal{S}_{i+1}$. We denote the open set $L_i(t) \subset \Omega^-(t)$ as the set enclosed by the curve $\mathfrak{c}_i(t)$ and the vertical segment $]\mathcal{Y}_i(t), \mathcal{X}_{i+1}(t)[$, as shown in Fig. 5. The set $L_i(t)$ is either to the left or to the right of the vertical path $\mathfrak{r}_1(t)$.

Below, we shall prove that the slope of the tangent vector to $\Gamma(t)$ at the points $\mathcal{X}_i(t)$ and $\mathcal{Y}_i(t)$ cannot be too large; specifically, we will show that

$$|\mathcal{T}(\mathcal{X}_i, t) \cdot \mathrm{e}_1| \geqq \frac{1}{\sqrt{2}} \quad \text{and} \quad |\mathcal{T}(\mathcal{Y}_i, t) \cdot \mathrm{e}_1| \geqq \frac{1}{\sqrt{2}}. \tag{6.10}$$

We now assume that (6.10) holds, and as shown in Fig. 5, we assume that $L_i(t)$ is to the left of $\mathfrak{r}_1(t)$. Let $\theta : [0, 1] \to \Gamma$ denote a $W^{4,\infty}$-class parameterization of the reference interface Γ. Let $\mathcal{P}_i(t) \in \mathfrak{c}_i(t)$ denote the left-most extreme point

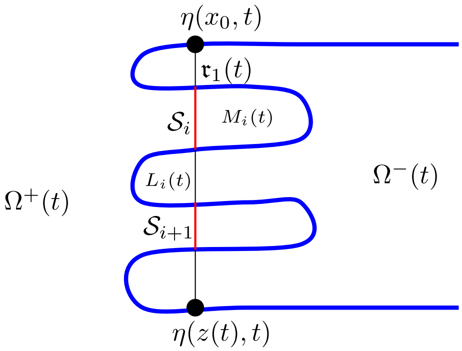

Fig. 5. If we suppose that the vertical line segment $\mathfrak{r}(t)$, connecting $\eta(x_0, t)$ to $\eta(z(t), t)$ is not contained in $\overline{\Omega^-(t)}$, then $\Omega^+(t) \cap \mathfrak{r}_1(t)$ consists of the union of finitely many open intervals $\mathcal{S}_i$ (shown in *red*) (color figure online)

on $\partial L_i(t)$; then, there exists $\alpha(t) \in [0, 1]$ such that $\mathcal{P}_i(t) = \eta(\theta(\alpha(t)), t)$, and $\mathcal{N}(\mathcal{P}_i(t), t) = -\mathrm{e}_1$. Let $\beta(t) \in [0, 1]$ be such that $\eta(\theta(\beta(t)), t) = \mathcal{Y}_i$.

Using Lemma 6.1 and the lower-bound (6.10),

$$\frac{1}{\sqrt{2}} \leqq \left|\mathcal{T}(\eta(\theta(\alpha(t)), t), t) - \mathcal{T}(\eta(\theta(\beta(t)), t), t)\right| \leqq \mathcal{M} \times \text{length of } \mathfrak{c}_i(t). \quad (6.11)$$

Since each loop $\mathfrak{c}_i(t)$ is of length greater than $\frac{1}{\sqrt{2}\mathcal{M}}$ and $\mathfrak{c}_i$ is disjoint from $\mathfrak{c}_j$ for $i \neq j$, the fact that $\Gamma(t)$ is of finite length, by (5.33), implies that the number of such loops $\mathfrak{c}_i(t)$ is bounded; hence, the intersection of $\mathfrak{r}_1(t)$ with $\Omega^+(t)$ consists of a finite number of segments $\mathcal{S}_i$.

Having established that this generic loop $\mathfrak{c}_i(t)$ (shown to the left of the vertical path $\mathfrak{r}_1(t)$ in Fig. 5) is of length greater than $\frac{1}{\sqrt{2}\mathcal{M}}$, we now turn our attention to the study of the subset $M_i(t) \subset \Omega^+(t)$ which is directly to the right of $\mathcal{S}_i$; that is, $M_i(t)$ is the open set whose boundary consists of that portion of $\Gamma(t)$ connecting $\mathcal{X}_i(t)$ with $\mathcal{Y}_i(t)$, which we call $\mathfrak{b}_i(t)$, and $\mathcal{S}_i$.

Next, let $\mathcal{Q}_i(t) \in \mathfrak{b}_i(t)$ denote the right-most extreme point of $M_i(t)$; then, similarly as for the case of $\mathfrak{c}_i(t)$, we find that the length of $\mathfrak{b}_i(t)$ is greater than $\frac{1}{\sqrt{2}\mathcal{M}}$.

We now explain why the projection of the set $M_i(t) \cup L_i(t)$ onto the horizontal axis spanned by e_1 has a vastly larger length than $T - t$. In the same way as we obtained the inequality (6.11), we have that for any $x = \eta(\theta(\kappa(t)), t) \in \Gamma(t)$ that

$$\left|\mathcal{T}(\eta(\theta(\beta(t)), t), t) - \mathcal{T}(\eta(\theta(\kappa(t)), t), t)\right| \leqq \mathcal{M} \times d_{\Gamma(t)}(x, \mathcal{Y}_i(t)).$$

Therefore, with $|\Gamma(t)|$ denoting the length of $\Gamma(t)$, for any $x \in \Gamma(t)$ such that $d_{\Gamma(t)}(x, \mathcal{Y}_i(t)) \leqq \min(\frac{1}{2}|\Gamma(t)|, \frac{1}{2\sqrt{2}\mathcal{M}})$, we have that

$$\left|\mathcal{T}(\eta(\theta(\kappa(t)), t), t) \cdot \mathrm{e}_1\right| \geqq \frac{1}{2\sqrt{2}}.$$

We can assume that

$$\mathcal{T}(\eta(\theta(\kappa(t)),t),t)\cdot \mathrm{e}_1 \geqq \frac{1}{2\sqrt{2}}, \tag{6.12}$$

for the case with the opposite sign can be treated in a similar fashion (as what follows).

Then, for any $\kappa(t) > \beta(t)$ such that $x = \eta(\theta(\kappa(t)),t)$ satisfies

$$d_{\Gamma(t)}(x, \mathcal{Y}_i(t)) = \min\left(\frac{1}{2}|\Gamma(t)|, \frac{1}{2\sqrt{2}\mathcal{M}}\right),$$

we have, by Lemma 6.1 and the inequality (6.12), that

$$(x - \mathcal{Y}_i(t))\cdot \mathrm{e}_1 \geqq \frac{1}{2\sqrt{2}} d_{\Gamma(t)}(x, \mathcal{Y}_i) = \frac{1}{2\sqrt{2}}\min\left(\frac{1}{2}|\Gamma(t)|, \frac{1}{2\sqrt{2}\mathcal{M}}\right) > 0, \tag{6.13}$$

which shows that $\mathfrak{b}_i(t)$ extends to the right of $\mathfrak{r}_1(t)$ by a distance of at least

$$\frac{1}{2\sqrt{2}}\min\left(\frac{1}{2}|\Gamma(t)|, \frac{1}{2\sqrt{2}\mathcal{M}}\right) > 0$$

in the e_1-direction. Using the identical argument, we can prove that $\mathfrak{c}_i(t)$ extends to the left of $\mathfrak{r}_1(t)$ by a distance of at least $\frac{1}{2\sqrt{2}}\min(\frac{1}{2}|\Gamma(t)|, \frac{1}{2\sqrt{2}\mathcal{M}}) > 0$ in the $-\mathrm{e}_1$-direction.

We now prove the inequalities in (6.10). We shall consider the tangent vector $\mathcal{T}$ at $\mathcal{Y}_i(t)$, as the proof for $\mathcal{T}$ at $\mathcal{X}_i(t)$ is identical. For the sake of contradiction, we assume that

$$|\mathcal{T}(\mathcal{Y}_i(t),t)\cdot \mathrm{e}_1| < \frac{1}{\sqrt{2}}, \tag{6.14}$$

so that

$$|\mathcal{T}(\mathcal{Y}_i(t),t)\cdot \mathrm{e}_2| \geqq \frac{1}{\sqrt{2}}.$$

We choose a point $x \in \mathfrak{b}_i(t)$ which is either to the left or to the right of $\mathcal{Y}_i(t)$ such that

$$d_{\Gamma(t)}(x, \mathcal{Y}_i(t)) = \min\left(\frac{1}{3}|\Gamma(t)|, \frac{1}{2\sqrt{2}\mathcal{M}}\right). \tag{6.15}$$

In the same way as we obtained (6.13), we see that if we choose x to be on the correct side of $\mathcal{Y}_i(t)$ (depending on the sign of $\mathcal{T}(\mathcal{Y}_i(t),t)\cdot \mathrm{e}_2$), we have that

$$(x - \mathcal{Y}_i(t))\cdot \mathrm{e}_2 \geqq \frac{1}{2\sqrt{2}}\, d_{\Gamma(t)}(x, \mathcal{Y}_i(t)), \tag{6.16}$$

as well as

$$|(x - \mathcal{Y}_i(t))\cdot \mathrm{e}_1| \leqq \frac{3}{2\sqrt{2}}\, d_{\Gamma(t)}(x, \mathcal{Y}_i(t)),$$

so that x is in the cone with vertex $\mathcal{y}_i(t)$ given by

$$(x - \mathcal{y}_i(t)) \cdot \mathrm{e}_2 \geqq \frac{1}{3} \, |(x - \mathcal{y}_i(t)) \cdot \mathrm{e}_1|. \tag{6.17}$$

Furthermore, using (6.16), we have that

$$(x - \mathcal{y}_i(t)) \cdot \mathrm{e}_2 \geqq \frac{1}{2\sqrt{2}} \min\left(\frac{|\Gamma(t)|}{3}, \frac{1}{2\sqrt{2}\mathcal{M}}\right). \tag{6.18}$$

Therefore, we have just established the existence of $\tilde{\mathfrak{c}}_i(t) \subset \Gamma(t)$, such that $\tilde{\mathfrak{c}}_i(t)$ is the shortest curve which connects $\mathcal{y}_i(t)$ to x and satisfies

$$\text{length of } \tilde{\mathfrak{c}}_i(t) = \min\left(\frac{|\Gamma(t)|}{3}, \frac{1}{2\sqrt{2}\mathcal{M}}\right),$$

which is bounded from below by a positive constant as $t \to T$. Moreover, the curve $\tilde{\mathfrak{c}}_i(t)$ is contained in the cone defined in (6.17), whose vertex $\mathcal{y}_i(t)$ satisfies

$$|\eta(x_0, t) - \mathcal{y}_i(t)| < |\eta(x_0, t) - \eta(z(t), t)|,$$

the right-hand side tending to zero as $t \to T$, since as $t \to T$, $\eta(z(t), t) \to \eta(x_1, T)$ which implies that $\lim_{t \to T} |\eta(x_0, t) - \eta(z(t), t)| = 0$. To sum up, $\tilde{\mathfrak{c}}_i(t)$ is a curve of length of order 1, of positive vertical extension above $\mathcal{y}_i(t)$ of order 1, and is contained in the cone (6.17) with vertex $\mathcal{y}_i(t)$ which is below $\eta(x_0, t)$ (and the distance between these two points converges to zero as $t \to T$).

On the other hand, since $\mathcal{T}(\eta(x_0, T), T) \cdot \mathrm{e}_2 = 0$, we have in the same manner that for $T - t$ sufficiently small, there exists a curve $\tilde{\Gamma}_0(t) \subset \Gamma(t)$ containing the point $\eta(x_0, t)$, and of length $\min\left(\frac{1}{2}|\Gamma(t)|, \frac{1}{200\mathcal{M}}\right)$ such that the curve $\tilde{\Gamma}_0(t)$ is contained in the two cones (that are almost horizontal from the definition below) defined by

$$|(x - \eta(x_0, t)) \cdot \mathrm{e}_2| \leqq \frac{1}{100} |(x - \eta(x_0, t)) \cdot \mathrm{e}_1|; \tag{6.19}$$

additionally, the curve $\tilde{\Gamma}_0(t)$ extends in the $\pm \mathrm{e}_1$ direction a distance of at least $\frac{1}{2} \min\left(\frac{1}{2}|\Gamma(t)|, \frac{1}{200\mathcal{M}}\right)$ on each side of $\eta(x_0, t)$. It is then elementary to show that the cone given by (6.17) intersects each of the four lines enclosing the cone (6.19) at a distance which less than $\frac{1}{2} \min\left(\frac{1}{2}|\Gamma(t)|, \frac{1}{200\mathcal{M}}\right)$ on each side of $\eta(x_0, t)$. Therefore, the cone given by (6.17) intersects $\tilde{\Gamma}_0(t)$. The same is true for the curve $\tilde{\mathfrak{c}}_i(t)$, as its starting point $\mathcal{y}_i(t)$ lies below $\tilde{\Gamma}_0(t)$ while, due to (6.18), its ending point x lies above $\tilde{\Gamma}_0(t)$ for t sufficiently close to T, and stays in the cone given by (6.17). Furthermore, this self-intersection occurs with different tangent vectors, since thanks to Lemma 6.1, any point z on $\tilde{\Gamma}_0(t)$ satisfies (for t close enough to T)

$$|\mathcal{T}(z, t) \cdot \mathrm{e}_2| \leqq \frac{1}{100},$$

while any point z on $\tilde{\mathfrak{c}}_i(t)$ will satisfy thanks to Lemma 6.1 that

$$|\mathcal{T}(z, t) \cdot \mathrm{e}_2| \geqq \frac{1}{2\sqrt{2}}.$$

As $\Gamma(t)$ cannot self-intersect for $t < T$ (particularly not with different tangent vectors), this then leads to a contradiction of (6.14), and hence proves (6.10).

Let $\bar{\gamma} \subset \Gamma$ be the preimage of η of the loops $\mathfrak{b}_i(\cdot, t_0(\varepsilon)) \cap \mathfrak{c}_i(\cdot, t_0(\varepsilon))$. It follows that for all $t \in (t_0(\varepsilon), T)$, $\eta(\bar{\gamma}, t)$ must continue to intersect the vertical path $\mathfrak{r}_1(t)$ at the finite set of points $\mathcal{X}_i(t)$ and $\mathcal{Y}_i(t)$. Since $\eta_2(x_0, t) > \mathcal{X}_i(t) > \eta_2(x_1, t)$ (the same being true for $\mathcal{Y}_i(t)$), we still have by continuity that $\eta_2(x_0, t') > \mathcal{X}_i(t') > \eta_2(x_1, t')$ (the same being true for $\mathcal{Y}_i(t')$) for $t' \in [t, T)$, as the case $\eta_2(x_0, t') = \mathcal{X}_i(t')$ or $\eta_2(x_1, t') = \mathcal{X}_i(t')$ correspond to a self-intersection of $\Gamma(t')$ at time $t' < T$, which is excluded from our definition of T.

This ensures that the already established finite number $\mathcal{N}$ of loops $\mathfrak{b}_i(t)$ and $\mathfrak{c}_i(t)$ stays constant for $t \in [t_0(\varepsilon), T]$ for $T - t_0(\varepsilon) > 0$ small enough. We then have the existence of a finite number of points $x_0, x_1, x_2, ..., x_n$ in Γ such that

$$\eta(x_0, T) = \eta(x_1, T) = \cdots = \eta(x_n, T)$$

and such that $\eta(x_i, t)$ for $i \in [2, \mathcal{N}]$ belongs to the image by the flow of the same loop (of length of at least $\frac{1}{\sqrt{2}} \min(\frac{1}{2}|\Gamma(t)|, \frac{1}{2\sqrt{2}\mathcal{M}})$ on each side of a corresponding point of intersection of $\mathfrak{r}_1(t)$ and $\Gamma(t)$) for all $t \in [t_0(\varepsilon), T]$). We can then, if necessary, replace the point x_1 by an appropriate $x_i \in \Gamma$ (with $\eta(x_i, t)$ such that $\eta(x_0, t)$ and $\eta(x_i, t)$ are on the same loop for all $t \in [t - t_0(\varepsilon), T]$). Therefore, the vertical path $\mathfrak{r}_1(t)$, connecting $\eta(x_0, t)$ to $\Gamma_1(t)$, is contained in $\Omega^-(t)$ for $T - t$ sufficiently small. In what follows, we assume that this substitution has been made so that $x_1 \in \Gamma$ is the point which is assumed to flow into self-intersection from below (by renaming x_0 and x_1 if necessary).

We can therefore define the unique point $z(t) \in \Gamma$ such that $\eta(z(t), t)$ is the vertical projection of $\eta(x_0, t)$ onto the curve $\mathfrak{r}_1(t)$ (as shown in Fig. 4). Specifically, we define $\mathfrak{r}_1(t)$ to be the vertical line segment connecting $\eta(x_0, t) \in \Gamma_0(t)$ to $\eta(z(t), t) \in \Gamma_1(t)$ (which is contained in $\overline{\Omega^-}(t)$ as we just have shown), and we define $\mathfrak{r}_2(t)$ to be the portion of $\Gamma_1(t)$ connecting $\eta(z(t), t)$ to $\eta(x_1, t)$.

We will rely on the following two claims:

Claim 1. For $t \in [t_0(\varepsilon), T)$, $\eta_2(x_1, t) - \eta_2(z(t), t) = b(t)\boldsymbol{\delta}\eta_1(t)\ [(T-t)+|\boldsymbol{\delta}\eta_1(t)|]$ for a bounded function $b(t)$.

Proof. Near the point $\eta(x_1, t)$, we consider $\mathfrak{r}_2(t)$ as a graph $(X, h(X, t))$ (see Fig. 4), such that $h(0, t) = \eta_2(x_1, t)$ with tangent vector $(1, h'(X, t))$, which at $X = 0$ must be close to horizontal, since $h'(0, T) = 0$. Since h is a C^2 function, we can write the Taylor series for $h(X, t)$ about $X = 0$ as

$$h(X, t) = h(0, t) + h'(0, t)X + \frac{1}{2}h''(\xi, t)X^2 \quad \text{for some } \xi \in (0, X). \tag{6.20}$$

Next,

$$\begin{aligned} |h'(0, t)| &= \left| h'(0, T) + \int_T^t h'_t(0, s)\mathrm{d}s \right| \\ &= \left| \int_T^t v_2^{-\prime}(x_1, s)\mathrm{d}s \right| \lesssim \mathcal{M}(T - t), \end{aligned} \tag{6.21}$$

the inequality following from the bound on v^- given by Lemma 5.1. On the other hand,

$$\begin{aligned}|h''(\xi,t)| &= |H(\xi,h(\xi,t))\,(1+h'^2(\xi,t))^{\frac{3}{2}}|\\ &= \left|H(\xi,h(\xi,t))\left(1+\left[\frac{\mathcal{T}(\xi,h(\xi,t))\cdot \mathrm{e}_2}{\mathcal{T}(\xi,h(\xi,t))\cdot \mathrm{e}_1}\right]^2\right)^{\frac{3}{2}}\right|\\ &\leqq |H(\xi,h(\xi,t))|\left(1+\frac{\varepsilon^2}{(1-\varepsilon)^2}\right)^{\frac{3}{2}} \lesssim \mathcal{M},\end{aligned} \tag{6.22}$$

where we have used (5.30) for the first inequality and (3.1) for the second. From (6.20), (6.21) and (6.22), we then have that

$$|h(X,t)-h(0,t)| \leqq C\mathcal{M}|X|(T-t+|X|), \tag{6.23}$$

for some constant $C>0$.

Next, we notice that $\eta_2(z(t),t)=h(\boldsymbol{\delta}\eta_1(t),t)$; hence, we set $X=\boldsymbol{\delta}\eta_1(t)$. By setting $b(t)=C\mathcal{M}\vartheta(t)$ with $\vartheta(t)\in(0,1)$, the proof is complete. □

Claim 2. $|\boldsymbol{\delta}\eta_1(t)| \lesssim \mathcal{M}(T-t) < \varepsilon$ for $t\in[t_0(\varepsilon),T)$.

Proof. By the fundamental theorem of calculus, $|\boldsymbol{\delta}\eta_1(t)| \leqq \int_T^t |\delta v(s)|\mathrm{d}s \lesssim \mathcal{M}(T-t)$ by Lemma 5.1. Then, we choose $T-t_0(\varepsilon)$ sufficiently small. □

Step 2. The case that $\eta_2(x_0,t)>\eta_2(x_1,t)$. We will first consider the geometry displayed on the right side of Fig. 4. With $\vec{r}_1(t)$ and $\vec{r}_2(t)$ denoting unit-speed parameterizations for $\mathfrak{r}_1(t)$ and $\mathfrak{r}_2(t)$,

$$\begin{aligned}u_1^-(\eta(x_0,t),t)-u_1^-(\eta(x_1,t),t) &= \left[u_1^-(\eta(x_0,t),t)-u_1^-(\eta(z(t),t),t)\right]\\ &\quad+\left[u_1^-(\eta(z(t),t),t)-u_1^-(\eta(x_1,t),t)\right]\\ &= \int_{\mathfrak{r}_1(t)}\nabla u_1^-\cdot \mathrm{d}\vec{r}_1+\int_{\mathfrak{r}_2(t)}\nabla u_1^-\cdot \mathrm{d}\vec{r}_2\\ &= \int_{\mathfrak{r}_1(t)}\frac{\partial u_2^-}{\partial x_1}\,\mathrm{d}x_2+\int_{\mathfrak{r}_2(t)}\nabla_{\mathcal{T}}u_1^-\,\mathrm{d}s,\end{aligned}$$

where we have used the fact that $\frac{\partial u_1^-}{\partial x_2}=\frac{\partial u_2^-}{\partial x_1}$ in the last equality, as curl $u^-=0$. We will evaluate these two integrals using the mean value theorem for integrals, together with our estimate (5.41) for $\nabla_{\mathcal{T}}u^-\cdot\mathcal{T}$, and hence for $\frac{\partial u_1^-}{\partial x_1}$ (which is equivalent to $\nabla_{\mathcal{T}}u^-$ for $T-t_0(\varepsilon)$ sufficiently small, as the ratio of the two quantities is close to 1), and estimate (5.10) for $\frac{\partial u_2^-}{\partial x_1}$. In particular,

$$\begin{aligned}&u_1^-(\eta(x_0,t),t)-u_1^-(\eta(x_1,t),t)\\ &\quad=\frac{\varepsilon_1(t)}{T-t}\left(\eta_2(x_0,t)-\eta_2(z(t),t)\right)-\varrho(t)\frac{\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\end{aligned}$$

$$
\begin{aligned}
&- \nu(t)\frac{\alpha_1(t)}{T-t}\left(\eta_2(x_1,t)-\eta_2(z(t),t)\right),\\
&= \frac{\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \frac{\varepsilon_1(t)}{T-t}\left(\eta_2(x_1,t)-\eta_2(z(t),t)\right) - \varrho(t)\frac{\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\\
&- \nu(t)\frac{\alpha_1(t)}{T-t}\left(\eta_2(x_1,t)-\eta_2(z(t),t)\right),
\end{aligned}
\tag{6.24}
$$

where $\varepsilon_1(t) \in [-\varepsilon, \varepsilon]$, and where we choose $\alpha_1(t) \in [-\varepsilon, 1 + c_9(T-t)]$, where $0 < \varepsilon \ll 1$ is defined in Step 4 of the proof of Theorem 5.1. The functions $\varrho(t)$ and $\nu(t)$ satisfy $|1-\varrho(t)| \ll 1$ and $0 \leqq \nu(t) \ll 1$; this follows since $\mathfrak{r}_2(t)$ is nearly flat near $\eta(x_0,t)$, so the vertical distance $|\eta_2(x_1,t)-\eta_2(z(t),t)|$ is nearly zero, while the horizontal distance $|\eta_1(x_1,t)-\eta_1(z(t),t)|$ is nearly the total distance $|\eta(x_1,t)-\eta(z(t),t)|$.

The negative sign in front of $\alpha_1(t)$ is determined by the limiting behavior of $\frac{\partial u_1^-}{\partial x_1}$ given by (5.16). From Claim 1 above, we then see that

$$
\begin{aligned}
&u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t)\\
&\quad = \frac{\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \frac{b(t)(|\boldsymbol{\delta}\eta_1(t)|+\delta t)\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \frac{\varrho\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\\
&\qquad - \frac{\nu b(t)(|\boldsymbol{\delta}\eta_1(t)|+\delta t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t),
\end{aligned}
$$

where $\delta t = T - t$. We set

$$
\beta_1(t) = \left[\varrho(t) + \nu(t)b(t)(|\boldsymbol{\delta}\eta_1(t)|+\delta t)\right]\alpha_1(t) - b(t)(|\boldsymbol{\delta}\eta_1(t)|+\delta t)\varepsilon_1(t).
$$

Then, with Claim 2, we see that $\beta_1(t) \in [-2\varepsilon, 1 + 2c_9(T-t)]$, and that

$$
u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) = -\frac{\beta_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_2(t). \tag{6.25}
$$

Similarly, for u_2^-, we have that

$$
\begin{aligned}
&u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t)\\
&\quad = \left[u_2^-(\eta(x_0,t),t) - u_2^-(\eta(z(t),t),t)\right] + \left[u_2^-(\eta(z(t),t),t) - u_2^-(\eta(x_1,t),t)\right]\\
&\quad = \int_{\mathfrak{r}_1(t)} \nabla u_2^- \cdot \mathrm{d}\vec{r}_1 + \int_{\mathfrak{r}_2(t)} \nabla u_2^- \cdot \mathrm{d}\vec{r}_2 = \int_{\mathfrak{r}_1(t)} \frac{\partial u_2^-}{\partial x_2}\,\mathrm{d}x_2 + \int_{\mathfrak{r}_2(t)} \nabla_{\mathcal{T}} u_2^-\,\mathrm{d}s,\\
&\quad = \frac{\alpha_2(t)}{T-t}\left(\eta_2(x_0,t)-\eta_2(z(t),t)\right) + \varrho(t)\frac{\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\\
&\qquad + \nu(t)\frac{\alpha_2(t)}{T-t}\left(\eta_2(x_1,t)-\eta_2(z(t),t)\right),\\
&\quad = \frac{\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_2(t) + \frac{b(t)(|\boldsymbol{\delta}\eta_1(t)|+\delta t)\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\\
&\qquad + \frac{\varrho(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\nu(t)b(t)(|\boldsymbol{\delta}\eta_1(t)|+\delta t)\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t),
\end{aligned}
$$

with $\varepsilon_2(t) \in [-\varepsilon, \varepsilon]$ and $\alpha_2(t) \in [-\varepsilon, 1+c_9(T-t)]$, and where $0 \leqq 1-\varrho(t) \ll 1$ and $0 \leqq \nu(t) \ll 1$. Setting

$$\mathcal{E}_2(t) = (b(t) + \nu(t)b(t))(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\alpha_2(t) + \varrho(t)\varepsilon_2(t) \tag{6.26}$$

we see that by Claim 2,

$$\mathcal{E}_2(t) \in [-2\varepsilon, 2\varepsilon], \tag{6.27}$$

and

$$u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t) = \frac{\mathcal{E}_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_2(t). \tag{6.28}$$

Equations (6.2), (6.25) and (6.28), then give the desired relation (6.6).

Step 3. The case that $\eta_2(x_0,t) \leqq \eta_2(x_1,t)$. We next consider the geometry displayed on the left side of Fig. 4. Again, using $\vec{r}_1(t)$ and $\vec{r}_2(t)$ to denote unit-speed parameterisations for $\mathfrak{r}_1(t)$ and $\mathfrak{r}_2(t)$, we see that once again

$$\begin{aligned} u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) &= \left[u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t),t)\right] \\ &\quad + \left[u_1^-(\eta(z(t),t),t) - u_1^-(\eta(x_1,t),t)\right] \\ &= \int_{\mathfrak{r}_1(t)} \frac{\partial u_2^-}{\partial x_1}\,\mathrm{d}x_2 + \int_{\mathfrak{r}_2(t)} \nabla_{\mathcal{T}} u_1^-\,\mathrm{d}s, \end{aligned}$$

where s denotes arc length. We again evaluate these two integrals using the mean value theorem for integrals:

$$\begin{aligned} u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) &= \frac{\varepsilon_1(t)}{T-t}\left(\eta_2(x_0,t) - \eta_2(z(t),t)\right) \\ -\frac{\varrho(t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) &- \frac{\nu(t)\alpha_1(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right), \end{aligned}$$

where once again $\alpha_1(t) \in [-\varepsilon, 1+c_9(T-t)]$ and $\varepsilon_1(t) \in [-\varepsilon, \varepsilon]$. For some $\theta(t) \in (0,1]$,

$$|\eta_2(x_0,t) - \eta_2(z(t),t)| = \theta(t)\,|\eta_2(x_1,t) - \eta_2(z(t),t)|\,.$$

Hence, by Claim 1,

$$\begin{aligned} &u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) \\ &\quad = \frac{\theta(t)b(t)(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\varepsilon_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) - \frac{\varrho(t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t) \\ &\qquad - \frac{b(t)(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\nu(t)\alpha_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t). \end{aligned}$$

With

$$\beta_1(t) = [\varrho(t) + b(t)(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\nu(t)]\alpha_1(t) - \theta(t)b(t)(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\varepsilon_1(t),$$

then $\beta_1(t) \in [-2\varepsilon, 1+2c_9(T-t)]$ and

$$u_1^-(\eta(x_0,t),t) - u_1^-(\eta(x_1,t),t) = -\frac{\beta_1(t)}{T-t}\boldsymbol{\delta}\eta_1(t).$$

Similarly, for u_2^-, we have that

$$
\begin{aligned}
&u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t)\\
&\quad = \big[u_2^-(\eta(x_0,t),t) - u_2^-(\eta(z(t),t),t)\big] + \big[u_2^-(\eta(z(t),t),t) - u_2^-(\eta(x_1,t),t)\big]\\
&\quad = \frac{\alpha_2(t)}{T-t}\left(\eta_2(x_0,t) - \eta_2(z(t),t)\right) + \frac{\varrho(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\\
&\qquad + \frac{\nu(t)\alpha_2(t)}{T-t}\left(\eta_2(x_1,t) - \eta_2(z(t),t)\right),
\end{aligned}
$$

with $\varepsilon_2(t) \in [-\varepsilon, \varepsilon]$ and $\alpha_2(t) \in [-\varepsilon, 1 + c_9(T-t)]$. Hence, from Claim 1, we see that

$$
\begin{aligned}
&u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t)\\
&\quad = \frac{\theta(t)b(t)(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t) + \frac{\varrho(t)\varepsilon_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t)\\
&\qquad + \frac{\nu(t)b(t)(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\alpha_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t).
\end{aligned}
$$

Setting

$$
\mathcal{E}_2(t) = [\theta(t) + \nu(t)]b(t)(|\boldsymbol{\delta}\eta_1(t)| + \delta t)\alpha_2(t) + \varrho(t)\varepsilon_2(t),
$$

we see that by Claim 2, $\mathcal{E}_2(t) \in [-2\varepsilon, 2\varepsilon]$, and

$$
u_2^-(\eta(x_0,t),t) - u_2^-(\eta(x_1,t),t) = \frac{\mathcal{E}_2(t)}{T-t}\boldsymbol{\delta}\eta_1(t).
$$

In this case, $\boldsymbol{\delta}\eta_t = \mathcal{M}\,\boldsymbol{\delta}\eta$ with

$$
\mathcal{M}(t) = \frac{1}{T-t}\begin{bmatrix} -\beta_1(t) & 0 \\ \mathcal{E}_2(t) & 0 \end{bmatrix},
$$

which is a special case of the matrix given (6.6) with $\varepsilon_1(t) = 0$ and $\alpha_2(t) = 0$. This completes the proof. □

7. Proof of the Main Theorem

We now give a proof of Theorem 3.1. We assume that either a splash or splat singularity does indeed occur, and then show that this leads to a contradiction.

We begin the proof with the case that a single splash singularity occurs at time $t = T$ and that there exist two points x_0 and x_1 in Γ, such that $\eta(x_0, T) = \eta(x_1, T)$, as we assumed in Section 6. (In Sections 7.2 and 7.3, we will also rule-out the case of multiple simultaneous splash singularities, as well as the splat singularity).

7.1. A Single Splash Singularity Cannot Occur in Finite Time

As we stated above, for $T - t_0$ sufficiently small and in a small neighborhood of $\eta(x_0, T)$, the interface $\Gamma(t)$, $t \in [t_0, T)$, consists of two curves $\Gamma_0(t)$ and $\mathfrak{r}_1(t)$ evolving towards one another, with $\eta(x_0, t) \in \Gamma_0(t)$ and $\eta(x_1, t) \in \mathfrak{r}_1(t)$.

We consider the two cases that either $|\nabla u^-(\cdot, t)|$ remains bounded or blows-up as $t \to T$.

7.1.1. The Case that $|\nabla u^-(\eta(x_0,t),t)| \to \infty$ as $t \to T$ We prove that both $\boldsymbol{\delta} u_1^-(T) \neq 0$ and $\boldsymbol{\delta} u_1^-(T) = 0$, where recall that $\boldsymbol{\delta} u^-(t)$ is given by (6.1).

Step 1. $\boldsymbol{\delta} u_1^- \neq 0$ *at the assumed splash singularity* $\eta(x_0, T)$.

The scalar product of (6.6) with $\boldsymbol{\delta}\eta(t)$ yields

$$\partial_t |\boldsymbol{\delta}\eta|^2 = -2\frac{\beta_1(t)}{T-t}|\boldsymbol{\delta}\eta_1|^2 + 2\frac{\varepsilon_1(t)+\mathcal{E}_2(t)}{T-t}\boldsymbol{\delta}\eta_1\,\boldsymbol{\delta}\eta_2 + 2\frac{\alpha_2(t)}{T-t}|\boldsymbol{\delta}\eta_2|^2, \tag{7.1}$$

where the constants $\beta_1(t), \alpha_2(t), \varepsilon_1(t), \mathcal{E}_2(t)$ are defined in Theorem 6.1. Therefore, since $T - t < \varepsilon \ll 1$,

$$\partial_t |\boldsymbol{\delta}\eta|^2 \geqq -\frac{2+C\varepsilon}{T-t}|\boldsymbol{\delta}\eta|^2,$$

from which we infer that

$$|\boldsymbol{\delta}\eta(t)|^2 \geqq |\boldsymbol{\delta}\eta(0)|^2 \frac{(T-t)^{2+C\varepsilon}}{T^{2+C\varepsilon}}. \tag{7.2}$$

We now assume that

$$\boldsymbol{\delta} u_1^-(T) = 0, \tag{7.3}$$

and now proceed to infer a contradiction from this assumption. Since $\boldsymbol{\delta}\eta(T) = 0$ (since we have assumed that a splash singularity occurs at $t = T$), we have that

$$\begin{aligned}
\boldsymbol{\delta}\eta_1(t) &= \int_T^t (\partial_t\eta(x_0,s) - \partial_t\eta(x_1,s))\,\mathrm{d}s \\
&= \int_T^t (v_1^-(x_0,s) - v_1^-(x_1,s))\mathrm{d}s \\
&= \int_T^t (v_1^-(x_0,s) - v_1^+(x_0,s))\mathrm{d}s + \int_T^t (v_1^+(x_0,s) - v_1^+(x_1,s))\mathrm{d}s \\
&\quad - \int_T^t (v_1^-(x_1,s) - v_1^+(x_1,s))\mathrm{d}s \\
&= -\int_T^t \delta v_1(x_0,s)\mathrm{d}s + \int_T^t (v_1^+(x_0,s) - v_1^+(x_1,s))\mathrm{d}s + \int_T^t \delta v_1(x_1,s)\mathrm{d}s \\
&= -\int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_0,s)\mathrm{d}s - \int_T^t \delta v\cdot\tau(x_0,s)\mathrm{d}s \\
&\quad + \int_T^t (v_1^+(x_0,s) - v_1^+(x_1,s))\mathrm{d}s + \int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_1,s)\mathrm{d}s \\
&\quad + \int_T^t \delta v\cdot\tau(x_1,s)\mathrm{d}s \\
&= -\int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_0,s)\mathrm{d}s - \int_T^t \left[\delta v\cdot\tau(x_0,T) + \int_T^s \partial_t(\delta v\cdot\tau)(x_0,l)\mathrm{d}l\right]\mathrm{d}s \\
&\quad + \int_T^t \left[(v_1^+(x_0,T) - v_1^+(x_1,T)) + \int_T^s \partial_t(v_1^+(x_0,l) - v_1^+(x_1,l))\mathrm{d}l\right]\mathrm{d}s \\
&\quad + \int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_1,s)\mathrm{d}s + \int_T^t \left[\delta v\cdot\tau(x_1,T) + \int_T^s \partial_t(\delta v\cdot\tau)(x_1,l)\mathrm{d}l\right]\mathrm{d}s.
\end{aligned} \tag{7.4}$$

Using the fact that $\tau(x_0, T) = \mathrm{e}_1 = \tau(x_1, T)$, (7.4) then becomes

$$\begin{aligned}\boldsymbol{\delta}\eta_1(t) = &-\int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_0,s)\mathrm{d}s - \int_T^t\int_T^s \partial_t(\delta v\cdot\tau)(x_0,l)\mathrm{d}l\mathrm{d}s\\ &+\int_T^t \Big[-\delta v_1(x_0,T)+v_1^+(x_0,T)-v_1^+(x_1,T)\\ &+\delta v_1(x_1,T)+\int_T^s \partial_t(v_1^+(x_0,l)-v_1^+(x_1,l))\mathrm{d}l\Big]\mathrm{d}s\\ &+\int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_1,s)\mathrm{d}s+\int_T^t\int_T^s \partial_t(\delta v\cdot\tau)(x_1,l)\mathrm{d}l\mathrm{d}s. \end{aligned}\tag{7.5}$$

Next, since $-\delta v_1(x_0,T)+v_1^+(x_0,T)-v_1^+(x_1,T)+\delta v_1(x_1,T)=\delta u_1^-(T)$, (7.5) and the assumption (7.3) then provide us with

$$\begin{aligned}\boldsymbol{\delta}\eta_1(t) = &-\int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_0,s)\mathrm{d}s - \int_T^t\int_T^s \partial_t(\delta v\cdot\tau)(x_0,l)\mathrm{d}l\mathrm{d}s\\ &+\int_T^t\int_T^s \partial_t(v_1^+(x_0,l)-v_1^+(x_1,l))\mathrm{d}l\mathrm{d}s\\ &+\int_T^t \delta v\cdot(\mathrm{e}_1-\tau)(x_1,s)\mathrm{d}s+\int_T^t\int_T^s \partial_t(\delta v\cdot\tau)(x_1,l)\mathrm{d}l\mathrm{d}s. \end{aligned}\tag{7.6}$$

Due to the L^∞ control of $(\delta v\cdot\tau)_t$ provided by (4.7), and by writing $\mathrm{e}_1 - \tau(x_i,s) = \int_s^T \tau_t(x_i,s)$ (for $i=0,1$), (7.6) allows us to conclude that

$$|\boldsymbol{\delta}\eta_1(t)| \lesssim \mathcal{M}(T-t)^2. \tag{7.7}$$

Note here that we used $\tau_t(x,t)=\mathcal{T}_t(\eta(x,t),t)+\nabla_{\mathcal{T}}\mathcal{T}(\eta(x,t),t)|\eta'(x,t)|$. By noticing that $\mathcal{T}_t$ can be computed from Remark 3, we then have $|\tau_t(\cdot,t)|_{L^\infty(\Gamma)}\lesssim\mathcal{M}$.

Therefore, $|\boldsymbol{\delta}\eta_1(t)|^2\lesssim\mathcal{M}(T-t)^4$, and from (7.2), this implies (since $0\leqq\varepsilon\ll 1$) that $(T-t)^{2+C\varepsilon}\leqq C|\boldsymbol{\delta}\eta_2(t)|^2$; hence, by choosing $t_1\in(0,T)$ sufficiently close to T, for any $t\in[t_1,T]$,

$$\begin{aligned}|\boldsymbol{\delta}\eta_1(t)| &\lesssim \mathcal{M}(T-t)^2=\mathcal{M}(T-t)^{1+C\varepsilon/2}(T-t)^{1-C\varepsilon/2}\\ &\lesssim (T-t)^{1-C\varepsilon/2}\,|\boldsymbol{\delta}\eta_2(t)|\leqq\sqrt{T-t}\,|\boldsymbol{\delta}\eta_2(t)|.\end{aligned}\tag{7.8}$$

Using (7.8) in (7.1) and the fact that $0<T-t<\varepsilon\ll 1$, we then obtain

$$\begin{aligned}\partial_t|\boldsymbol{\delta}\eta|^2 &\geqq -[2(2c_9+\frac{1}{\sqrt{T-t}})+\frac{8\varepsilon}{\sqrt{T-t}}+\frac{4\varepsilon}{T-t}]|\boldsymbol{\delta}\eta_2|^2\\ &\geqq -[2(2c_9+\frac{1}{\sqrt{T-t}})+\frac{8\varepsilon}{\sqrt{T-t}}+\frac{4\varepsilon}{T-t}]|\boldsymbol{\delta}\eta|^2.\end{aligned}\tag{7.9}$$

Thus,

$$\partial_t\left(|\boldsymbol{\delta}\eta|^2\mathrm{e}^{\int_{t_1}^t[2(2c_9+\frac{1+4\varepsilon}{\sqrt{T-s}})]\,\mathrm{d}s}(T-t)^{-4\varepsilon}\right)\geqq 0.$$

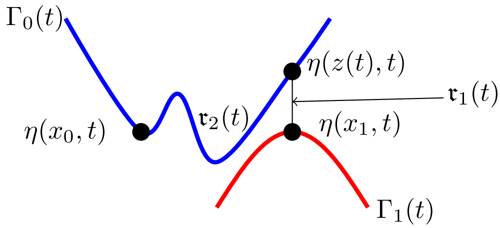

Fig. 6. The portion of the interface $\Gamma_0(t)$, near $\eta(x_0, t)$, is shown to have an oscillation that may only disappear in the limit as $t \to T$

Hence,

$$|\boldsymbol{\delta}\eta|^2(t) \geqq |\boldsymbol{\delta}\eta|^2(t_1) \mathrm{e}^{-\int_{t_1}^{T} [2(2c_9 + \frac{1+4\varepsilon}{\sqrt{T-s}})] \, \mathrm{d}s} \frac{(T-t)^{4\varepsilon}}{(T-t_1)^{4\varepsilon}} > C(T-t)^{4\varepsilon},$$

with $C > 0$ finite, since $(T-s)^{-\frac{1}{2}}$ is integrable. This is then in contradiction with our assumption of a splash singularity occurring at time $t = T$ which implies that

$$|\delta\eta|^2(t) \lesssim \mathcal{M}(T-t)^2;$$

therefore, the assumption (7.3) was wrong as it lead to a contradiction, leading us to conclude that, in fact,

$$|\boldsymbol{\delta} u_1^-(T)| > 0. \tag{7.10}$$

Step 2. $\boldsymbol{\delta} u_1^- = 0$ *at the assumed splash singularity* $\eta(x_0, T)$. Having shown that $\boldsymbol{\delta} u_1^- \neq 0$ at the splash singularity, in order to arrive at a contradiction, we shall next prove that we also have $\boldsymbol{\delta} u_1^- = 0$ at the splash singularity (Fig. 6).

We now define the following two curves. The first curve $\mathfrak{r}_1(t)$ is the vertical segment joining $\eta(x_1, t) \in \mathfrak{r}_1(t)$ to a point $\eta(z(t), t) \in \Gamma_1(t)$. This segment is contained in full in the closure of $\Omega^-(t)$ (for $T - t$ sufficiently small), as we have shown in Step 1 of the proof of Theorem 6.1, by simply switching the role of x_0 and x_1 in the definition of this vertical segment.

The second curve $\mathfrak{r}_2(t)$ is the portion of $\Gamma_0(t)$ linking $\eta(z(t), t)$ to $\eta(x_0, t)$.

We now simply write

$$\begin{aligned} \boldsymbol{\delta} u_1^-(t) &= u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t),t) + u_1^-(\eta(z(t),t),t) - u_1^-(\eta(x_1,t),t) \\ &= u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t),t) + \int_{\mathfrak{r}_1(t)} \nabla u_1^- \cdot \tau \, \mathrm{d}l \\ &= u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t),t) + \int_{\mathfrak{r}_1(t)} \frac{\partial u_1^-}{\partial x_2} \, \mathrm{d}x_2, \end{aligned} \tag{7.11}$$

where we have used that e_2 is the tangent vector to $\mathfrak{r}_1(t)$ in the last equality of (7.11).

Next, we estimate the length of the vertical segment $\mathfrak{r}_1(t)$, by simply noticing that

$$|\eta(x_0,t)-\eta(x_1,t)|^2 = |\eta(x_0,t)-\eta(z(t),t)|^2 + |\eta(z(t),t)-\eta(x_1,t)|^2 + 2|\eta(x_0,t)-\eta(z(t),t)||\eta(z(t),t)-\eta(x_1,t)|\cos\theta, \tag{7.12}$$

where θ denotes the angle between the two vectors $\eta(x_0,t)-\eta(z(t),t)$ and $\eta(z(t),t)-\eta(x_1,t)$. Due to (5.28), the direction of the tangent vector $\mathcal{T}$ on $\eta(\gamma_0(\varepsilon),t)$ in a small neighborhood of $\eta(x_0,t)$ is very close to horizontal; in particular, $|\mathcal{T}(\eta(x,t),t)\cdot \mathrm{e}_2| < \varepsilon$ for all $x \in \gamma_0(\varepsilon)$ and $t \in [t_0(\varepsilon), T)$. Hence, we have that $\eta(x_0,t)-\eta(z(t),t)$ is in direction close to horizontal. On the other hand, $\eta(z(t),t)-\eta(x_1,t)$ is in the vertical direction. Therefore, θ is very close to $\frac{\pi}{2}$ which then, in turn, implies from (7.12) that

$$\begin{aligned}|\eta(x_0,t)-\eta(x_1,t)|^2 &\geqq |\eta(x_0,t)-\eta(z(t),t)|^2 + |\eta(z(t),t)-\eta(x_1,t)|^2\\ &\quad - \frac{1}{2}|\eta(x_0,t)-\eta(z(t),t)||\eta(z(t),t)-\eta(x_1,t)|\\ &\geqq \frac{3}{4}|\eta(x_0,t)-\eta(z(t),t)|^2 + \frac{3}{4}|\eta(z(t),t)-\eta(x_1,t)|^2,\end{aligned}$$

which shows that the square of the length of the vertical segment satisfies

$$\begin{aligned}|\eta(x_1,t)-\eta(z(t),t)|^2 &\leqq \frac{4}{3}|\eta(x_0,t)-\eta(x_1,t)|^2\\ &\leqq \frac{4}{3}|\eta(x_0,t)-\eta(x_0,T)-\eta(x_1,t)+\eta(x_1,T)|^2\\ &\leqq \frac{4}{3}\left|\int_T^t v^-(x_0,s)\,\mathrm{d}s - \int_T^t v^-(x_1,s)\,\mathrm{d}s\right|^2\\ &\leqq \frac{16}{3}(T-t)^2\|v^-\|^2_{L^\infty(\Gamma)}\\ &\lesssim \mathcal{M}^2(T-t)^2,\end{aligned} \tag{7.13}$$

thanks to Lemma 5.1.

Then, with our estimate (5.10) on $\frac{\partial u_2^-}{\partial x_1}$ and the fact that $\operatorname{curl} u^- = 0$, we then have with (7.13) that

$$\left|\int_{\mathfrak{r}_1(t)} \nabla u_1^- \cdot \tau \,\mathrm{d}l\right| \lesssim \mathcal{M}\,(T-t)\frac{\varepsilon}{T-t} = \varepsilon\mathcal{M}. \tag{7.14}$$

It remains to estimate the difference $u_1^-(\eta(x_0,t),t) - u_1^-(\eta(z(t),t)$ appearing on the right-hand side of (7.11). Recall that $\Gamma_0(t) = \eta(\gamma_0(\varepsilon),t)$, for $\varepsilon > 0$ small enough fixed. From Lemma 5.1, v^- is continuous along Γ_0. Next, we have that η is continuous and injective from $\overline{\gamma_0(\varepsilon)} \times [0,T]$, into its image $\mathcal{K}$. Since η is continuous and injective, and $\overline{\gamma_0(\varepsilon)}$ is closed, $\mathcal{K}$ is closed (as the sequential definition of a closed set is straightforwardly satisfied). As a result, η^{-1} is also continuous and injective from $\mathcal{K}$ into $\overline{\gamma_0(\varepsilon)} \times [t_0(\varepsilon), T]$, as the sequential definition of continuity is straightforwardly satisfied. By composition, $u^- = v^- \circ \eta^{-1}$ is also continuous on $\mathcal{K}$. Since $z(t) \in \gamma_0(\varepsilon)$ by step 1 of the proof of Theorem 6.1 (by switching the roles of x_0 and x_1), and $z(t)$ converges to x_0 as $t \to T$, we then have that $\eta(z(t), t)$ belongs to $\mathcal{K}$ and satisfies

$$\lim_{t \to T} (\eta(z(t), t) - \eta(x_0, t)) = 0.$$

Since we just established the continuity of u^- on $\mathcal{K}$, and henceforth its uniform continuity in the compact set $\mathcal{K}$, we can infer from the previous limit and this uniform continuity that $u_1(\eta(x_0, t), t) - u_1(\eta(z(t), t)$ converges to zero as $t \to T$.

With this fact, we can infer from (7.11) and (7.13) that as $t \to T$,

$$|\boldsymbol{\delta} u_1^-(T)| \leqq \varepsilon \mathcal{M},$$

this being true for any $\varepsilon > 0$. Therefore,

$$|\boldsymbol{\delta} u_1^-(T)| = 0,$$

which is a contradiction with (7.10).

We shall next explain why a non-singular gradient of the velocity u^- also does not allow for a splash singularity, which will finish the proof of our main result in the case of a single self-intersection.

7.1.2. The Case that $|\nabla u^-(x, t)|$ Remains Bounded If $\|\nabla u^-(\cdot, t)\|_{L^\infty(\Omega^-(t))}$ is bounded on $[0, T]$, we can still obtain the differential equation $\boldsymbol{\delta}\eta_t(t) = \mathcal{M}(t)\boldsymbol{\delta}\eta(t)$ using the same path integral that we used in the proof of Theorem 6.1, with paths shown in Fig. 4; in this case, however, the components of the matrix $\mathcal{M}$ are bounded on $[0, T]$. From $\boldsymbol{\delta}\eta_t(t) = \mathcal{M}(t)\boldsymbol{\delta}\eta(t)$, we see that

$$\partial_t |\boldsymbol{\delta}\eta|^2 = 2\mathcal{M}_{11}|\boldsymbol{\delta}\eta_1|^2 + 2\left(\mathcal{M}_{12}(t) + \mathcal{M}_{21}(t)\right)\boldsymbol{\delta}\eta_1\ \boldsymbol{\delta}\eta_2 + 2\mathcal{M}_{22}|\boldsymbol{\delta}\eta_2|^2,$$

with $\mathcal{M}_{ij}$ bounded for $i, j = 1, 2$. Therefore,

$$\partial_t |\boldsymbol{\delta}\eta|^2 \geqq -C(\mathcal{M})|\boldsymbol{\delta}\eta|^2,$$

which then provides

$$|\boldsymbol{\delta}\eta(t)|^2 \geqq |\boldsymbol{\delta}\eta(0)|^2 \mathrm{e}^{-C(\mathcal{M})t}.$$

Since $\boldsymbol{\delta}\eta(0) \neq 0$, we then cannot have $\boldsymbol{\delta}\eta(T) = 0$ for any finite T.

7.1.3. The Case that the Region Between x_0 and x_1 is Ω^+. In this case, we can still proceed with the same geometric constructions as before. The difference is that in this case, the matrix $\mathcal{M}(t)$ has bounded coefficients (since ∇u^+ is bounded in $L^\infty(\Omega^+(t))$, and therefore, we are in the same situation as the case treated previously where $|\nabla u^-(x, t)|$ remains bounded, which leads to the impossibility of a splash singularity at time T.

7.2. An Arbitrary Number (Finite or Infinite) of Splash Singularities at Time T is not Possible

We assume that an arbitrary number of simultaneous splash singularities occur at time $T > 0$. We now focus on one of the many possible self-intersection points. To this end, let x_0 and x_1 be two points in Γ such that $\eta(x_0, T) = \eta(x_1, T)$. Let $\Gamma_0 \subset \Gamma$ be a local neighborhood of x_0 and let $\Gamma_1 \subset \Gamma$ be a local neighborhood of x_1.

Then, there exists a sequence of points $x_0^n \in \Gamma_0$ converging to x_0, and of a sequence of points $x_1^n \in \Gamma_1$ converging to x_1 such that

$$d_0^n := d(\eta(x_0^n, T), \eta(\Gamma_1, T)) \neq 0, \quad d_1^n := d(\eta(x_1^n, T), \eta(\Gamma_0, T)) \neq 0 \ \ \forall n \in \mathbb{N}, \tag{7.15}$$

where d denotes the distance function; otherwise, if (7.15) did not hold, then we would have non trivial neighborhoods γ_0 of x_0 and γ_1 of x_1 such that $\eta(\gamma_0, T) = \eta(\gamma_1, T)$, which means a splat singularity occurs at $t = T$, and we treat that case below in Section 7.3.

We continue to let e_1 denote a tangent unit vector to $\Gamma(T)$ at the splash contact point $\eta(x_0, T)$. We then have, by the continuity of the tangent vector $\mathcal{T}$ to the interface, that for both sequences of points,

$$\left|\mathrm{e}_1 - \mathcal{T}(\eta(x_0^n, T), T)\right| \leqq \varepsilon, \tag{7.16}$$

for $\varepsilon > 0$ fixed and n large enough. We now call z_1^n the orthogonal projection of $\eta(x_0^n, T)$ onto $\eta(\Gamma_1, T)$. We then have from (7.15) that

$$\left|\eta(x_0^n, T) - z_1^n\right| = d_0^n > 0. \tag{7.17}$$

Furthermore, we denote by the unit vector e_0^n the direction of the vector $\eta(x_0^n, T) - z_1^n$ (with base point at z_1^n and "arrow" at $\eta(x_0^n, T)$). By definition, e_0^n points in the normal direction to $\eta(\Gamma_1, T)$ at z_1^n and by (7.16), e_0^n is close to e_2. For each point x_0^n, the segment $(\eta(x_0^n, T), z_1^n)$ is contained in $\eta(\Omega^-, T)$.

By continuity of η on $\Gamma \times [0, T]$ we also infer from (7.17) that there exists a connected neighborhood γ_0^n of x_0^n on Γ, of length $L_n > 0$, such that for any $x \in \gamma_0^n$ we have

$$d(\eta(x, T), \eta(\Gamma_1, T)) \geqq \frac{d_0^n}{2}; \tag{7.18}$$

moreover, the direction of the vector $\eta(x, T) - P_{\eta(\Gamma_1, T)}(\eta(x, T))$, normal to $\eta(\Gamma_1, T)$ at $P_{\eta(\Gamma_1, T)}(\eta(x, T))$, is close to e_2, where $P_{\eta(\Gamma_1, T)}$ denotes the orthogonal projection onto $\eta(\Gamma_1, T)$.

Note that for each $x \in \gamma_0^n$, the segment $(\eta(x, T), P_{\eta(\Gamma_1, T)}(\eta(x, T)))$ is contained in $\eta(\Omega^-, T)$. By continuity of the direction of these vectors, we then have that

$$\omega_n = \cup_{x \in \gamma_0^n} (\eta(x, T), P_{\eta(\Gamma_1, T)}(\eta(x, T))), \tag{7.19}$$

is an open set contained in $\eta(\Omega^-, T)$. Furthermore, $\partial\omega_n$ contains the set $\eta(\gamma_0^n, T)$ of length $L_n > 0$ (as its top boundary), and by continuity of the directions, $\partial\omega_n$ also contain a connected subset $\eta(\gamma_1^n, T)$ of $\eta(\Gamma_1, T)$, of length greater than $\frac{L_n}{2}$ (as

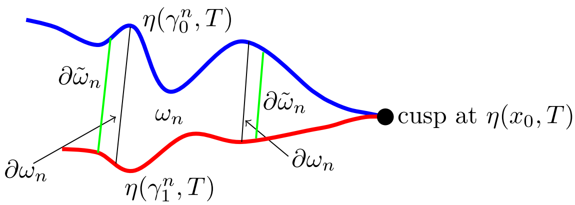

Fig. 7. The open set ω_n is contained in the larger open set $\tilde{\omega}_n$

its bottom boundary). Because ω_n does not intersect the cusp which occurs at the contact point, we define the open set $\tilde{\omega}_n \supset \omega_n$, such that the lateral part of $\partial\tilde{\omega}_n$ is parallel to the lateral part of $\partial\omega_n$ and connects $\eta(\Gamma_0, T)$ and $\eta(\Gamma_1, T)$ as shown in Fig. 7.

Next, we introduce the stream functions $\psi^{\pm}$ such that $u^{\pm}(\cdot, T) = \nabla^{\perp}\psi^{\pm}$, and we recall that u^+ (and hence ψ^+) has the good regularity on $\Gamma(t)$ for $t \in [0, T]$, given by (3.1). Let $\mathcal{W}_n$ be an open set such that $\omega_n \subset \mathcal{W}_n \subset \tilde{\omega}_n$. Let $0 \leqq \vartheta_n \leqq 1$ denote a C^{∞} cut-off function which is equal to 1 in $\overline{\omega_n}$ and equal to 0 on $\overline{\tilde{\omega}_n}/\mathcal{W}_n$.

We have that ψ^- is an $H^1(\Omega^-(T))$ weak solution of $\Delta\psi^- = 0$ in $\Omega^-(T)$ and $\psi^- = \psi^+$ on $\partial\Omega^-(T)$. Then $\vartheta_n\psi^-$ satisfies

$$\begin{aligned} -\Delta(\vartheta_n\psi^-) &= -\psi^-\Delta\vartheta_n - 2\nabla\vartheta_n \cdot \nabla\psi^-, && \text{in } \tilde{\omega}_n, \\ \vartheta_n\psi^- &= \psi^+ && \text{on } \eta(\Gamma_0, T) \cup \eta(\Gamma_1, T) \cap \partial\tilde{\omega}_n, \end{aligned}$$

and as $\psi^+ \in H^{3.5}(\eta(\Gamma_0, T)) \cup H^{3.5}(\eta(\Gamma_1, T))$, standard elliptic regularity shows that

$$\psi^- \in H^4(\omega_n),$$

and therefore that

$$\nabla u^-(\cdot, T) \in H^3(\omega_n) \subset L^{\infty}(\omega_n). \tag{7.20}$$

Let $\mathcal{D}_n^r$ denote the pre-image of ω_n under the map $\eta(\cdot, T)$. Let us assume that $\partial\mathcal{D}_n^r \cap \Gamma_0$ lies to the right of x_0. Since $\overline{\omega_n}$ does not intersect the splash singularity at time T, $\eta(\cdot, T)$ is bijective and continuous from $\mathcal{D}_n^r$ into ω_n, and therefore $\mathcal{D}_n^r$ is an open connected set.

Furthermore, $\nabla u^- \circ \eta$ is also continuous on $\overline{\mathcal{D}_n^r} \times [0, T]$ which, thanks to (7.20), shows that for all $t \in [0, T]$,

$$\|\nabla u^-(\cdot, t)\|_{L^{\infty}(\eta(\mathcal{D}_n^r, t))} \leqq \mathcal{M}_n^r. \tag{7.21}$$

We can also choose the sequence x_0^n to lie on the left of x_0 (otherwise, we would have a splat singularity). This similarly gives an open neighborhood $\mathcal{D}_n^l$ of the same type as $\mathcal{D}_n^r$ satisfying for all $t \in [0, T]$,

$$\|\nabla u^-(\cdot, t)\|_{L^{\infty}(\eta(\mathcal{D}_n^l, t))} \leqq \mathcal{M}_n^l. \tag{7.22}$$

We now denote by $\mathcal{C}_n^r$ (respectively $\mathcal{C}_n^l$) the lateral part of $\partial\mathcal{D}_n^r$ (respectively $\partial\mathcal{D}_n^l$) joining Γ_0 to Γ_1, and we denote by $\mathcal{K}_n$ the open set delimited by $\mathcal{C}_n^r$; the subset of Γ_0 containing x_0 linking $\mathcal{C}_n^r$ to $\mathcal{C}_n^l$; $\mathcal{C}_n^l$; and the subset of Γ_1 containing x_1 linking $\mathcal{C}_n^l$ to $\mathcal{C}_n^r$.

For n large enough, we will have estimate (7.16) satisfied at any point of $\partial\mathcal{K}_n \cap \Gamma$, with moreover the length of $\partial\mathcal{K}_n \cap \Gamma$ being of order ε. This then implies, in a way similar to Step 4 of Theorem 5.1, that

$$\left\| \frac{\partial u_2^-}{\partial x_1}(\cdot, t) \right\|_{L^\infty(\eta(\partial\mathcal{K}_n \cap \Gamma, t))} \leqq \frac{\varepsilon}{T-t} \tag{7.23}$$

for any $t < T$. Moreover, for t close enough to T, the maximum of the two constants $\mathcal{M}_n^r$ and $\mathcal{M}_n^l$ of (7.21) and (7.22) will become smaller than $\frac{\varepsilon}{T-t}$. Thus, for any $t < T$ close enough to T,

$$\left\| \frac{\partial u_2^-}{\partial x_1}(\cdot, t) \right\|_{L^\infty(\eta(\partial\mathcal{K}_n, t))} \leqq \frac{\varepsilon}{T-t},$$

which by application (for each fixed $t < T$ close enough to T) of the maximum and minimum principle for the harmonic function $\frac{\partial u_2^-}{\partial x_1}(\cdot, t)$ on the open set $\eta(\mathcal{K}_n, t)$ provides

$$\left\| \frac{\partial u_2^-}{\partial x_1}(\cdot, t) \right\|_{L^\infty(\eta(\mathcal{K}_n, t))} \leqq \frac{\varepsilon}{T-t}. \tag{7.24}$$

We can then apply the same arguments as in the Sections 6 and 7.1 to exclude a splash singularity associated with x_0 and x_1 simply by working in the neighborhood of size $C\varepsilon$ (C bounded from below away from 0) where (7.24) holds.

7.3. A Splat Singularity is not Possible

We now assume the existence of a splat singularity: there exists two disjoint closed subsets of Γ, which we denote by Γ_0 and Γ_1, with non-zero measure, such that contact occurs at time $t = T$ and $\eta(\Gamma_0, T) = \eta(\Gamma_1, T)$. We furthermore assume that the set

$$\mathcal{S}_0 = \left\{ x \in \Gamma_0 \ : \lim_{t \to T} |\nabla u^-(\eta(x,t), t)| = \infty \right\} \tag{7.25}$$

has a non-empty interior, and denote by x_0 and y_0 two distinct points on $\mathcal{S}_0$ such that the curve $\gamma_0 \subset \Gamma_0$, which connects the points x_0 to y_0, is contained in $\mathcal{S}_0$. We denote by $L(t)$ the length of the curve $\eta(\gamma_0, t)$, which is given by

$$L(t) = \int_{\gamma_0} |\eta'(x,t)| \, \mathrm{d}l. \tag{7.26}$$

By Lemma 5.2, for any $x \in \mathcal{S}_0$, $\lim_{t\to T} \eta'(x,t) = 0$, and from Lemma 5.1, we have the uniform bound $\sup_{t\in[0,T]} |\eta'|_{L^\infty(\Gamma)} \leqq \mathcal{M}$ where $\mathcal{M}$ is independent of $t < T$. Therefore, by the dominated convergence theorem,

$$\lim_{t\to T} L(t) = 0, \tag{7.27}$$

which shows that $\eta(x_0, T) = \eta(y_0, T)$, which is a contradiction with the fact that η is injective on $\Gamma_0 \times [0, T]$. Therefore our assumption that $\mathcal{S}_0$ has non-empty interior was wrong, which shows that this set has empty interior. Therefore the set

$$\mathcal{B}_0 = \left\{ x \in \Gamma_0 \ : \lim_{t \to T} |\nabla u^-(\eta(x,t),t)| < \infty \right\}, \tag{7.28}$$

is dense in Γ_0. Furthermore, by Lemma 5.1, $|v'(\cdot, t)|_{L^\infty(\Gamma)} \leqq \mathcal{M}$ where $\mathcal{M}$ is independent of $t < T$. Hence, by Lemma 5.2, $\mathcal{B}_0$ is defined equivalently as

$$\mathcal{B}_0 = \left\{ x \in \Gamma_0 \ : |\eta'(x, T)| > 0 \right\},$$

which shows that this set is open in Γ_0. Therefore, $\mathcal{B}_0$ is an open and dense subset of Γ_0.

Now since η is continuous and injective from $\Gamma_0 \times [0, T]$ onto its image, it also is a homeomorphism from $\Gamma_0 \times [0, T]$ onto its image, which shows that $\eta(\mathcal{B}_0, T)$ is open and dense in $\eta(\Gamma_0, T)$. With

$$\mathcal{B}_1 = \left\{ x \in \Gamma_1 \ : \lim_{t \to T} |\nabla u^-(\eta(x,t),t)| < \infty \right\}, \tag{7.29}$$

the same argument shows that $\eta(\mathcal{B}_1, T)$ is also open and dense in $\eta(\Gamma_1, T)$. Our assumption of a splat singularity at $t = T$ means that $\eta(\Gamma_0, T) = \eta(\Gamma_1, T)$, showing that $\eta(\mathcal{B}_0, T)$ and $\eta(\mathcal{B}_1, T)$ are two open and dense sets in $\eta(\Gamma_0, T) = \eta(\Gamma_1, T)$. They, therefore, have an open and dense intersection.

Let $\mathcal{Z}$ be a point in this intersection with tangent direction given by e_1. By definition, there exists $z_0 \in \mathcal{B}_0$ and $z_1 \in \mathcal{B}_1$ such that $\eta(z_0, T) = \eta(z_1, T)$. We are therefore back to the case where interface self-intersection occurs with non-singular ∇u^- (from the definition of the sets $\mathcal{B}_0$ and $\mathcal{B}_1$), except that we do not have an estimate for ∇u^- valid for the entire interface $\Gamma(t)$.

We now consider two open connected curves $\gamma_0 \subset \mathcal{B}_0$ and $\gamma_1 \subset \mathcal{B}_1$ such that for any point $z_0 \in \gamma_0$ there exist a point $z_1 \in \gamma_1$ such that $\eta(z_0, T) = \eta(z_1, T)$. For $t \in [T_0, T)$, T_0 being very close to T, the two curves $\eta(\gamma_0, t)$ and $\eta(\gamma_1, t)$ are very close to each other, and at each point, have tangent vector close to e_1 (to ensure this, if necessary, we take a sufficiently small subset of each of these two curves).

Furthermore, from the definition of $\mathcal{B}_0$, we have that the length of the curve $\eta(\gamma_0, t)$ for $t \in [T_0, T)$, T_0 being very close to T, is close to a number $L_0 > 0$ (which is the length of $\eta(\gamma_0, T) = \eta(\gamma_1, T)$). Similarly, the length of the curve $\eta(\gamma_1, t)$ for $t \in [T_0, T)$, is close to L_0.

We now fix two distinct and close-by points $\eta(z_0, T_0)$ and $\eta(\tilde{z}_0, T_0)$ on $\eta(\gamma_0, T_0)$ such that $|\eta(z_0, T_0) - \eta(\tilde{z}_0, T_0)| < \frac{L_0}{200}$, and the distance between each of these points and the complement of $\eta(\gamma_0, T_0)$ in $\eta(\Gamma_0, T_0)$ is greater than $\frac{L_0}{4}$. By taking T_0 closer to T if necessary, we can assume that for any $t \in [T_0, T]$ the distance between $\eta(z_0, t)$ (or $\eta(\tilde{z}_0, t)$) and the complement of $\eta(\gamma_0, t)$ in $\eta(\Gamma_0, t)$ is greater than $\frac{L_0}{5}$.

As shown in Fig. 8, we now define $\eta(z_1, T_0)$ as being the intersection of the vertical line passing through $\eta(z_0, T_0)$ and $\eta(\gamma_1, T_0)$. This defines a unique point since the tangent vector to $\eta(\gamma_1, T_0)$ is close to e_1, and furthermore the segment

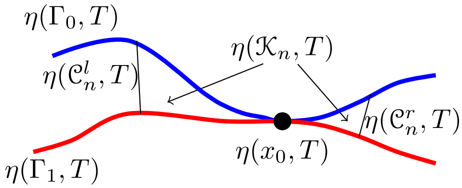

Fig. 8. The region in which we apply the maximum and minimum principle

$(\eta(z_0,T_0),\eta(z_1,T_0))$ is contained in $\eta(\Omega^-,T_0)$. Similarly, we define $\eta(\tilde z_1,T_0)$ as being the intersection of the vertical line passing through $\eta(\tilde z_0,T_0)$ and $\eta(\gamma_1,T_0)$. This defines a unique point, with the segment $(\eta(\tilde z_0,T_0),\eta(\tilde z_1,T_0))$ contained in $\eta(\Omega^-,T_0)$.

By taking T_0 closer to T if necessary, we can assume that for any $t\in[T_0,T]$ the distance between $\eta(z_1,t)$ (or $\eta(\tilde z_1,t)$) and the complement of $\eta(\gamma_1,t)$ in $\eta(\Gamma_1,t)$ is greater than $\frac{L_0}{5}$. By further taking T_0 closer to T, if necessary, we can also assume that

$$\operatorname{dist}(\eta(\gamma_0,T_0),\eta(\gamma_1,T_0))\leqq\frac{L_0}{100},\tag{7.30}$$

and also that

$$\left(1+\sup_{[0,T]}\|v^-(\cdot,t)\|_{L^\infty(\Omega^-)}\right)(T-T_0)<\frac{L_0}{12}.\tag{7.31}$$

We denote by $\eta(\omega,T_0)$ the domain enclosed by the two vertical segments $[\eta(z_0,T_0),\eta(z_1,T_0)]$, $[\eta(\tilde z_0,T_0),\eta(\tilde z_1,T_0)]$, the portion of the curve $\eta(\gamma_0,T_0)$ linking $\eta(z_0,T_0)$ to $\eta(\tilde z_0,T_0)$, and the portion of the curve $\eta(\gamma_1,T_0)$ linking $\eta(z_1,T_0)$ to $\eta(\tilde z_1,T_0)$. This domain is contained in $\eta(\Omega^-,T_0)$ (which justifies its name $\eta(\omega,T_0)$, for $\omega\subset\Omega^-$), and has a non-zero area A_0 (since its boundary contains two distinct vertical lines and two near horizontal and distinct curves).

By incompressibility, for any $t\in[T_0,T)$, the area of $\eta(\omega,t)$ remains a constant which we call A_0. Now, as $t\to T$, the two curves $\eta(\gamma_0,t)$ and $\eta(\gamma_1,t)$ get close to a splat contact (which occurs at $t=T$); therefore, the domain $\mathcal{D}(t)$ between these two curves and the two short lateral segments joining them has an area converging to zero (see Fig. 9). Therefore for $t<T$ close enough to T we cannot have $\eta(\omega,t)\subset\mathcal{D}(t)$, as points on the lateral edges of $\eta(\omega,t)$ would be pushed-out of the lateral boundaries of $\mathcal{D}(t)$.

Therefore, we have at least a point (in fact a subset of non zero area) $\eta(z,t)$ $(z\in\omega)$ such that

$$|\eta(z_0,t)-\eta(z,t)|\geqq\frac{L_0}{5}.\tag{7.32}$$

From (7.30), and from the fact that the boundary of $\eta(\omega,T_0)$ is comprised of two vertical segments of length less than $\frac{L_0}{100}$ and of two near horizontal curves of length less than $\frac{L_0}{100}$, we have that

$$|\eta(z_0,T_0)-\eta(z,T_0)|\leqq\frac{L_0}{50}.\tag{7.33}$$

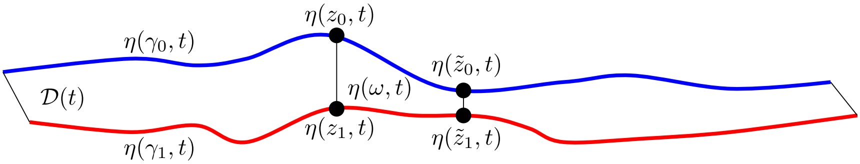

Fig. 9. That portion of $\Omega^-(t)$ being squeezed together by the approaching splat singularity

From (7.32) and (7.33) we then have

$$\left|\int_{T_0}^{t}[v(z_0,s)-v(z,s)]\mathrm{d}s\right| \geqq \frac{L_0}{5}-\frac{L_0}{50} \geqq \frac{L_0}{6}. \tag{7.34}$$

Using (7.34), we infer that

$$2(T-T_0)\sup_{[0,T]}\|v\|_{L^\infty(\Omega^-)} \geqq \frac{L_0}{6},$$

which is in contradiction to (7.31). This establishes the impossibility of a splat singularity at time $t=T$.

As our analysis was reduced to a local neighborhood of any assumed splat singularity, as shown in Fig. 9, this means that any combination of splat and splash singularities at time $t=T$ can be analyzed in the same way. This finishes the proof of the exclusion of splat or splash singularities in finite time.

Acknowledgments. Daniel Coutand was supported by the Centre for Analysis and Nonlinear PDEs funded by the UK EPSRC grant EP/E03635X and the Scottish Funding Council. Steve Shkoller was supported by the National Science Foundation under grants DMS-1001850 and DMS-1301380, by OxPDE at the University of Oxford, and by the Royal Society Wolfson Merit Award. We are grateful to the referees for the time and care in reading our manuscript and for the excellent suggestions that have improved the presentation greatly.

References

1. Alazard, T., Burq, N., Zuily, C.: On the Cauchy problem for gravity water waves. *Invent. Math.* (2014). arXiv:1212.0626
2. Alazard, T., Delort, J.-M.: *Global solutions and asymptotic behavior for two dimensional gravity water waves* (2013, preprint). arXiv:1305.4090
3. Alazard, T., Delort, J.-M.: *Sobolev estimates for two dimensional gravity water waves* (2013, preprint). arXiv:1307.3836
4. Alvarez-Samaniego, B., Lannes, D.: Large time existence for 3D water-waves and asymptotics. *Invent. Math.* **171**, 485–541 (2008)
5. Ambrose, D.M., Masmoudi, N.: The zero surface tension limit of two-dimensional water waves. *Commun. Pure Appl. Math.* **58**, 1287–1315 (2005)
6. Ambrose, D.M., Masmoudi, N.: Well-posedness of 3D vortex sheets with surface tension. *Commun. Math. Sci.* **5**(2), 391–430 (2007)
7. Ambrose, D., Masmoudi, N.: The zero surface tension limit of three-dimensional water waves. *Indiana Univ. Math. J.* **58**, 479–521 (2009)

8. Beale, J.T., Hou, T., Lowengrub, J.: Growth rates for the linearized motion of fluid interfaces away from equilibrium. *Commun. Pure Appl. Math.* **46**, 1269–1301 (1993)
9. Castro, Á., Córdoba, D., Fefferman, C., Gancedo, F., López-Fernández, M.: Rayleigh–Taylor breakdown for the Muskat problem with applications to water waves. *Ann. Math.* (2)**175**(2), 909–948 (2012)
10. Castro, Á., Córdoba, D., Fefferman, C., Gancedo, F., López-Fernández, M.: Turning waves and breakdown for incompressible flows. *Proc. Natl. Acad. Sci.* **108**, 4754–4759 (2011)
11. Castro, A., Córdoba, D., Fefferman, C., Gancedo, F., Gómez-Serrano, M.: Finite time singularities for the free boundary incompressible Euler equations. *Ann. Math.* **178**, 1061–1134 (2013)
12. Craig, W.: An existence theory for water waves and the Boussinesq and Korteweg–de Vries scaling limits. *Commun. Partial Differ. Equ.* **10**(8), 787–1003 (1985)
13. Cheng, C.H.A., Coutand, D., Shkoller, S.: On the motion of vortex sheets with surface tension in the 3D Euler equations with vorticity. *Commun. Pure Appl. Math.* **61**(12), 1715–1752 (2008)
14. Cheng, C.H.A., Coutand, D., Shkoller, S.: On the limit as the density ratio tends to zero for two perfect incompressible 3-D fluids separated by a surface of discontinuity. *Commun. Partial Differ. Equ.* **35**, 817–845 (2010)
15. Christodoulou, D., Lindblad, H.: On the motion of the free surface of a liquid. *Commun. Pure Appl. Math.* **53**, 1536–1602 (2000)
16. Coutand, D., Shkoller, S.: Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Am. Math. Soc.* **20**, 829–930 (2007)
17. Coutand, D., Shkoller, S.: A simple proof of well-posedness for the free-surface incompressible Euler equations. *Discrete Contin. Dyn. Syst. Ser. S* **3**, 429–449 (2010)
18. Coutand, D., Shkoller, S.: On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. *Commun. Math. Phys.* **325**, 143–183 (2014)
19. Dacorogna, B., Moser, J.: On a partial differential equation involving the Jacobian determinant. *Ann. Inst. Henri Poincaré (C) Anal. Nonlinéaire* **7**, 1–26 (1990)
20. Disconzi, M., Ebin, D.: On the limit of large surface tension for a fluid motion with free boundary. *Commun. Partial Differ. Equ.* **39**, 740–779 (2014)
21. Disconzi, M., Ebin, D.: *The free boundary Euler equations with large surface tension* (2015). arXiv:1506.02094
22. Fefferman, C., Ionescu, A.D., Lie, V.: *On the absence of "splash" singularities in the case of two-fluid interfaces* (2013, preprint). arXiv:1312.2917
23. Gancedo, F., Strain, R.: *Absence of splash singularities for SQG sharp fronts and the Muskat problem* (2013, preprint). arXiv:1309.4023
24. Germain, P., Masmoudi, N., Shatah, J.: Global solutions for the gravity water waves equation in dimension 3. *C. R. Math. Acad. Sci. Paris* **347**, 897–902 (2009)
25. Hunter, J., Ifrim, M., Tataru, D.: *Two dimensional water waves in holomorphic coordinates* (2014). arXiv:1401.1252
26. Ifrim, M., Tataru, D.: *Two dimensional water waves in holomorphic coordinates II: global solutions* (2014). arXiv:1404.7583
27. Ifrim, M., Tataru, D.: *The lifespan of small data solutions in two dimensional capillary water waves* (2014). arXiv:1406.5471
28. Ionescu, A.D., Pusateri, F.: Global solutions for the gravity water waves system in 2d. *Invent. Math.* (2014). arXiv:1303.5357
29. Lannes, D.: Well-posedness of the water-waves equations. *J. Am. Math. Soc.* **18**, 605–654 (2005)
30. Lindblad, H.: Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. Math.* **162**, 109–194 (2005)
31. Nalimov, V.I.: The Cauchy–Poisson problem (in Russian). *Dynamika Splosh. Sredy* **18**, 104–210 (1974)
32. Pusateri, F.: On the limit as the surface tension and density ratio tend to zero for the two-phase Euler equations. *J. Hyperbolic Differ. Equ.* **8**, 347–373 (2011)

33. Rayleigh, L.: On the instability of jets. *Proc. Lond. Math. Soc.* **s1–s10**(1), 4–13 (1878)
34. Shatah, J., Zeng, C.: Geometry and a priori estimates for free boundary problems of the Euler equation. *Commun. Pure Appl. Math.* **61**, 698–744 (2008)
35. Shatah, J., Zeng, C.: Local well-posedness for fluid interface problems. *Arch. Rational Mech. Anal.* **199**, 653–705 (2011)
36. Taylor, G.: The instability of liquid surfaces when accelerated in a direction perpendicular to their planes I. *Proc. R. Soc. Lond. Ser. A.* **201**, 192–196 (1950)
37. Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.* **130**, 39–72 (1997)
38. Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Am. Math. Soc.* **12**, 445–495 (1999)
39. Wu, S.: Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.* **177**, 45–135 (2009)
40. Wu, S.: Global wellposedness of the 3-D full water wave problem. *Invent. Math.* **184**, 125–220 (2011)
41. Yosihara, H.: Gravity waves on the free surface of an incompressible perfect fluid. *Publ. RIMS Kyoto Univ.* **18**, 49–96 (1982)
42. Zhang, P., Zhang, Z.: On the free boundary problem of three-dimensional incompressible Euler equations. *Commun. Pure Appl. Math.* **61**, 877–940 (2008)

CANPDE, Maxwell Institute for Mathematical Sciences,
Department of Mathematics,
Heriot-Watt University,
Edinburgh,
EH14 4AS, UK
e-mail: D.Coutand@ma.hw.ac.uk

and

Mathematical Institute,
University of Oxford,
Andrew Wiles Building,
Radcliffe Observatory Quarter,
Woodstock Road,
Oxford, OX2 6GG, UK

and

Present address:
Department of Mathematics,
University of California,
Davis, CA 95616,
USA
e-mail: shkoller@math.ucdavis.edu

(*Received September 15, 2014 / Accepted February 6, 2016*)
Published online February 24, 2016 –