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REGULARITY OF THE VELOCITY FIELD FOR EULER VORTEX PATCH EVOLUTION

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ABSTRACT. We consider the vortex patch problem for both the 2-D and 3-D incompressible Euler equations. In 2-D, we prove that for vortex patches with $H^{k-0.5}$ Sobolev-class contour regularity, $k \geq 4$, the velocity field on both sides of the vortex patch boundary has H^k regularity for all time. In 3-D, we establish existence of solutions to the vortex patch problem on a finite-time interval $[0,T]$, and we simultaneously establish the $H^{k-0.5}$ regularity of the two-dimensional vortex patch boundary, as well as the H^k regularity of the velocity fields on both sides of vortex patch boundary, for $k \geq 3$.

1. INTRODUCTION

1.1. **The incompressible Euler equations.** Global existence for the Euler 2-D vortex patch problem was first established by Chemin [4,5], Bertozzi & Constantin [3], and Serfati [18]; see also [1,2,9,11,14] for further results on the 2-D vortex patch. Local existence for the 3-D vortex patch problem was first proved by Gamblin & Saint Raymond [13]; see also [12,15,22,23]. A very nice summary of results on vortex patch problems can be found in [20].

We are interested in the regularity properties of the velocity field associated to the vortex patch evolution. In particular, we analyze the incompressible Euler equations on $\mathbb{R}^{\mathrm{n}}$, $\mathrm{n} = 2, 3$, written as

$$u_t + D_u u + Dp = 0, \tag{1.1a}$$

$$\operatorname{div} u = 0, \tag{1.1b}$$

where $u(x,t)$ is the velocity vector field and $p(x,t)$ is the pressure function, where the advection term $D_u u$ denotes $\sum_{j=1}^{\mathrm{n}} \frac{\partial u}{\partial x_j} u^j$.

1.2. **The 2-D vortex patch problem.** Letting $D^\perp = (-\partial_{x_2}, \partial_{x_1})$, we define the 2-D vorticity function $\omega(x,t) = D^\perp \cdot u(x,t) = u^2,_1 - u^1,_2$. The vorticity ω is transported and satisfies

$$\omega_t + D_u \omega = 0\,. \tag{1.2}$$

Letting $\psi(x,t)$ denote the stream function, given by $u = D^\perp \psi$, we have that $\Delta\psi = \omega$, so that $\psi(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\omega(y)dy$. Thanks to the Biot-Savart kernel

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$K(x) = \frac{1}{2\pi} D^\perp \log|x|$,

$$u(x,t) = \int_{\mathbb{R}^2} K(x-y)\omega(y)dy\,. \tag{1.3}$$

For each time $t \in [0,\infty)$, let $\Omega^+(t)$ denote an open, simply-connected, and bounded subset of $\mathbb{R}^2$ with boundary $\Gamma(t) := \partial\Omega^+(t)$ given by a closed curve which is diffeomorphic to the circle $\mathbb{S}^1$. Let $\Omega^-(t)$ denote $\overline{\Omega^+(t)}^c$. The *2-D vortex patch problem* consists of the following initial data for the Euler equations:

$$\omega_0(x) = \begin{cases} 1, & x \in \overline{\Omega^+(0)}, \\ 0, & x \in \Omega^-(0)\,. \end{cases} \tag{1.4}$$

The time-dependent open set $\Omega^+(t)$ is thus termed the *vortex patch*; the vortex patch boundary $\Gamma(t) := \partial\Omega^+(t)$ moves with the velocity of the fluid, given by $u(x,t) = \int_{\Omega^+(t)} K(x-y)dy$. It follows that

$$Du(x,t) = \int_{\Omega^+(t)} DK(x-y)dy\,. \tag{1.5}$$

Given an initial 2-D vortex patch boundary $\Gamma(0)$ of Hölder-class $\mathcal{C}^{k,\alpha}$, it was established by Chemin [4] and Bertozzi & Constantin [3] that a unique solution exists for all time, that the $\mathcal{C}^{k,\alpha}$ contour regularity propagates, and that the gradient of the velocity remains bounded for all time. Their proof of $\mathcal{C}^{k,\alpha}$ contour regularity (in 2-D) can also be used to establish H^k contour regularity (we provide a proof for the n-dimensional case, n = 2 or 3 in Section 5), and we state one of their fundamental results as follows: Given an initial vortex patch boundary $\Gamma(0)$ of class $H^{k-0.5}$, $k \geq 3$, for all $t \in [0,\infty)$, there exists a unique solution to the vortex patch problem, with non-self-intersecting boundary $\Gamma(t)$, and satisfying the following estimate:

$$\frac{1}{|z|_*(t)} + \|z(\cdot,t)\|_{H^{k-0.5}(\mathbb{S}^1)} + \|Du(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \leq F(t), \tag{1.6}$$

where $z(\cdot,t) : \mathbb{S}^1 \to \Gamma(t)$ denotes an $H^{k-0.5}$-class parameterization of the vortex patch boundary $\Gamma(t)$,

$$|z|_*(t) = \inf_{\theta_1 \neq \theta_2} \frac{|z(\theta_1,t) - z(\theta_2,t)|}{|\theta_1 - \theta_2|}, \tag{1.7}$$

and $0 < F(t) < \infty$ for any $t < \infty$. We see that (1.6) provides a strictly positive lower-bound on $|z|_*(t)$ which, in turn, provides a strictly positive lower-bound for the metric $|\partial_\theta z(\theta)|$ and ensures that $\Gamma(t)$ does not self-intersect (see, for example, Majda & Bertozzi [17]). We identity $\mathbb{S}^1$ with the interval $[0,2\pi]$.

1.3. **The 3-D vortex patch problem.** In three space dimensions, the 3-D vorticity $\omega = \operatorname{curl} u$ is a vector field, and satisfies the vector equation

$$\omega_t + D_u\omega = D_\omega u, \tag{1.8}$$

where in components and for each $i = 1,2,3$, $[D_u\omega]^i = \sum_{j=1}^3 \frac{\partial \omega^i}{\partial x_j} u^j$ and $[D_\omega u]^i = \sum_{j=1}^3 \frac{\partial u^i}{\partial x_j}\omega^j$.

Letting $\psi(x,t)$ denote the vector stream function, given by $u = -\operatorname{curl}\psi$, we have that $\Delta\psi = \omega$, and hence $\psi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\omega(y)}{|x-y|}dy$. It follows that

$$u(x,t) = \int_{\mathbb{R}^2} \mathcal{K}(x-y)\omega(y)dy, \tag{1.9}$$

where $\mathcal{K}(x) = \frac{1}{4\pi}\frac{x\times\cdot}{|x|^3}$ is the Biot-Savart 3x3 matrix kernel.

What type of vortex evolution in three space dimension is analogous to the 2-D vortex patch problem? The answer is as follows: we suppose that at time $t=0$, $\Omega^+(0)$ denotes an open bounded subset of $\mathbb{R}^3$ which is diffeomorphic to a $\mathcal{C}^\infty$, connected, bounded, open set B (so that the boundary ∂B is a smooth surface, which can be a sphere, a donut, etc.). We then let $\Gamma(0) = \partial\Omega^+(0)$, and define $\Omega^-(0) = \overline{\Omega^+(0)}^c$. We choose an initial divergence-free velocity field $u_0(x) = u_0^+(x)\mathbf{1}_{\overline{\Omega^+(0)}} + u_0^-(x)\mathbf{1}_{\Omega^-(0)}$ such that the initial vorticity vector $\omega_0 = \operatorname{curl} u_0 \in L^\infty(\mathbb{R}^3)$ and satisfies

$$\omega_0(x) = \begin{cases} \operatorname{curl} u_0^+(x), & x \in \overline{\Omega^+(0)}, \\ \operatorname{curl} u_0^-(x), & x \in \Omega^-(0), \end{cases} \tag{1.10a}$$

$$[\![\omega_0 \cdot n(\cdot,0)]\!] = 0, \tag{1.10b}$$

where $n(\cdot,0)$ denotes the outward unit normal to $\partial\Omega^+(0)$. If $[\![\omega_0(x)\times n(x,0)]\!] \neq 0$ for some $x\in\Gamma(0)$, then the tangential components of ω_0 are discontinuous, while the velocity u_0 is continuous across $\Gamma(0)$. The 3-D analogue of a 2-D vortex patch amounts to choosing u_0 in such a way that $\operatorname{curl} u_0^- = 0$ on $\Omega^-(0)$ and hence, necessarily, $\operatorname{curl} u_0^+ \cdot n(0) = 0$ so that ω_0 is tangent to $\Gamma(0)$.

To explain this analogy, we first state the following existence theorem for the Euler equations (1.1) with initial data $u(x,0) = u_0(x)$. Gamblin & Saint Raymond [13] proved that whenever $\Gamma(0)$ is $\mathcal{C}^{1,\alpha}$, $\alpha\in(0,1)$, $u_0 \in L^p(\mathbb{R}^3)$, $1<p<\infty$, and $\omega_0 \in L^q(\mathbb{R}^3)$, $1\le q<3$ such that ω_0 has $\mathcal{C}^\alpha$ regularity in directions tangent to $\Gamma(0)$; then there exists a unique solution $u \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^3)) \cap W^{1,\infty}(0,T;L^p(\mathbb{R}^3))$ to (1.1). Furthermore, letting $\eta(x,t)$ denote the Lagrangian flow of u, so that

$$\partial_t\eta(x,t) = u(\eta(x,t),t) \quad \text{for} \quad t>0, \tag{1.11a}$$

$$\eta(x,0) = x, \tag{1.11b}$$

and for each $t\in(0,T]$, setting $\Gamma(t) = \eta(\Gamma(0),t)$; then $\Gamma(t)$ is a closed surface of class $\mathcal{C}^{1,\alpha}$ and $\omega(t)\in L^q(\mathbb{R}^3)$ such that $\omega(t)$ has $\mathcal{C}^\alpha$ regularity in directions tangent to $\Gamma(t)$.

For each $t\in[0,T]$, the Lagrangian flow $\eta(\cdot,t)$ is a diffeomorphism with Jacobian determinant $\det D\eta(x,t) = 1$. We set $\Omega^+(t) = \eta(\Omega^+(0),t)$ and $\Omega^-(t) = \eta(\Omega^-(0),t)$. Integrating the vorticity equation (1.8), we see that

$$\omega(\eta(x,t),t) = D\eta(x,t)\cdot\omega_0(x), \tag{1.12}$$

where in components, $[D\eta\cdot\omega_0]^i = \sum_{j=1}^3 \frac{\partial\eta^i}{\partial x_j}\omega_0^j$.

We will set the 3-D vortex patch problem inside of a periodic box. We let Ω denote a periodic box $[-\ell,\ell]^3$ in $\mathbb{R}^3$ with opposite sides of the box identified with one another, and with ℓ taken sufficiently large so that $\overline{\Omega^+(0)} \subset \Omega$. Functions defined on Ω are 2ℓ-periodic in each of the three coordinate directions, i.e.,

$$u(x+2\ell e_i) = u(x) \quad \forall x\in\mathbb{R}^3, i=1,2,3,$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

The *3-D vortex patch problem* has the following initial data:

$$\Gamma(0) \text{ is a closed surface diffeomorphic to } \partial B, \tag{1.13a}$$

$$\Omega^+(0) \text{ is an open set diffeomorphic to } B \subset \mathbb{R}^3, \tag{1.13b}$$

$$\Omega^-(0) = \Omega - \overline{\Omega^+(0)}, \tag{1.13c}$$

$$u_0(x) = u_{0+}(x)\mathbf{1}_{\overline{\Omega^+(0)}} + u_{0-}(x)\mathbf{1}_{\Omega^-(0)}, \tag{1.13d}$$

$$\operatorname{div} u_0 = 0, \tag{1.13e}$$

$$\omega_0 = \operatorname{curl} u_0, \tag{1.13f}$$

$$\omega_0(x) = \begin{cases} \operatorname{curl} u_0^+(x), & x \in \overline{\Omega^+(0)}, \\ 0, & x \in \Omega^-(0), \end{cases} \tag{1.13g}$$

$$\operatorname{curl} u_0^+ \cdot n(\cdot, 0) = 0 \text{ on } \Gamma(0), \tag{1.13h}$$

$$\int_\Omega u_0(x)dx = 0\,. \tag{1.13i}$$

We then call $\Omega^+(0)$ the initial *vortex patch* and $\Gamma(0)$ the initial vortex patch boundary. The identity (1.12) shows that for each $t \in [0,T]$, $\omega(\cdot,t) = 0$ in $\Omega^-(t)$ and that $\omega(\cdot,t)\cdot n(\cdot,t) = 0$ on $\Gamma(t)$. In particular, if the initial vorticity is supported in a set which is diffeomorphic to B, then the vorticity stays supported in a set diffeomorphic to B for all time $t \in [0,T]$ for which the solution exists. In (1.13), we could instead set $\Omega^-(0) = \mathbb{R}^3 - \overline{\Omega^+(0)}$.

Of particular interest are those solutions for which $\operatorname{curl} u_0^+(x) \times n(x,0) \neq 0$ for almost all points $x \in \Gamma(0)$.

1.4. **Statement of the main result.** Because of the singular nature of DK, it is difficult to establish regularity for higher-order derivatives of u with the formula (1.5). By taking a different approach, however, we shall prove that the velocity field indeed enjoys higher-order Sobolev regularity on both sides of the vortex patch boundary. In particular, for the 2-D vortex patch problem defined in Section 1.2, we have the following

Theorem 1 (Regularity of velocity field in 2-D)**.** *Given initial data* (1.4) *and a global-in-time solution to the 2-D vortex patch problem satisfying*

$$\frac{1}{|z|_*(t)} + \|z(\cdot,t)\|_{H^{k-0.5}(\mathbb{S}^1)} + \|Du(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \le F(t)$$

for $t \in [0,\infty)$ and $k \ge 4$, the velocity field satisfies $u^+(\cdot,t) \in H^k(\Omega^+(t))$ and $u^-(\cdot,t) \in H^k_{\mathrm{loc}}(\Omega^-(t))$, and

$$\|u^+(\cdot,t)\|_{H^k(\Omega^+(t))} + \|u^-(\cdot,t)\|_{H^k(\Omega^-(t))\cup B(0,R(t)))} \le G(t),$$

where $B(0,R(t))$ is a ball centered at 0 with radius $R(t) > 0$ such that $\Gamma(t) \subset B(0,R(t))$, and $G(t) > 0$ is a function of $F(t)$, defined in (1.6)*, with $G(t) < \infty$ for any $t < \infty$.*

Remark 1. Notice that both velocity vector fields u^+ and u^- gain a half-derivative of regularity with respect to the regularity of the vortex patch boundary $\Gamma(t)$. This is very natural in Sobolev spaces H^k, but requires us to locally extend our 1-D parameterization $z(\cdot,t)$ to a 2-D local diffeomorphism $\theta^+(\cdot,t)$ and $\theta^-(\cdot,t)$ which also gains a half-derivative of regularity. This is accomplished by a specially chosen elliptic extension which we describe in Section 3. On the other hand, if we had assumed instead that the parameterization $z(\cdot,t) \in H^k(\mathbb{S}^1)$, then a standard local "graph" extension would have sufficed. More specifically, if $z(\cdot,t)$ is given locally by the graph $(x_1, h(x_1))$, then $(x_1, x_2 + h(x_1))$ provides a local extension to a diffeomorphism, but does not gain a half-derivative of regularity.

Remark 2. Without any change to our proof, the initial data (1.4) can be replaced by the more general initial data

$$\omega_0(x) = \begin{cases} \omega_0^+(x), & x \in \overline{\Omega^+(0)}, \\ \omega_0^-(x), & x \in \Omega^-(0), \end{cases}$$

for any functions $\omega_0^+ \in H^{k-1}(\Omega_0^+)$ and $\omega_0^- \in H^{k-1}_{\text{loc}}(\Omega_0^-)$, $k \geq 4$.

Remark 3. In fact, Theorem 1 is true for $k \geq 3$, but the proof requires one less regularization step for $k \geq 4$.

Whereas Chemin [5] and Bertozzi & Constantin [3] have established regularity of the contour $\Gamma(t)$ for the 2-D vortex patch problem, the regularity of the 3-D vortex patch boundary $\Gamma(t)$ is considered in $\mathcal{C}^{1,\alpha}$ in the analysis of Gamblin & Saint Raymond [13] and in Besov spaces by Danchin [10] for fluids in dimension $d \geq 2$. As our final result, we simultaneously establish an existence theory in Sobolev spaces for the 3-D vortex patch problem, as well as the Sobolev-class regularity of the 2-D closed surface $\Gamma(t)$ and the velocity fields u^+ and u^-.

Theorem 2 (Existence and regularity for the 3-D vortex patch boundary and velocity fields)**.** *For $k \geq 3$, if $\Gamma(0)$ is a closed surface of Sobolev-class $H^{k-0.5}$, and $u_0 \in H^1(\Omega)$ with $u_0^+ \in H^k(\Omega^+(0))$, $u_0^- \in H^k(\Omega^-(0))$ and satisfying* (1.13)*, then there is a time $T > 0$ such that there exists a unique solution to the 3-D vortex patch problem, and for each $t \in [0,T]$, the vortex patch boundary $\Gamma(t)$ is in $H^{k-0.5}$, $u^+(\cdot,t) \in H^k(\Omega^+(t))$, and $u^-(\cdot,t) \in H^k(\Omega^-(t))$.*

Remark 4. The more general initial data (1.10) can replace (1.13) in Theorem 2.

Notation. We will denote the partial derivative $\frac{\partial f}{\partial x_j}$ by $f_{,j}$ for $j = 1$, 2, or 3. We will use the Einstein summation convention, wherein repeated indices are summed from 1 to n, with n equaling either 2 or 3.

1.5. **Outline of the paper.** In Section 2, we define the strong form of the two-phase elliptic problem that the two-dimensional stream function must satisfy, and we also define the associated variational formulation. In Section 3, we define the local diffeormorphisms that we use to locally flatten the vortex patch boundary; these diffeomorphisms gain one-half derivative of interior regularity in H^k spaces relative to the regularity of the vortex patch boundary. Section 4 is devoted to the Sobolev regularity theory of the fluid velocities $u^+(\cdot,t)$ and $u^-(\cdot,t)$ in the 2-D vortex patch problem.

The 3-D vortex patch problem is studied in Section 5. After defining the two-phase elliptic problem for the fluid velocity, we simultaneously prove existence of solutions and establish the regularity theory for both the vortex patch boundary $\Gamma(t)$ and the velocity fields $u^+(\cdot,t)$ and $u^-(\cdot,t)$; this is done in the Lagrangian framework. Finally, in Section 6, we establish the fundamental regularity estimates for the two-phase elliptic problem with Sobolev-class coefficients in n-dimensions (which arises in many applications, including the vortex patch problem). For completeness, we include a short appendix with some basic inequalities that are used in Section 6.

2. A two-phase elliptic problem for the 2-D stream function

The 2-D vortex patch problem has been previously studied using the evolution equation for the parameterization of the contour $z(\cdot,t)$ ([3, 4]); see also [6] for perturbations of circular patches and [7] for elliptical patches. We will take a different approach.

While not necessary, it is convenient to introduce the stream function formulation of the problem. Let $\psi(x,t) = \psi^+(x,t)\mathbf{1}_{\overline{\Omega^+(t)}} + \psi^-(x,t)\mathbf{1}_{\Omega^-(t)}$. We set $[\![F]\!] = F^+ - F^-$ on $\Gamma(t)$, and let $n(\cdot,t)$ denote the outward unit normal to $\Gamma(t)$, and $\tau(\cdot,t)$ denote the unit tangent vector to $\Gamma(t)$.

For each time $t\in[0,\infty)$, the bounds (1.6) show that $Du(\cdot,t)\in L^\infty(\mathbb{R}^2)$; thus, $u(\cdot,t)\in H^1_{\rm loc}(\mathbb{R}^2)$ and so the stream function $\psi(\cdot,t)\in H^2_{\rm loc}(\mathbb{R}^2)$ is a solution to the following two-phase elliptic problem for each fixed $t\in[0,\infty)$:

$$-\Delta\psi^+(\cdot,t) = -1 \quad \text{in} \quad \Omega(t)^+, \tag{2.1a}$$

$$\Delta\psi^-(\cdot,t) = 0 \quad \text{in} \quad \Omega(t)^-, \tag{2.1b}$$

$$[\![\psi(\cdot,t)]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{2.1c}$$

$$\Big[\!\Big[\frac{\partial\psi}{\partial n}(\cdot,t)\Big]\!\Big] = 0 \quad \text{on} \quad \Gamma(t)\,. \tag{2.1d}$$

The fact that $\psi(\cdot,t)\in H^2_{\rm loc}(\mathbb{R}^2)$ means that the interface jump condition (2.1d) holds in $H^{0.5}(\Gamma(t))$.

For each time $t\in[0,\infty)$, (2.1) has the following weak formulation:

$$\int_{\Omega^+(t)} D\psi^+(\cdot,t)\cdot D\phi\,dx + \int_{\Omega^-(t)} D\psi^-(\cdot,t)\cdot D\phi\,dx = -\int_{\Omega^+(t)}\phi\,dx \quad \forall\phi\in H^1(\mathbb{R}^2)\,. \tag{2.2}$$

From the bounds (1.6), the stream-function satisfies

$$\|\psi(\cdot,t)\|_{H^2(B(0,R(t)))} \le F(t), \tag{2.3}$$

where $B(0,R(t))$ is a ball centered at 0 with radius $R(t)>0$ such that $\Gamma(t)\subset B(0,R(t))$.

3. Locally flattening the boundary $\Gamma(t)$

We construct local diffeomorphisms in small neighborhoods of $\Gamma(t)$ which locally "flatten" the vortex patch boundary, and which gain one-half derivative of regularity in the interior with respect to the regularity of the parameterization $z(\cdot,t)$. There are other methods to construct regularizing diffeomorphisms (see, for example, [8, 16, 19]), but the method we present appears quite natural for arbitrary geometries.

Let $D^+ = \{x\in\mathbb{R}^2 : |x|<1\}$ denote the open unit ball in $\mathbb{R}^2$ with boundary $\mathbb{S}^1 = \{x\in\mathbb{R}^2 : |x|=1\}$, the unit circle. For each $t\in[0,\infty)$, we solve the following elliptic equation for $Z(r,\theta,t)$:

$$\Delta^2 Z^+ = 0 \qquad \text{in } D^+, \tag{3.1a}$$

$$Z^+ = z \qquad \text{on } \mathbb{S}^1, \tag{3.1b}$$

$$\frac{\partial Z^+}{\partial r} = \frac{\partial z^\perp}{\partial\theta} \qquad \text{on } \mathbb{S}^1\,. \tag{3.1c}$$

The unique solution $Z^+(r,\theta,t)$ to (3.1) satisfies the estimate

$$\|Z^+(\cdot,\cdot,t)\|_{H^k(D^+)} \le C\|z(\cdot,t)\|_{H^{k-0.5}(\mathbb{S}^1)}, \tag{3.2}$$

and we are considering integers $k \geq 4$. The boundary conditions (3.1b,c) show that

$$\det DZ^+(1,\theta,t) = |\partial_\theta z(\theta,t)|^2 .$$

From the definition (1.7) of $|z|_*(t)$ and its lower-bound given by (1.6), it is proven in [17] that there exists a function $\alpha(t) > 0$ such that $\alpha(t) \leq \min_{\theta\in\mathbb{S}^1} |\partial_\theta z(\theta,t)|^2$. Hence, $\det DZ^+(1,\theta,t) \geq \alpha(t) > 0$. This shows that Z^+ is locally injective around each point on $\mathbb{S}^1$.

Next, we define $D^- = \{x \in \mathbb{R}^2 \ : \ 1 < |x| < R(t)\}$, where $R(t) > 0$ is chosen sufficiently large so that the ball $B(0,R(t))$ contains $\Gamma(t)$. We let $Z^-(r,\theta,t)$ solve

$$\Delta^2 Z^- = 0 \qquad \text{in } D^-, \tag{3.3a}$$

$$Z^- = z \qquad \text{on } \mathbb{S}^1, \tag{3.3b}$$

$$Z^- = \mathrm{Id} \qquad \text{on } \{r = R(t)\}, \tag{3.3c}$$

$$\frac{\partial Z^-}{\partial r} = \frac{\partial z^\perp}{\partial \theta} \qquad \text{on } \mathbb{S}^1, \tag{3.3d}$$

$$\frac{\partial Z^-}{\partial r} = e_r \qquad \text{on } \{r = R(t)\}, \tag{3.3e}$$

where e_r denotes the unit basis vector $(\cos\theta, \sin\theta)$. Again, we see that the unique solution $Z^-(r,\theta,t)$ to (3.3) satisfies the estimate

$$\|Z^-(\cdot,\cdot,t)\|_{H^k(D^-)} \leq C\|z(\cdot,t)\|_{H^{k-0.5}(\mathbb{S}^1)} . \tag{3.4}$$

We define the map $Z = Z^+\mathbf{1}_{\overline{D^+}} + Z^-\mathbf{1}_{D^-}$. Due to the boundary conditions (3.1b,c) and (3.3b,d) and the Sobolev embedding theorem, the map $(r,\theta) \mapsto Z(r,\theta,t)$ is $\mathcal{C}^1$, and for any point $\theta \in \mathbb{S}^1$, there exists a ball $B(\theta,\varepsilon(t)) \subset \mathbb{R}^2$, centered at θ with radius $\varepsilon(t) > 0$ taken sufficiently small, such that $Z(\cdot,\cdot,t)$ is injective on $B(\theta,\varepsilon(t))$.

Next, we show that for $\epsilon > 0$ sufficiently small, the image $Z^+(1-\epsilon,\theta,t)$ is contained in $\Omega^+(t)$, and similarly, that the image $Z^-(r,\theta,t)$ is contained in $\Omega^-(t)$. To that end, let $\theta_0(t)$ denote the point in $[0,2\pi]$ at which the maximum value of $z(\theta,t)\cdot e_2$ occurs. We assume that the tangent vector $\partial_\theta z(\theta_0(t),t) = \beta(t)\, e_1$ for some $\beta(t) > 0$ (for, otherwise, we can reverse the orientation of the parameterization). Hence, $\partial_\theta z^\perp(\theta_0(t),t) = \beta(t)\, e_2$. This shows that $\frac{\partial Z_2^+}{\partial x_2}(1,\theta_0(t),t) > 0$, which in turn implies that $Z_2^+(1-\epsilon,\theta_0(t),t) < Z_2^+(1,\theta_0(t),t)$ which proves that, for $\epsilon > 0$ sufficiently small, for all $r \in [1-\epsilon,1)$ and $\theta \in [\theta_0(t)-\epsilon, \theta_0(t)+\epsilon]$,

$$Z^+(r,\theta,t)\cdot e_2 < z(\theta_0(t),t)\cdot e_2 .$$

Therefore, Z^+ maps a local neighborhood of $\theta_0(t)$ (in D^+) into $\Omega^+(t)$. Since Z^+ is locally injective around $\mathbb{S}_1$, this means that the image of any $Z^+(1-\epsilon,\cdot,t)$ (for $\epsilon > 0$ small enough) stays in $\Omega^+(t)$, otherwise it would intersect $\Gamma(t)$, which we shall next prove cannot occur. Similarly, the image of any $Z^-(1+\epsilon,\cdot,t)$ stays in $\Omega^-(t)$.

We next prove that for $\epsilon > 0$ sufficiently small,

$$Z^+(1-\epsilon,\theta,t) \cap \Gamma(t) = \emptyset \quad \forall\theta \in \mathbb{S}^1 .$$

Since $\det DZ^+(1,\theta,t) \geq \alpha(t) > 0$ for all $\theta \in \mathbb{S}^1$, by the inverse function theorem, there exists a small ball $B(\theta,\mathcal{R}(\theta)) \subset \mathbb{R}^2$, centered at $\theta \in \mathbb{S}^1$ with radius $\mathcal{R}(\theta) > 0$, such that $Z^+(\cdot,\cdot,t)$ is a $\mathcal{C}^1$-diffeomorphism between $D^+ \cap B(\theta,\mathcal{R}(\theta))$ and $Z^+(D^+ \cap B(\theta,\mathcal{R}(\theta)),t)$, as well as a homeomorphism between $\overline{D^+ \cap B(\theta,\mathcal{R}(\theta))}$ and $Z^+(\overline{D^+ \cap B(\theta,\mathcal{R}(\theta))},t)$. Since the compact set $\mathbb{S}^1$ is covered by $\bigcup_{\theta\in\mathbb{S}^1} B(\theta,\mathcal{R}(\theta))$, we can extract a finite subcover $\bigcup_{i=1}^N B(\theta_i, \mathcal{R}(\theta_i))$, where θ_i, $i = 1, \ldots, N$ are points in $\mathbb{S}^1$.

Let $\mathfrak{A}^\epsilon = \{x \in \mathbb{R}^2 \;:\; 1 - \epsilon \leq |x| < 1\}$ denote an annulus. We choose $\epsilon > 0$ small enough so that $\mathfrak{A}^\epsilon \subset \bigcup_{i=1}^N B(\theta_i, \mathcal{R}(\theta_i))$. With $(r, \theta) \in \mathcal{A}^\epsilon$ fixed, we choose $i \in \{1, \ldots, N\}$ such that $(r, \theta) \in B(\theta_i, \mathcal{R}(\theta_i)) \cap D^+$. Since $Z^+(\cdot, \cdot, t)$ is a $\mathcal{C}^1$ diffeomorphism between $D^+ \cap B(\theta_i, \mathcal{R}(\theta_i))$ and $Z^+(D^+ \cap B(\theta_i, \mathcal{R}(\theta_i)), t)$, then $Z^+(r, \theta, t) \in Z^+(D^+ \cap B(\theta_i, \mathcal{R}(\theta_i)), t)$. Furthermore, as Z^+ is a homeomorphism between $\overline{D^+ \cap B(\theta_i, \mathcal{R}(\theta_i))}$ and $Z^+(\overline{D^+ \cap B(\theta_i, \mathcal{R}(\theta_i))}, t)$, then $Z^+(r, \theta, t) \notin Z^+(\partial[D^+ \cap B(\theta_i, \mathcal{R}(\theta_i))], t)$, which implies that

$$Z^+(r, \theta, t) \notin z([\theta_i - \mathcal{R}(\theta_i), \theta_i + \mathcal{R}(\theta_i)], t) \subset \Gamma(t).$$

In summary, we have shown that for $(r, \theta) \in B(\theta_i, \mathcal{R}(\theta_i)) \cap D^+$,

$$Z^+(r, \theta, t) \text{ is in the interior of } Z^+(\overline{D^+ \cap B(\theta_i, \mathcal{R}(\theta_i))}, t)$$

with

$$\text{diameter}\left(Z^+(\overline{D^+ \cap B(\theta_i, \mathcal{R}(\theta_i))}, t)\right) \leq 2\|DZ^+\|_{L^\infty(D^+)} \mathcal{R}(\theta_i)\,.$$

From the positive lower-bound (1.6) on the function $|z|_*(t)$ in (1.7), there exists $\epsilon_0 > 0$ such that for any $x \in \Gamma(t)$, $B(x, \epsilon_0) \cap \Omega^+(t)$ does not contain any point of $\Gamma(t)$; therefore, choosing the radius $\mathcal{R}(\theta)$ such that

$$2\|DZ^+\|_{L^\infty(D^+)} \mathcal{R}(\theta_i) < \epsilon_0,$$

(and increasing N if necessary) we have that $Z^+(D^+ \cap B(\theta_i, \mathcal{R}(\theta_i)), t)$ does not contain any point of $\Gamma(t)$, which shows that $Z^+(r, \theta, t) \notin \Gamma(t)$ as desired. A similar argument shows that for $(r, \theta) \in B(\theta_i, \mathcal{R}(\theta_i)) \cap D^-$, $Z^-(r, \theta, t)$ is contained in $\Omega^-(t)$.

Thus, for each $\theta_i \in \mathbb{S}^1$, $i \in \{1, \ldots, N\}$, let $\mathcal{U}_i(t) = B(\theta_i, \mathcal{R}(\theta_i)) \subset \mathbb{R}^2$, and let $\mathcal{V}_i(t) = Z(\mathcal{U}_i(t), t)$. The map Z is then a $\mathcal{C}^1$ diffeomorphism of $\mathcal{U}_i(t)$ onto $\mathcal{V}_i(t)$, and due to the estimates (3.2) and (3.4),

$$Z^\pm(\cdot, \cdot, t) : D^\pm \cap \mathcal{U}_i(t) \to \Omega^\pm(t) \cap \mathcal{V}_i(t) \text{ is an } H^k \text{ diffeomorphism}\,.$$

Next, we flatten the boundary of $\mathcal{U}_i(t) \cap \mathbb{S}^1$. For each $i \in \{1, \ldots, N\}$, $\mathcal{U}_i(t) \cap \mathbb{S}^1$ is a graph given by $(x_1, h_i(x_1, t))$ where each $h_i(\cdot, t)$ is $\mathcal{C}^\infty$. We define the C^∞ local diffeomorphisms $\vartheta_i^\pm(t)(x_1, x_2) = (x_1, x_2 \pm h_i(x_1, t))$ with $\det D\vartheta_i^\pm(t) = 1$, and we set

$$B_\pm^i = [\vartheta_i^\pm(t)]^{-1}(\mathcal{U}_i(t) \cap D^\pm) \quad \text{and} \quad B_0^i = [\vartheta_i^+(t)]^{-1}(\mathcal{U}_i(t) \cap \mathbb{S}^1)\,.$$

The set $B_0^i \subset \{x_2 = 0\}$ is a flat boundary.

Finally, we define $\theta_i^\pm(t) = Z^\pm(t) \circ \vartheta_i^\pm(t)$. Then

$$\theta_i^\pm(t) : B^\pm \to \Omega^\pm \cap \mathcal{V}_i(t) \text{ is an } H^k \text{ diffeomorphism,} \tag{3.5}$$

and thanks to (3.2), (3.4), and (1.6), for each $i \in \{1, \ldots, N\}$,

$$\left\| \frac{1}{\det D\theta_i^\pm(t)} \right\|_{L^\infty(B_\pm)} + \|\theta_i^\pm(t)\|_{H^k(B_\pm)} \leq \mathcal{P}(F(t)), \tag{3.6}$$

where $\mathcal{P}(F(t))$ denotes a generic polynomial function of $F(t)$. Furthermore, if we set $\theta_i(t) = \theta_i^+(t)\mathbf{1}_{\overline{B_+^i}} + \theta_i^-(t)\mathbf{1}_{B_-^i}$, then each $\theta_i(t) \in \mathcal{C}^1(B)$, where $B = B_+ \cup B_- \cup B_0$.

4. Regularity of the velocity field for 2-D vortex patches: Proof of Theorem 1

We first use the weak formulation (2.2) to build regularity of the stream-function $\psi^\pm$. Interior regularity of $\Psi^\pm$ on sets away from the patch boundary $\Gamma(t)$ is classical, so we focus our attention on regularity of $\Psi^\pm$ near $\Gamma(t)$. We will use the change-of-variables $\theta_i(t)$ given in (3.5).

Step 1 (The elliptic problem for $\Psi^\pm$ set on $B_\pm$). The weak formulation (2.2) can be written as

$$\int_{\mathcal{V}_i(t)\cap\Omega^+(t)} D\psi^+(\cdot,t)\cdot D\phi\, dx + \int_{\mathcal{V}_i(t)\cap\Omega^-(t)} D\psi^-(\cdot,t)\cdot D\phi\, dx = -\int_{\mathcal{V}_i(t)\cap\Omega^+(t)} \phi\, dx$$

for all test functions $\phi \in H^1_0(\mathcal{V}(t))$ and each $i \in \{1,\dots,N\}$.

With the collection of diffeomorphisms $\{\theta_i\}_{i=1}^N$ given in (3.5) for each $t\in[0,\infty)$, we define

$$A_i^\pm = [D\theta_i^\pm(t)]^{-1} \quad\text{and}\quad J_i^\pm(t) = \det D\theta_i^\pm(t),$$

and set

$$\mathcal{A}_i^\pm = J_i^\pm [A_i^\pm][A_i^\pm]^T\,.$$

It follows from (3.6), and (1.6) that for all $t\in[0,\infty)$, there exists a function $0<\lambda_i(t)$ such that

$$w^T\,\mathcal{A}_i^\pm(x)\,w \ge \lambda_i(t)|w|^2\ \forall w\in\mathbb{R}^2,\quad x\in B^i_\pm\,. \tag{4.1}$$

To establish (4.1), we drop the i subscript (and superscript), and let $\tilde w_\pm = J_\pm^{1/2} A_\pm\, w$. The left-hand side of (4.1) is simply $|\tilde w_\pm|^2$, and $w = J_\pm^{-1/2} D\theta^\pm \tilde w_\pm$; therefore,

$$\frac{|w|^2}{\|J_\pm^{-1/2} D\theta^\pm\|^2_{L^\infty(B^+)}} \le |\tilde w_\pm|^2,$$

so that $\lambda(t) = \|J_\pm^{-1/2}(t) D\theta^\pm\|^{-2}_{L^\infty(B_\pm)}$, which has a strictly positive lower-bound since $\lambda(t)^{-1} = \|J_\pm^{-1/2} D\theta^\pm\|^2_{L^\infty(B^+)} \le \mathcal{P}(F(t))$ by (3.6). Additionally, from (3.6),

$$\|\mathcal{A}_\pm\|_{H^{k-1}(B_\pm)} \le C\mathcal{P}(F(t))\,. \tag{4.2}$$

We set

$$\Psi^\pm = \psi^\pm\circ\theta,\quad \Phi = \phi\circ\theta\,.$$

Since $\phi\in H^1_0(\mathcal{V}(t))$ and each $\theta_i(t)\in\mathcal{C}^1(B)$, it follows that $\Phi\in H^1_0(B)$, and can thus be used as a test function. By another application of the change-of-variables formula, we then have that

$$\int_{B_+} \mathcal{A}_+^{kj}\Psi^+,_k(\cdot,t)\,\Phi,_j\ dx + \int_{B_-} \mathcal{A}_-^{kj}\Psi^-,_k(\cdot,t)\,\Phi,_j\ dx = -\int_{B_+} \Phi\, J_+\, dx\quad \forall\Phi\in H^1_0(B)\,. \tag{4.3}$$

Step 2 (H^3 regularity for ψ^+ and ψ^-). We set $k=4$ so that $\theta^\pm\in H^4(B_\pm)$ and first establish that each $\psi^\pm$ is H^3. We let $\{\zeta_i\}_{i=1}^N$ denote a smooth partition-of-unity, subordinate to the open cover $\mathcal{U}_i(t)$; in particular, $0\le\zeta_i\le1$ in $\mathcal{C}_c^\infty(\mathcal{U}_i(t))$

denote a smooth cut-off function, $\sum_{i=1}^N \zeta_i = 1$, and let $\xi_i = \zeta_i \circ \theta_i(t)$. We define the horizontal convolution operator as follows: for $\epsilon > 0$ sufficiently small,

$$\Lambda_\epsilon F = \int_{\mathbb{R}^{n-1}} \rho_\epsilon(x_1 - y_1) F(y_1, x_2) dy_1,$$

where $\rho_\epsilon(x_1) = \epsilon^{-1}\rho(x_1/\epsilon)$, and ρ is the standard mollifier on $\mathbb{R}$. We again drop the i subscript, and substitute

$$\Phi = \xi^2 \Lambda_\epsilon^2 \partial_1^4(\xi^2 \Psi) \in H_0^1(B), \quad \Psi = \mathbf{1}_{\overline{B_+}} \Psi^+ + \mathbf{1}_{B_-} \Psi^-,$$

into (4.3). Since differentiation commutes with convolution, we have that

$$\Phi,_j = \Lambda_\epsilon^2 \partial_1^4(\xi^2\Psi),_j + 2\xi\xi,_j \Lambda_\epsilon^2 \partial_1^4(\xi^2\Psi)\,.$$

The variational formulation (4.3) then takes the following form:

$$\mathcal{I}_1^\pm + \mathcal{I}_2^\pm = -\int_{B_+} \partial_1^2 J_+ \; \xi^2 \Lambda_\epsilon^2 \partial_1^4(\xi^2\Psi)\; dx, \tag{4.4}$$

where

$$\mathcal{I}_1^\pm = \int_{B_\pm} \Lambda_\epsilon(\xi^2 \mathcal{A}_\pm^{kj} \Psi^\pm,_k),_{11} \; \Lambda_\epsilon^2(\xi^2\Psi^\pm),_{j11}\; dx,$$
$$\mathcal{I}_2^\pm = -2\int_{B_\pm} (\xi\xi,_j \mathcal{A}_\pm^{kj} \Psi^\pm,_k),_1 \; \Lambda_\epsilon^2(\xi^2\Psi^\pm),_{11}\; dx\,.$$

Next, we see that

$$\begin{aligned}\mathcal{I}_1^\pm = & \underbrace{\int_{B_\pm} \mathcal{A}_\pm^{kj} \Lambda_\epsilon(\xi^2\Psi^\pm),_{k11}\; \Lambda_\epsilon(\xi^2\Psi^\pm),_{j11}\; dx}_{\mathcal{I}_1{}_a^\pm} \\ & + \underbrace{\int_{B_\pm} \left(\{\Lambda_\epsilon, \mathcal{A}_\pm^{kj}\}(\xi^2\Psi^\pm),_{k11}\right) \Lambda_\epsilon(\xi^2\Psi^\pm),_{j11}\; dx}_{\mathcal{I}_1{}_b^\pm} \\ & + \underbrace{\int_{B_\pm} \Lambda_\epsilon \Big[2\mathcal{A}_\pm^{kj},_1 (\xi^2\Psi),_{k1} + \mathcal{A}_\pm^{kj},_{11} (\xi^2\Psi),_k - 2(\xi\xi,_k \mathcal{A}_\pm^{kj}\Psi),_{11}\Big] \Lambda_\epsilon(\xi^2\Psi^\pm),_{j11}\; dx}_{\mathcal{I}_1{}_c^\pm},\end{aligned}$$

where

$$\{\Lambda_\epsilon, \mathcal{A}_\pm^{kj}\}(\xi^2\Psi^\pm),_{k11} = \Lambda_\epsilon(\mathcal{A}_\pm^{kj}(\xi^2\Psi^\pm),_{k11}) - \mathcal{A}_\pm^{kj}\, \Lambda_\epsilon(\xi^2\Psi^\pm),_{k11} \tag{4.5}$$

denotes the commutator of the horizontal convolution operator and multiplication by $\mathcal{A}_\pm^{kj}$. Using the lower-bound (4.1), we see that

$$\lambda(t) \|\partial_1^2 \Lambda_\epsilon D(\xi^2\Psi^\pm)\|^2_{L^2(B_+)} \le \mathcal{I}_1{}_a^\pm\,. \tag{4.6}$$

We let $0 < \delta \ll 1$; we will make use of the Cauchy-Young inequality $a\,b \le \delta\lambda(t) a^2 + \frac{1}{4\delta\lambda(t)} b^2$ for $a, b \ge 0$.

Using Hölder's inequality together with the Sobolev inequality $\|f\|_{L^p(B_\pm)} \le C\|f\|_{H^1(B_\pm)}$ for all $f \in H^1(B_\pm)$ and all $p \in [1, \infty)$, we have that

$$|\mathcal{I}_1{}_c^\pm| \le C \|\mathcal{A}_\pm^{kj},_{11}\|_{H^1(B_\pm)} \|\Psi^\pm,_k\|_{H^1(B_\pm)} \|\Lambda_\epsilon(\xi^2\Psi^\pm),_{j11}\|_{L^2(B_\pm)}\,.$$

Thanks to (4.2) and (2.3), we then infer that

$$\begin{aligned}|\mathcal{I}_{1c}^{\pm}| &\le \mathcal{P}(F(t))\|\Lambda_\epsilon(\xi^2\Psi^\pm),_{j11}\|_{L^2(B_\pm)}\\ &\le \delta\lambda(t)\|\Lambda_\epsilon(\xi^2\Psi^\pm),_{j11}\|^2_{L^2(B_\pm)} + \frac{\mathcal{P}(F(t))}{4\delta\lambda(t)}\\ &\le \delta\lambda(t)\|\Lambda_\epsilon(\xi^2\Psi^\pm),_{j11}\|^2_{L^2(B_\pm)} + (1+(\delta\lambda(t))^{-1})\mathcal{P}(F(t)),\end{aligned}$$

where we continue to use $\mathcal{P}(F(t))$ to denote a generic polynomial function of $F(t)$. A similar estimate can be established for the integral $\mathcal{I}_2^{\pm}$, which provides us with

$$|\mathcal{I}_{1c}^{\pm}| + |\mathcal{I}_2^{\pm}| \le [1+(\delta\lambda(t))^{-1}]\mathcal{P}(F(t)) + \delta\lambda(t)\|\Lambda_\epsilon(\xi^2\Psi^\pm),_{11}\|^2_{H^1(B_\pm)}\,. \tag{4.7}$$

Also, the integral on the right-hand side of (4.4) has the same upper bound.

It remains to establish such an upper-bound for $|\mathcal{I}_{1b}|$. We set $g_k^\pm = (\xi^2\Psi^\pm),_{k11}$; then,

$$\begin{aligned}&[\Lambda_\epsilon(\mathcal{A}_\pm^{kj}\, g_k^\pm) - \mathcal{A}_\pm^{kj}\,\Lambda_\epsilon g_k^\pm](x_1,x_2)\\ &\quad= \int_{x_1-\epsilon}^{x_1+\epsilon} \rho_\epsilon(x_1-y_1)[\mathcal{A}_\pm^{kj}(y_1,x_2) - \mathcal{A}_\pm^{kj}(x_1,x_2)]g_k^\pm(y_1,x_2)dy_1\,.\end{aligned} \tag{4.8}$$

From Morrey's inequality, for all $y_1 \in B(x_1,\epsilon)$,

$$|\mathcal{A}_\pm^{kj}(x_1,x_2) - \mathcal{A}_\pm^{kj}(y_1,x_2)| \le C\epsilon \sup_{y_1\in(x_1-\epsilon,x_1+\epsilon)} |\mathcal{A}_\pm^{kj},_1(y_1,x_2)| \le C\epsilon\|\mathcal{A}\|_{H^3(B_\pm)}\,. \tag{4.9}$$

Substituting (4.9) into (4.8) and using Young's inequality for convolution, together with (4.2), we see that

$$\begin{aligned}\|\{\Lambda_\epsilon,\mathcal{A}_\pm^{kj}\}(\xi^2\Psi^\pm),_{k11}\|_{L^2(B_\pm)} &\le C\epsilon\|\mathcal{A}_\pm\|_{H^3(B_\pm)}\|(\xi^2 D\Psi^\pm),_{11}\|_{L^2(B_\pm)}\\ &\le C\epsilon\mathcal{P}(F(t))\|D(\xi^2\Psi^\pm),_{11}\|_{L^2(B_\pm)},\end{aligned}$$

so that

$$|\mathcal{I}_{1b}| \le C\epsilon\mathcal{P}(F(t))\|D(\xi^2\Psi^\pm),_{11}\|^2_{L^2(B_\pm)}\,.$$

We choose ϵ sufficiently small so that $C\epsilon\mathcal{P}(F(t)) < \lambda(t)/2$. By choosing $\delta > 0$ sufficiently small, we obtain from (4.4), (4.6) and (4.7) the estimate

$$\int_{B_\pm} |D\Lambda_\epsilon(\xi^2\Psi^\pm),_{11}|^2\,dx \le [1+\lambda(t)^{-1}]\mathcal{P}(F(t))\,.$$

Passing to the limit as $\epsilon \to 0$, we find that

$$\int_{B_\pm} \xi^2|D\Psi^\pm,_{11}|^2\,dx \le [1+\lambda(t)^{-1}]\mathcal{P}(F(t))\,. \tag{4.10}$$

From (2.1a,b), we have the following identity holding at any point of the interior of $B_\pm$:

$$\begin{aligned}-\mathcal{A}_\pm^{22}\Psi^\pm,_{221} &= 2\mathcal{A}_\pm^{21}\Psi^\pm,_{211} + \mathcal{A}_\pm^{11}\Psi^\pm,_{111} + 2\mathcal{A}_\pm^{21},_1\Psi^\pm,_{21} + \mathcal{A}_\pm^{11},_1\Psi^\pm,_{11}\\ &\quad+ \mathcal{A}_\pm^{22},_1\Psi^\pm,_{22} + \mathcal{A}_\pm^{jk},_{j1}\Psi^\pm,_k + \mathcal{A}_\pm^{jk},_j\Psi^\pm,_{k1}\,.\end{aligned}$$

The lower-bound (4.1) shows that $\mathcal{A}_\pm^{22} \ge \lambda(t)$; hence from (4.10), (4.2), and (2.3),

$$\int_{B_\pm} \xi^2|\Psi^\pm,_{221}|^2\,dx \le [\lambda(t)^{-1} + \lambda(t)^{-2}]\mathcal{P}(F(t))\,. \tag{4.11}$$

Then, since

$$-\mathcal{A}_{\pm}^{22}\Psi^{\pm},_{222} = 2\mathcal{A}_{\pm}^{21}\Psi^{\pm},_{221} + \mathcal{A}_{\pm}^{11}\Psi^{\pm},_{211} + 2\mathcal{A}_{\pm}^{21},_2\Psi^{\pm},_{21} + \mathcal{A}_{\pm}^{11},_2\Psi^{\pm},_{11} + \mathcal{A}_{\pm}^{22},_2\Psi^{\pm},_{22} + \mathcal{A}_{\pm}^{jk},_{j2}\Psi^{\pm},_k + \mathcal{A}_{\pm}^{jk},_j\Psi^{\pm},_{k2}\,,$$

we use (4.11) to conclude that

$$\int_{B_\pm} \xi^2|\Psi^{\pm},_{222}|^2\,dx \le [\lambda(t)^{-2} + \lambda(t)^{-3}]\mathcal{P}(F(t))\,. \tag{4.12}$$

Given the interior estimates, we sum (4.10), (4.11), and (4.12) over our finite cover index $i = 1, \dots, N$, and find that

$$\|\psi^+(\cdot,t)\|^2_{H^3(\Omega^+(t))} + \|\psi^-(\cdot,t)\|^2_{H^3(\Omega^-(t)\cap B(0,R(t)))} \le \mathcal{P}(F(t)), \tag{4.13}$$

where we have used the fact that $\lambda(t)^{-1} \le \mathcal{P}(F(t))$. Then since $u^\pm = D^\perp\psi^\pm$, (4.13) shows that

$$\|u^+(\cdot,t)\|^2_{H^2(\Omega^+(t))} + \|u^-(\cdot,t)\|^2_{H^2(\Omega^-(t)\cap B(0,R(t)))} \le \mathcal{P}(F(t))\,. \tag{4.14}$$

Note that the estimate (4.14) has been obtained for the case that $\Gamma(t)$ is of Sobolev-class $H^{3.5}$ so that we can indeed build further regularity for $u^\pm$.

Step 3 (H^3 regularity for u^+ and u^-). We will now use estimate (4.14) to build the H^3 regularity for u^+ and u^-. On $\Gamma(t)$, we let $D_\tau u$ denote the directional derivative of u in the direction τ and similarly, we let $D_n u$ denote the directional derivative of u in the direction n; for example, in components $D_\tau u^i = u^i,_j \tau^j$. We make use of the following identities on $\Gamma(t)$:

$$\operatorname{div} u = D_\tau u\cdot\tau + D_n u\cdot n, \tag{4.15a}$$

$$\operatorname{curl} u = D_\tau u\cdot n - D_n u\cdot\tau\,. \tag{4.15b}$$

Since $u(\cdot,t)$ is continuous across $\Gamma(t)$, it follows that

$$[\![D_n u\cdot\tau]\!] = -[\![\operatorname{curl} u]\!] = -1\,.$$

Then, using (4.15a), and the identity

$$[\![D_n u]\!] = [\![D_n u\cdot\tau]\!]\tau + [\![D_n u\cdot n]\!]n,$$

we see that

$$[\![D_n u]\!] = -\tau\,.$$

From (4.14), the velocity field u is a solution to the following two-phase elliptic problem:

$$\begin{aligned} \Delta u^\pm &= 0 &&\text{in } \ \Omega(t)^\pm,\\ [\![u]\!] &= 0 &&\text{on } \ \Gamma(t),\\ [\![D_n u]\!] &= -\tau &&\text{on } \ \Gamma(t), \end{aligned}$$

with variational form given by

$$\int_{\Omega^+(t)} Du^+(\cdot,t) : Dw\,dx + \int_{\Omega^-(t)} Du^-(\cdot,t) : Dw\,dx = \int_{\Gamma(t)} \tau(\cdot,t)\cdot w\,dS(t) \quad \forall w \in H^1(\mathbb{R}^2;\mathbb{R}^2), \tag{4.16}$$

where $A : B = A^i_j B^i_j$ for any 2x2 matrices A and B.

Again dropping the subscript i, we write (4.16) locally as

$$\int_{\mathcal{V}(t)\cap\Omega^+(t)} Du^+(\cdot,t) : Dw\,dx + \int_{\mathcal{V}(t)\cap\Omega^-(t)} Du^-(\cdot,t) : Dw\,dx \\ = \int_{\mathcal{V}(t)\cap\Gamma(t)} \tau(\cdot,t)\cdot w\,dS(t)\,. \tag{4.17}$$

for all $w \in H^1_0(\mathcal{V}(t);\mathbb{R}^2)$. We set $U = u\circ\theta$ and $W = w\circ\theta$. By the change-of-variables formula, (4.17) becomes

$$\int_{B_+} \mathcal{A}^{kj}_+ U^+_{,k}(\cdot,t)\cdot W_{,j}\,dx + \int_{B_-} \mathcal{A}^{kj}_- U^-_{,k}(\cdot,t)\cdot W_{,j}\,dx \\ = -\int_{B_0} \theta_{,1}\cdot W\,dS \quad \forall W\in H^1_0(B;\mathbb{R}^2)\,. \tag{4.18}$$

We then substitute

$$W = \xi^2\Lambda_\epsilon^2\partial_1^4(\xi^2 U)\in H^1_0(B),$$

into (4.18). By repeating the identical argument of Step 2 above, we find that

$$\|u^+(\cdot,t)\|^2_{H^3(\Omega^+(t))} + \|u^-(\cdot,t)\|^2_{H^3(\Omega^-(t)\cap B(0,R(t)))} \le \mathcal{P}(F(t))\,. \tag{4.19}$$

Step 4 (H^4 regularity for u^+ and u^-). We continue to assume that $k=4$ so that the boundary $\Gamma(t)$ is of Sobolev-class $H^{3.5}$ and our change-of-variables $\theta^\pm_i(t)\in H^4(B_\pm)$. We will now show that u^+ and u^- have H^4 regularity.

To do so, we let the test function $W = -\xi^2\Lambda_\epsilon^2\partial_1^6(\xi^2U)$ in (4.18). By a slight modification of Step 3, we find that

$$\|u^+(\cdot,t)\|^2_{H^4(\Omega^+(t))} + \|u^-(\cdot,t)\|^2_{H^4(\Omega^-(t)\cap B(0,R(t)))} \le \mathcal{P}(F(t))\,. \tag{4.20}$$

There are new types of integrals that arise in establishing the H^4 regularity; namely, integrals that have highest-order derivatives on both $U^\pm$ and $\theta^\pm$.

One of these integrals is analogous to one of the integrals in ${\mathcal{I}_1}^\pm_c$ defined in Step 1 and is written as

$$\mathcal{J}^\pm = \int_{B_\pm} \Lambda_\epsilon\left[\mathcal{A}^{kj}_\pm{}_{,111}\,(\xi^2U^\pm)_{,k}\right]\,\Lambda_\epsilon(\xi^2U^\pm)_{,j111}\,dx\,.$$

We estimate the integral $|\mathcal{J}^\pm|$ using an L^2-L^∞-L^2 Hölder's inequality:

$$|\mathcal{J}^\pm| \le \|\mathcal{A}^{kj}_\pm{}_{,111}\|_{L^2(B_\pm)}\|(\xi^2U)^\pm{}_{,k}\|_{L^\infty(B_\pm)}\,\|\Lambda_\epsilon(\xi^2U)^\pm{}_{,j111}\|_{L^2(B_\pm)},$$

which, with the Sobolev embedding of $H^2(B_\pm)$ into $L^\infty(B_\pm)$, shows that

$$|\mathcal{J}^\pm| \le C\|\mathcal{A}^{kj}_\pm{}_{,111}\|_{L^2(B_\pm)}\|U^\pm{}_{,k}\|_{H^2(B_\pm)}\,\|\Lambda_\epsilon(\xi^2U)^\pm{}_{,j111}\|_{L^2(B_\pm)}\,.$$

Using the estimate (3.6) with $k=4$ together with the previous lower-order estimate (4.19) of $u^\pm$ in H^3, we obtain that

$$|\mathcal{J}^\pm| \le \mathcal{P}(F(t))\|\Lambda_\epsilon(\xi^2U^\pm)_{,j111}\|_{L^2(B_\pm)},$$

which is just a linear term in $\|\Lambda_\epsilon(\xi^2U^\pm)_{,j111}\|_{L^2(B_\pm)}$, easily controlled by the *energy integral*

$$\mathcal{I}^\pm_{1_a,4} = \lambda(t)\int_{B_\pm} |\Lambda_\epsilon D(\xi^2U)^\pm{}_{,111}|^2\,dx,$$

analogous to the term ${\mathcal{I}_1}^\pm_a$ in Step 2 above.

Other integral terms set on $B_\pm$ of this type arise and can be treated similarly. There is one slight variation: the boundary integral term

$$\mathcal{I}_\partial = \int_{B_0} \partial_1^3 \Lambda_\epsilon(\xi^2 \theta,_1) \cdot \partial_1^3 \Lambda_\epsilon(\xi^2 U) \, dS,$$

for which we simply notice that

$$\begin{aligned} |\mathcal{I}_\partial| &\le \|\partial_1^3 \Lambda_\epsilon(\xi^2 \theta,_1)\|_{H^{-0.5}(B_0)} \|\partial_1^3 \Lambda_\epsilon(\xi^2 U)\|_{H^{0.5}(B_0)} \\ &\le C \|\theta,_1\|_{H^{3.5}(B_0)} \|\partial_1^3 \Lambda_\epsilon(\xi^2 U)\|_{H^1(B_+)}, \end{aligned}$$

where we have used the properties of the convolution operator for the first norm on the right-hand side, and the trace theorem for the second norm. This then provides us with

$$|\mathcal{I}_\partial| \le \mathcal{P}(F(t)) \|\Lambda_\epsilon(\xi^2 U^\pm),_{111}\|_{H^1(B_+)},$$

which is a linear term controlled in a similar manner as $\mathcal{I}^\pm_{1_a,4}$.

Step 5 (H^k regularity for u^+ and u^-). Letting $W = (-1)^{k-1} \xi^2 \Lambda_\epsilon^2 \partial_1^{2(k-1)}(\xi^2 U)$ in (4.18) and repeating Step 3, concludes the proof. □

5. Existence and regularity of the 3-D vortex patch boundary $\Gamma(t)$ and $u_\pm$: Proof of Theorem 2

5.1. **The two-phase elliptic problem for velocity.** As defined in Section 1.3, the vortex patch boundary $\Gamma(t)$ is a closed 2-D surface which is diffeomorphic to a $\mathcal{C}^\infty$ closed surface ∂B, and that $\Omega^+(t)$ is an open subset of $\mathbb{R}^3$ such that $\partial\Omega^+(t) = \Gamma(t)$, and $\Omega^-(t) = \mathbb{T}^3 - \overline{\Omega^+(0)}$. We denote $\mathbb{T}^3$ by Ω in what follows, and we set $\Omega^\pm = \Omega^\pm(0)$.

We let $\tau_1(\cdot, t)$ and $\tau_2(\cdot, t)$ denote an orthonormal basis of the tangent plane to each point of $\Gamma(t)$, so that (τ_1, τ_2, n) is a direct orthonormal frame of $\mathbb{R}^3$. We let $D_{\tau_\alpha} u$ ($\alpha = 1, 2$) denote the directional derivative of u in the direction τ_α and similarly, we let $D_n u$ denote the directional derivative of u in the direction n; for example, in components $D_{\tau_\alpha} u^i = u^i,_j (\tau_\alpha)^j$. We make use of the following identities on $\Gamma(t)$:

$$\operatorname{div} u = D_{\tau_\alpha} u \cdot \tau_\alpha + D_n u \cdot n, \tag{5.1a}$$

$$\operatorname{curl} u = (D_{\tau_2} u \cdot n - D_n u \cdot \tau_2)\tau_1 - (D_{\tau_1} u \cdot n - D_n u \cdot \tau_1)\tau_2, \tag{5.1b}$$

where we have used the fact that $\operatorname{curl} u^+ \cdot n = 0$ on $\Gamma(t)$ by (1.13h). Since $u(\cdot, t)$ is continuous across $\Gamma(t)$, it follows that

$$[\![D_n u \cdot \tau_1]\!] = [\![\operatorname{curl} u \cdot \tau_2]\!] = \operatorname{curl} u^+ \cdot \tau_2, \tag{5.2}$$

$$-[\![D_n u \cdot \tau_2]\!] = [\![\operatorname{curl} u \cdot \tau_1]\!] = \operatorname{curl} u^+ \cdot \tau_1 \,. \tag{5.3}$$

Then, using (5.1a), and the identity

$$[\![D_n u]\!] = [\![D_n u \cdot \tau_\alpha]\!]\tau_\alpha + [\![D_n u \cdot n]\!]n, \tag{5.4}$$

we see that

$$[\![D_n u]\!] = \operatorname{curl} u^+ \cdot \tau_2 \, \tau_1 - \operatorname{curl} u^+ \cdot \tau_1 \, \tau_2 \,. \tag{5.5}$$

The velocity field $u = u_+ \mathbf{1}_{\overline{\Omega^+(t)}} + u_- \mathbf{1}_{\Omega^-(t)}$ is a weak solution to the following two-phase elliptic problem:

$$-\Delta u^+ = \operatorname{curl}\operatorname{curl} u^+ \quad \text{in} \quad \Omega(t)^+, \tag{5.6a}$$

$$\Delta u^- = 0 \quad \text{in} \quad \Omega(t)^-, \tag{5.6b}$$

$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{5.6c}$$

$$[\![D_n u]\!] = \operatorname{curl} u^+ \cdot \tau_2\ \tau_1 - \operatorname{curl} u^+ \cdot \tau_1\ \tau_2 \quad \text{on} \quad \Gamma(t), \tag{5.6d}$$

with variational (or weak) form given as follows: For all vector test-functions $w \in H^1(\Omega)$ given by

$$\begin{aligned}\int_{\Omega^+(t)} Du^+(\cdot,t) : Dw\,dx + \int_{\Omega^-(t)} Du^-(\cdot,t) : Dw\,dx &= \int_{\Omega^+(t)} \operatorname{curl} u^+(\cdot,t)\cdot \operatorname{curl} w\,dx \\ &\quad + \int_{\Gamma(t)} [n\times \operatorname{curl} u^+]\cdot w\,dS(t) \\ &\quad + \int_{\Gamma(t)} [\operatorname{curl} u^+(\cdot,t)\cdot\tau_2\ \tau_1 - \operatorname{curl} u^+(\cdot,t)\cdot \tau_1\ \tau_2]\cdot w\,dS(t),\end{aligned} \tag{5.7}$$

where $A : B = A^i_j B^i_j$ for any 3x3 matrices A and B. Next we notice that

$$n\times \operatorname{curl} u^+ = n\times[\operatorname{curl} u^+\cdot\tau_1\ \tau_1 + \operatorname{curl} u^+\cdot\tau_2\ \tau_2] = \operatorname{curl} u^+\cdot\tau_1\ \tau_2 - \operatorname{curl} u^+\cdot\tau_2\ \tau_1, \tag{5.8}$$

so that the boundary integral terms of (5.7) cancel each other, and we are left with

$$\begin{aligned}\int_{\Omega^+(t)} Du^+(\cdot,t) : Dw\,dx + \int_{\Omega^-(t)} Du^-(\cdot,t) : Dw\,dx \\ = \int_{\Omega^+(t)} \operatorname{curl} u^+(\cdot,t)\cdot \operatorname{curl} w\,dx \quad \forall w \in H^1(\Omega)\,.\end{aligned} \tag{5.9}$$

We let $\eta(x,t)$ denote the Lagrangian flow of u, as defined in (1.11). We set $v^\pm = u^\pm \circ \eta$ and we define $A(x,t) = [D\eta(x,t)]^{-1}$. Then, letting $\phi = w\circ\eta$, (5.9) can be written as

$$\begin{aligned}\int_{\Omega^+} \mathcal{A}^{jk} \frac{\partial v^+}{\partial x_j}\cdot\frac{\partial \phi}{\partial x_k}\,dx + \int_{\Omega^-} \mathcal{A}^{jk} \frac{\partial v^-}{\partial x_j}\cdot\frac{\partial \phi}{\partial x_k}\,dx \\ = \int_{\Omega^+} \left[\operatorname{curl} u^+\right]\circ\eta\cdot\left[\operatorname{curl}(\phi\circ\eta^{-1})\right]\circ\eta \,\det D\eta dx\end{aligned} \tag{5.10}$$

for all $\phi \in H^1(\Omega)$, where

$$\mathcal{A}^{jk} = A^j_i A^k_i \det D\eta\,.$$

For solutions to the Euler equations (1.1), $\operatorname{div} u = 0$ so that $\det D\eta = 1$, but the general form (5.10) will be necessary for our fixed-point scheme.

5.2. **The fixed-point procedure for existence of solutions to the vortex patch problem.** In Section 6, we will establish the fundamental elliptic regularity results for a Lagrangian variational formulation as in (5.9). Using that regularity theory, we now prove the existence and regularity of solutions to the 3-D vortex patch problem; our solutions have smooth Sobolev regularity on both sides of the vortex patch boundary $\Gamma(t)$ and are globally in $H^1(\Omega)$.

5.2.1. *The functional framework.* We remind the reader that we use Ω to denote a periodic box $[-\ell,\ell]^3$ in $\mathbb{R}^3$ with opposite sides of the box identified with one another, and with ℓ taken sufficiently large so that $\overline{\Omega^+(0)} \subset \Omega$. Functions defined on Ω are 2ℓ-periodic in each of the three coordinate directions, i.e.,

$$u(x+2\ell e_i) = u(x) \quad \forall x \in \mathbb{R}^3, i = 1,2,3,$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Functions in $H^1(\Omega)$ satisfy periodic boundary conditions, and $H^1(\Omega)$ can be identified with $H^1(\mathbb{T}^3)$.

Given $T > 0$ and $M > 0$ assumed fixed, we work in the Lagrangian framework and define the bounded closed convex and non-empty set

$$\begin{aligned}(5.11) \quad \mathbf{V}^k_M = \{v \in L^2(0,T;H^1(\Omega)) \cap L^2(0,T;H^k(\Omega^\pm));\\ \|v\|_{L^2(0,T;H^1(\Omega))} + \|v^\pm\|_{L^2(0,T;H^k(\Omega^\pm))} \le M\}\end{aligned}$$

for integers $k \ge 3$. For any $v \in \mathbf{V}^k_M$, we define the Lagrangian flow

$$\eta(x,t) = x + \int_0^t v(x,s)\, ds, \tag{5.12}$$

which therefore, from (5.11), satisfies $\eta \in \mathcal{C}^0(0,T;H^1(\Omega)) \cap \mathcal{C}^0(0,T;H^k(\Omega^\pm))$. Note, also, that since the vortex patch boundary is transported by the fluid velocity, we have that

$$\Gamma(t) = \eta(\Gamma,t)\,.$$

Hence, the regularity of the velocity field in Ω^+ provides us with the regularity of η in Ω^+; the trace theorem then provides the regularity of η on Γ, and this in turn provides the regularity of the vortex patch boundary $\Gamma(t)$.

Since Ω is a periodic box, and hence convex, any two distinct points x and y in $\overline{\Omega}$ can be connected by the straight-line segment (x,y); therefore, by splitting the segment (x,y) into a finite union of subsegments (x_i, x_{i+1}), we can assume that each subsegment (x_i,x_{i+1}) is contained in either Ω^+ or Ω^-. It follows from (5.12) that

$$\begin{aligned}\eta(x,t) - \eta(y,t) =& x - y + \int_0^t v(x,s) - v(y,s)\, ds\\ =& x - y + \int_0^t v(x_1,s) - v(x_K,s)\, ds\\ =& x - y + \sum_{i=1}^{K-1} \int_0^t v(x_i,s) - v(x_{i+1},s)\, ds,\end{aligned}$$

which therefore shows by the fundamental theorem of calculus, that since each (x_i,x_{i+1}) is either contained in Ω^+ or Ω^-, that

$$\begin{aligned}&|\eta(x,t) - \eta(y,t) - x - y|\\ &\quad \le C \sum_{i=1}^{n-1} |x_i - x_{i+1}| \int_0^t \|Dv(\cdot,s)\|_{L^\infty(\Omega^+)} + \|Dv(\cdot,s)\|_{L^\infty(\Omega^-)}\, ds,\end{aligned}$$

and from the Sobolev embedding theorem,

$$|\eta(x,t)-\eta(y,t)-x-y| \\ \le C\sum_{i=1}^{n-1}|x_i-x_{i+1}|\int_0^t \|Dv(\cdot,s)\|_{H^2(\Omega^+)}+\|Dv(\cdot,s)\|_{H^2(\Omega^-)}\,ds\,.$$

From the definitions (5.11) and (5.12), it follows that

$$|\eta(x,t)-\eta(y,t)-x-y| \le \sum_{i=1}^{n-1}|x_i-x_{i+1}|2\sqrt{t}M \le 2\sqrt{T}MC|x-y|\,. \tag{5.13}$$

We now choose T such that

$$0<T\le \frac{1}{16M^2C^2}, \tag{5.14}$$

so that for any x and y in $\overline{\Omega}$,

$$|\eta(x,t)-\eta(y,t)|\ge \frac{1}{2}|x-y|,$$

which establishes the injectivity of η in $\overline{\Omega}$. Furthermore, since

(5.15)

$$|D\eta(x,s)-\mathrm{Id}| \le \left|\int_0^t Dv(x,s)\;ds\right| \\ \le \left|\int_0^t \|Dv^+(\cdot,s)\|_{L^\infty(\Omega^+)}+\|Dv^-(\cdot,s)\|_{L^\infty(\Omega^-)}\;ds\right| \le 2C\sqrt{T}M,$$

due to the continuity of the determinant at Id in $\mathbb{R}^9$, we can choose $T>0$ small enough, so that for all $x\in\Omega$ and $0\le t\le T$,

$$\frac{3}{2}\ge \det D\eta(x,t)\ge \frac{1}{2}, \tag{5.16}$$

which shows, with the previously established injectivity, that $\eta(\cdot,t)$ is an H^4 diffeomorphism from $\Omega^\pm$ onto the image $\eta(\Omega^\pm,t)$, and a homeomorphism from Ω onto $\eta(\Omega,t)$. Finally, by choosing T sufficiently small we can ensure the strict positivity of the coeffiicient matrix $\mathcal{A}$: for all $t\in[0,T]$,

$$w^T\,\mathcal{A}_i^\pm(x,t)\,w\ge \frac{1}{4}|w|^2\ \forall w\in\mathbb{R}^2,\quad x\in\Omega\,. \tag{5.17}$$

5.2.2. *The fixed-point procedure.* We define the Lagrangian curl operator curl_η as follows: if $u(y,t)$ is an Eulerian vector, and $v=u\circ\eta$, then we $\mathrm{curl}_\eta\, v=[\mathrm{curl}\, u]\circ\eta$ where for any differential vector field F, and for $i=1,2,3$,

$$[\mathrm{curl}_\eta\, F]_i=\varepsilon_{ijk}\frac{\partial F^k}{\partial x_r}A_j^r, \tag{5.18}$$

where ε_{ijk} denotes the permutation symbol, so that $\varepsilon_{ijk}=1$ for even permutations, $\varepsilon_{ijk}=-1$ for odd permutations, and $\varepsilon_{ijk}=0$ otherwise. We will employ a fixed-point procedure on the variational equation (5.10), which we write as

(5.19)

$$\int_{\Omega^+}\mathcal{A}^{jk}\frac{\partial v^+}{\partial x_j}\cdot\frac{\partial\phi}{\partial x_k}\,dx+\int_{\Omega^-}\mathcal{A}^{jk}\frac{\partial v^-}{\partial x_j}\cdot\frac{\partial\phi}{\partial x_k}\,dx=\int_{\Omega^+}\left[\mathrm{curl}\, u^+\right]\circ\eta\cdot\mathrm{curl}_\eta\,\phi\;\det D\eta dx$$

for all $\phi \in H^1(\Omega)$. From (1.12),

$$\operatorname{curl} u \circ \eta = D\eta \cdot \omega_0, \quad \omega_0 = \operatorname{curl} u_0^+ \mathbf{1}_{\Omega^+}\,. \tag{5.20}$$

Since $\operatorname{div} \omega_0 = 0$, using the formula (5.20), we see that

$$\int_{\Omega^+} \operatorname{curl} u^+ \circ \eta \; dx = \int_\Gamma \eta \; (\operatorname{curl} u_0^+ \cdot n(\cdot, 0))\, dS(0) = 0, \tag{5.21}$$

where the last equality follows from (1.13h).

Now, given v in our convex set $\mathbf{V}_M^k$ and letting η denote the homeomorphism defined in (5.12), we define

$$\mathcal{C}(v)(x,t) = D\eta(x,t) \cdot \omega_0(x) \quad \text{in} \quad \Omega\,. \tag{5.22}$$

Notice that for any $x \in \Gamma$, the trace on Γ of $\mathcal{C}(v)(x,t) \cdot n(\eta(x,t),t)$ (the trace taken from Ω^+) is zero, and is thus equal to the trace of $\mathcal{C}(v)(x,t) \cdot n(\eta(x,t),t)$ evaluated from Ω^-. To see this, we use an important geometric property of the inverse deformation matrix $A(x,t) = [D\eta(x,t)]^{-1}$; namely, if $N(x) := n(0,x)$ denotes the outward unit normal to $\partial\Omega^+$ and if $n(\eta(x,t),t)$ denotes the outward unit normal to $\partial\Omega^+(t)$, then

$$n_i(\eta(x,t),t) = \frac{A_i^k N_k}{|A^T N|}\,.$$

Hence, it follows that

$$\mathcal{C}(v) \cdot n \circ \eta = \mathcal{C}(v)^i \frac{A_i^k N_k}{|A^T N|} = \frac{1}{|A^T N|} N_k A_i^k \frac{\partial \eta^i}{\partial x_l} \omega_0^l = \frac{1}{|A^T N|} N_k \omega_0^k = 0, \tag{5.23}$$

where we have again used (1.13h) for the last equality.

Furthermore, the same computation as in (5.21) shows that

$$\int_{\Omega^+} \mathcal{C}(v) \; dx = \int_\Gamma \eta \; (\operatorname{curl} u_0^+ \cdot N)\, dS(0) = 0\,. \tag{5.24}$$

Now, for each time $t \in [0,T]$, we construct a solution $\bar{v}(\cdot,t)$ to the following variational problem:

$$\begin{aligned} &\int_{\Omega^+} \mathcal{A}^{jk} \frac{\partial v^+}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k}\, dx + \int_{\Omega^-} \mathcal{A}^{jk} \frac{\partial v^-}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k}\, dx \\ &\qquad = \int_{\Omega^+} \mathcal{C}(v) \cdot \operatorname{curl}_\eta \phi \; \det D\eta dx \quad \forall \phi \in H^1(\Omega)\,. \end{aligned} \tag{5.25}$$

From (5.17) and the Lax-Milgram theorem, there exists a unique periodic solution $\bar{v}(\cdot,t) \in H^1(\Omega)$ for each fixed $t \in [0,T]$, satisfying

$$\int_\Omega \bar{v} \; dx = 0. \tag{5.26}$$

Furthermore, since $\mathcal{C}(v) \in H^k\Omega^+$, $k \geq 2$, we may integration-by-parts on the right-hand side of (5.25). We use the fact that the cofactor matrix $a(x,t)$, defined by $a = \det D\eta\, A$, satisfies the Piola identity $\frac{\partial}{\partial x_k} a_i^k = 0$ for $i = 1,2,3$. Thus, we see that (5.25) can be written as follows: for all $\phi \in H^1(\Omega)$,

$$\begin{aligned} \int_{\Omega^+} \mathcal{A}^{jk} \frac{\partial v^+}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k}\, dx + \int_{\Omega^-} \mathcal{A}^{jk} \frac{\partial v^-}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k}\, dx &= \int_{\Omega^+} \operatorname{curl}_\eta \mathcal{C}(v) \cdot \phi \det D\eta \, dx \\ &\quad + \int_{\partial\Omega^+} \mathcal{C}(v) \times (a^T N)\, \phi \, dS(0), \end{aligned}$$

which is the variational form of the general elliptic system (6.1) studies in Section 6, with forcing functions

$$\mathbf{f}_- = 0, \quad \mathbf{f}_+ = \operatorname{curl}_\eta \mathcal{C}(v) \det D\eta, \quad \text{and} \quad \mathbf{g} = \mathcal{C}(v) \times (a^T N),$$

for which our regularity result Theorem 4 applies. We therefore have that (for $k \geq 2$)

(5.27)

$$\begin{aligned}&\|\bar v^+\|_{H^{k+1}(\Omega^+)} + \|\bar v^-\|_{H^{k+1}(\Omega^-)}\\ &\leq C\Big[\|\boldsymbol{f}_\pm\|_{H^{k-1}(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{k-0.5}(\Gamma)} + \mathcal{P}\big(\|\mathcal{A}_\pm\|_{H^k(\Omega^\pm)}\big)\Big(\|\boldsymbol{f}_\pm\|_{L^2(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{-0.5}(\Gamma)}\Big)\Big],\end{aligned}$$

where $\mathcal{P}$ is a polynomial function and the constant C depends on $\Omega^\pm$.

From (5.15) and for

$$\sqrt{T}\ M \leq \epsilon_0, \tag{5.28}$$

with $0 < \epsilon_0 \ll 1$ denoting a sufficiently small constant (which is independent of M), that for any $v \in \mathbf{V}_M^k$

$$\|\eta\|_{H^{k+1}(\Omega^+)} \leq C|\Omega|\,. \tag{5.29}$$

Since from the definition (5.22),

$$\|\mathcal{C}(v)\|_{H^k(\Omega^+)} \leq \|u_0\|_{H^{k+1}(\Omega^+)}(1 + C\sqrt{T}M), \tag{5.30}$$

we then infer from (5.30), (5.29) and (5.27) that

$$\|\bar v^+\|_{H^{k+1}(\Omega^+)} + \|\bar v^-\|_{H^{k+1}(\Omega^-)} \leq C\Big[C|\Omega|\|u_0\|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|))\Big]\,.$$

Therefore,

$$\begin{aligned}&\|\bar v^+\|_{L^2(0,T;H^{k+1}(\Omega^+))} + \|\bar v^-\|_{L^2(0,T;H^{k+1}(\Omega^-))}\\ &\qquad \leq 2C\Big[C|\Omega|\|u_0\|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|))\Big]\sqrt{T},\end{aligned}$$

which thanks to (5.28) shows that

$$\begin{aligned}&\|\bar v^+\|_{L^2(0,T;H^{k+1}(\Omega^+))} + \|\bar v^-\|_{L^2(0,T;H^{k+1}(\Omega^-))}\\ &\qquad \leq 2C\Big[C|\Omega|\|u_0\|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|))\Big]\frac{\epsilon_0}{M}\,.\end{aligned}$$

This inequality then proves that $\bar v \in \mathbf{V}_M^k$ for

$$M^2 = 2C\Big[C|\Omega|\|u_0\|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|))\Big]\epsilon_0\,.$$

Moreover, it is easy to check that the map $\Theta: \ v \mapsto \bar v$ is sequentially weakly lower semi-continuous; that is, if $v_j \rightharpoonup v$ in the weak topology of the norm defining the closed convex set $\mathbf{V}_M^k$, then $\Theta v_j \rightharpoonup \Theta v$. Therefore, by Schauder's second fixed-point theorem (see [21, p. 452]), which is itself a corollary of Tyhonov's fixed-point theorem, we then have that Θ has a fixed point in $\mathbf{V}_M^k$.

5.2.3. *The fixed point is a solution to the Euler equations.* We now explain why this fixed point, $v = \bar{v}$, is indeed a solution of the Euler equations with initial data u_0, and hence a solution to the 3-D vortex patch boundary. At a fixed point $v = \bar{v}$, (5.25) becomes the following variational problem:

$$\int_{\Omega^+} \mathcal{A}^{jk} \frac{\partial v^+}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k}\, dx + \int_{\Omega^-} \mathcal{A}^{jk} \frac{\partial v^-}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k}\, dx = \int_{\Omega^+} \mathcal{C}(v) \cdot \operatorname{curl}_\eta \phi \, \det D\eta \, dx \quad \forall \phi \in H^1(\Omega), \tag{5.31}$$

where the operator curl_η is defined (5.18). We define the following Eulerian quantities associated to our Lagrangian velocity v and test function ϕ:

$$u = v \circ \eta^{-1}, \quad \mathfrak{C} = \mathcal{C}(v) \circ \eta^{-1}, \quad \text{and} \quad w = \phi \circ \eta^{-1}.$$

The change-of-variables theorem shows that (5.31) can be written as[1]

$$\int_{\eta(\Omega^+,t)} Du^+ \cdot Dw \; dy + \int_{\eta(\Omega^-,t)} Du^- \cdot Dw \; dy = \int_{\eta(\Omega^+,t)} \mathfrak{C} \cdot \operatorname{curl} w \; dy\,. \tag{5.32}$$

Our goal is to show that $\operatorname{div} u = 0$ and that $\operatorname{curl} u = \mathfrak{C}$. To do so, we use integration-by-parts on the left-hand side of (5.32); we see that

$$\begin{aligned}
&\int_{\eta(\Omega,t)} Du \cdot Dw \; dy \\
&\quad = -\int_{\eta(\Omega,t)} \Delta u \cdot w \; dy + \int_{\eta(\Gamma,t)} [\![D_n u]\!] \cdot w \, dS(t) \\
&\quad = \int_{\eta(\Omega,t)} \operatorname{curl}\operatorname{curl} u \cdot w \; dy - \int_{\eta(\Omega,t)} D \operatorname{div} u \cdot w \; dy + \int_{\eta(\Gamma,t)} [\![D_n u]\!] \cdot w \, dS(t) \\
&\quad = \int_{\eta(\Omega,t)} \operatorname{curl} u \cdot \operatorname{curl} w \; dy + \int_{\eta(\Omega,t)} \operatorname{div} u \cdot \operatorname{div} w \; dy \\
&\qquad + \int_{\eta(\Gamma,t)} ([\![D_n u]\!] + [\![n \times \operatorname{curl} u]\!] - [\![\operatorname{div} u]\!]\, n \cdot) \cdot w \, dS(t)\,.
\end{aligned}$$

The identities (5.4) and (5.8) show that for $u \in H^1(\eta(\Omega,t))$, so that $[\![u]\!] = 0$ on $\eta(\Gamma,t)$, we have that

$$[\![D_n u]\!] + [\![n \times \operatorname{curl} u]\!] - [\![\operatorname{div} u]\!]\, n = 0 \quad \text{on} \quad \eta(\Gamma,t),$$

so that

$$\int_{\eta(\Omega,t)} Du \cdot Dw \; dy = \int_{\eta(\Omega,t)} [\operatorname{curl} u \cdot \operatorname{curl} w + \operatorname{div} u \;\; \operatorname{div} w] \; dy\,. \tag{5.33}$$

Comparing (5.32) and (5.33), we have that for all test function $w \in H^1(\eta(\Omega,t))$,

$$\int_{\eta(\Omega,t)} [\operatorname{curl} u \cdot \operatorname{curl} w + \operatorname{div} u \;\; \operatorname{div} w] \; dy = \int_{\eta(\Omega,t)} \mathbf{1}_{\eta(\Omega^+,t)} \mathfrak{C} \cdot \operatorname{curl} w \; dy\,. \tag{5.34}$$

We now choose the test function w to have the potential form

$$w = D\psi,$$

[1] Note that $\eta(\Omega,t)$ is the image of the 2ℓ-periodic box, and hence functions defined on $\eta(\Omega,t)$ are periodic.

for some function periodic function $\psi \in H^2(\eta(\Omega,t))$. Then,

$$\operatorname{curl} w = 0,$$

and (5.34) reduces to

$$\int_{\eta(\Omega,t)} \operatorname{div} u \; \Delta\psi \; dx = 0\,. \tag{5.35}$$

Since $u \in H^1(\eta(\Omega,t))$ and is periodic, there exists a periodic function $\psi_0 \in H^2(\eta(\Omega,t))$, such that

$$\operatorname{div} u = \Delta\psi_0 \text{ in } \eta(\Omega,t)\,. \tag{5.36}$$

Letting $\psi = \psi_0$ in (5.35) then shows that

$$0 = \int_{\eta(\Omega,t)} (\operatorname{div} u)^2 \; dx,$$

and thus

$$\operatorname{div} u = 0\,. \tag{5.37}$$

This being true for all time $t \in [0,T]$, since $\eta(x,0) = x$, we then infer that

$$\det D\eta = 1\,. \tag{5.38}$$

Using (5.38) and (5.37) in (5.34), we see that for all $w \in H^1(\eta(\Omega,t))$,

$$0 = \int_{\eta(\Omega,t)} (\mathfrak{C} - \operatorname{curl} u) \cdot \operatorname{curl} w \; dy, \tag{5.39}$$

and from (5.22) we see that $\mathcal{C}(v)(x,t) = 0$ for all $x \in \Omega^-$, since $\omega_0^- = 0$. Next, we note that

$$\partial_t \mathcal{C}(v) = \frac{\partial v}{\partial x_k}\omega_0^k = \frac{\partial v}{\partial x_r} A^r_j \frac{\partial \eta^j}{\partial x_k}\omega_0^k = \frac{\partial v}{\partial x_r} A^r_j \mathcal{C}(v)^j = Du(\eta) \cdot \mathcal{C}(v),$$

where $Du(\eta)$ denotes $Du \circ \eta$. Hence, since $\mathcal{C}(v) = \mathfrak{C} \circ \eta$, it follows that $\mathfrak{C}$ satisfies

$$\mathfrak{C}_t + D_u\mathfrak{C} - Du \cdot \mathfrak{C} = 0 \;\text{ in }\; \eta(\Omega^+,t), \tag{5.40}$$

and $\mathfrak{C}(y,t) = 0$ for all $y \in \eta(\Omega^-,t)$. Since $\mathfrak{C} \in H^k(\eta(\Omega^+,t))$, $k \geq 2$, we take the divergence of equation (5.40) and find that

$$\operatorname{div} \mathfrak{C}_t + D_u \operatorname{div} \mathfrak{C} - D_{\mathfrak{C}} \operatorname{div} u + (u^i,_j \mathfrak{C}^j,_i - u^j,_i \; \mathfrak{C}^i,_j) = 0\,. \tag{5.41}$$

From (5.37) and the symmetry of the last two terms, we conclude that

$$\operatorname{div} \mathfrak{C}_t + D_u \operatorname{div} \mathfrak{C} = 0,$$

and thus

$$\operatorname{div} \mathfrak{C}(\eta(x,t),t) = \operatorname{div} \mathfrak{C}(x,0)\,. \tag{5.42}$$

Since $\mathfrak{C}(0) = \operatorname{curl} u_0$ we then have from (5.42) that

$$\operatorname{div} \mathfrak{C}(\eta(x,t),t) = 0\,. \tag{5.43}$$

From (5.24) and (5.38)

$$\int_{\eta(\Omega^+,t)} \mathfrak{C}(y,t) \; dy = 0\,.$$

We note that $\mathfrak{C}(\cdot,t) \in L^2(\eta(\Omega,t))$. Next, we define the periodic vector field $\psi \in H^2(\eta(\Omega,t))$ as the solution, modulo constants, of

$$\begin{aligned} -\Delta\psi^+ =& \mathfrak{C} \quad \text{in} \quad \eta(\Omega^+,t), \\ -\Delta\psi^- =& 0 \quad \text{in} \quad \eta(\Omega^-,t), \end{aligned}$$

with the continuity conditions, which follow from the fact that $D\psi \in H^1(\eta(\Omega,t))$,

$$[\![\psi]\!] = 0 \quad \text{and} \quad [\![D_n\Psi]\!] = 0 \quad \text{on } \eta(\Gamma,t)\,. \tag{5.44}$$

Theorem 4 shows that $\psi \in H^{k+2}(\eta(\Omega^\pm,t))$, $k \geq 2$. Moreover, from (5.42), $\operatorname{div}\psi$ is harmonic in both $\eta(\Omega^+,t)$ and $\eta(\Omega^-,t)$ and is a periodic function; furthermore, $[\![D\operatorname{div}\psi\cdot n]\!] = 0$ on $\Gamma(t)$, for

$$\begin{aligned} D\operatorname{div}\psi^\pm\cdot n =& \operatorname{curl}(\operatorname{curl}\psi^\pm)\cdot n + \Delta\psi^\pm\cdot n \\ =& \operatorname{curl}(\operatorname{curl}\psi^\pm)\cdot n + \mathfrak{C}\cdot n \\ =& \operatorname{curl}(\operatorname{curl}\psi^\pm)\cdot n \\ =& D_{\tau_1}(\operatorname{curl}\psi^\pm)\cdot\tau_2 - D_{\tau_2}(\operatorname{curl}\psi^\pm)\cdot\tau_1, \end{aligned}$$

where we have used $\mathfrak{C}\cdot n = 0$ on $\Gamma(t)$ in the third equality, so that

$$[\![D\operatorname{div}\psi\cdot n]\!] = [\![D_{\tau_1}\operatorname{curl}\psi\cdot\tau_2 - D_{\tau_2}\operatorname{curl}\psi\cdot\tau_1]\!]\,. \tag{5.45}$$

Using (5.44),

$$[\![\operatorname{curl}\psi]\!] = 0 \quad \text{on } \eta(\Gamma,t),$$

so that

$$[\![D_{\tau_\alpha}\operatorname{curl}\psi]\!] = 0 \quad \text{on } \eta(\Gamma,t),$$

and from (5.45),

$$[\![D\operatorname{div}\psi\dot{}n]\!] = 0 \quad \text{on } \eta(\Gamma,t)\,. \tag{5.46}$$

We now set $\Omega^\pm(t) = \eta(\Omega^\pm,t)$ Using (5.46) and the fact that $\operatorname{div}\psi \in H^1(\Omega(t)) \cap H^{k+1}(\Omega^\pm(t))$, $k \geq 2$, is harmonic in $\Omega^\pm(t)$ and is a periodic function, we find that

$$\begin{aligned} 0 =& \int_{\Omega^+(t)} \Delta\operatorname{div}\psi\ \operatorname{div}\psi\ dy + \int_{\Omega^-(t)} \Delta\operatorname{div}\psi\ \operatorname{div}\psi\ dy \\ =& -\int_{\Omega^+(t)} |D\operatorname{div}\psi|^2 dy - \int_{\Omega^-(t)} |D\operatorname{div}\psi|^2 dy + \int_{\Gamma(t)} [\![D\operatorname{div}\psi\dot{}n]\!]\operatorname{div}\psi\ dS(t) \\ =& -\int_{\Omega^+(t)} |D\operatorname{div}\psi|^2 dy - \int_{\Omega^-(t)} |D\operatorname{div}\psi|^2 dy \end{aligned}$$

which shows that $\operatorname{div}\psi(\cdot,t)$ is a constant.

Therefore,

$$\Delta\psi = -\operatorname{curl}(\operatorname{curl}\psi), \tag{5.47}$$

so that $-\operatorname{curl}(\operatorname{curl}\psi) = \mathfrak{C}$. Substituting this into (5.39), we see that for all test functions $w \in H^1(\eta(\Omega,t))$,

$$0 = \int_{\eta(\Omega,t)} (-\operatorname{curl}(\operatorname{curl}\psi) - \operatorname{curl}u)\cdot\operatorname{curl}w\ dy\,. \tag{5.48}$$

Next, we set $w = -\operatorname{curl}\psi + u$ in (5.48), which satisfies the condition of being a test function, and obtain that

$$0 = \int_{\eta(\Omega,t)} |\operatorname{curl}(\operatorname{curl}\psi + u)|^2\ dx,$$

and thus

$$\operatorname{curl} u^+ = -\operatorname{curl}(\operatorname{curl}\psi^+) = \mathfrak{C} \text{ in } \Omega^+(t),$$

and

$$\operatorname{curl} u^- = -\operatorname{curl}(\operatorname{curl}\psi^-) = 0 \text{ in } \Omega^-(t).$$

Thanks to (5.40), we have that in $\Omega^+(t)$,

$$\operatorname{curl} u_t + D_u \operatorname{curl} u - Du \cdot \operatorname{curl} u = 0,$$

which is the same as

$$\operatorname{curl}(u_t + D_u u) = 0,$$

from which we infer the existence of a pressure function p such that

$$u_t + D_u u + Dp = 0.$$

Therefore, u is the solution of the incompressible Euler equations (1.1), as we have already proven that $\operatorname{div} u = 0$.

It remains only to show that $u(x,0) = u_0(x)$. To this end, we notice that from (5.22),

$$\mathcal{C}(v)(x,0) = \operatorname{curl} u_0(x),$$

and thus

$$\operatorname{curl} u(\cdot,0) = \operatorname{curl} u_0,$$

which coupled with the fact that $\operatorname{div} u(\cdot,0) = 0 = \operatorname{div} u_0$ and the periodicity of u, provides us with

$$u(\cdot,0) = u_0 + c,$$

where c is a constant vector. From (5.26),

$$\int_\Omega u(x,0)\,dx = 0,$$

which coupled with

$$\int_\Omega u_0(x)\,dx = 0,$$

then shows that $c = 0$, so that $u(\cdot,0) = u_0$, which completes our proof that u is the solution of the vortex patch problem on $[0,T]$, with the desired regularity properties. In particular, by (5.12) and (5.29), we see that $\eta \in \mathcal{C}^0([0,T]; H^{k+1}(\Omega^+))$ and hence by the trace theorem, $\eta \in \mathcal{C}^0([0,T]; H^{k+1/2}(\Gamma))$. Since the vortex patch boundary $\Gamma(t) = \eta(\Gamma,t)$ for each $t \in [0,T]$, we see that $\Gamma(t)$ is of Sobolev-class $H^{k+1/2}$. To explain why $\Gamma(t)$ is indeed $\eta(\Gamma,t)$, we use the identity $\operatorname{curl} u \circ \eta = D\eta \cdot \omega_0$, where we recall that $\omega_0 = \operatorname{curl} u_0$ and satisfies (1.13). Next, we choose a local coordinate system at a point $x \in \Gamma$, such that $n(x,0) = e_3$ and the two tangent vectors are $\tau_1 = e_1$ and $\tau_2 = e_2$. By conditions (1.13g,h), we can write $\omega_0^+ = \sum_{\alpha=1}^2 \omega_0^+ \cdot e_\alpha\, e_\alpha$. This means that $\operatorname{curl} u^+(\eta(x,t),t) = \eta,_\alpha\, \omega_0^+ \cdot e_\alpha$, and as we have shown already, $\operatorname{curl} u^+(\eta(x,t),t) \cdot n(\eta(x,t),t) = \omega_0^+ \cdot e_\alpha \eta,_\alpha \cdot \frac{(\eta,_1 \times \eta,_2)}{|\eta,_1 \times \eta,_2|} = 0$. Since for $\alpha = 1,2$, $\eta,_\alpha$ is a tangent vector to $\eta(\Gamma,t)$ at the point $\eta(x,t)$ and hence continuous, then

$$[\![\operatorname{curl} u]\!] \circ \eta = \eta,_\alpha [\![\omega_0^+ \cdot e_\alpha]\!].$$

This shows that the set $\Gamma(t)$, on which $\operatorname{curl} u(\cdot,t)$ has a jump discontinuity, is propagated by the Lagrangian flow map $\eta(\cdot,t)$.

Uniqueness of solutions has been shown by Gamblin & Saint Raymond [13].

6. Elliptic regularity

6.1. **A two-phase elliptic problem.** For $k \geq 2$, let $\Omega^+ \subseteq \mathbb{R}^{\rm n}$ denote an open, bounded H^{k+1}-domain which is diffeomorphic to a $\mathcal{C}^\infty$, connected, open, and bounded domain B. We set $\Gamma := \partial\Omega^+$, which is then an $H^{k+1/2}$-class closed surface. Let Ω denote a periodic box $[-\mathcal{L}, \mathcal{L}]^{\rm n}$ in $\mathbb{R}^{\rm n}$ with opposite sides identified, and with $\mathcal{L}$ sufficiently large so that $\overline{\Omega^+}$ is properly contained in Ω. Functions defined on Ω are $2\mathcal{L}$-periodic in each of the n coordinate directions, i.e.,

$$u(x + 2\ell e_i) = u(x) \quad \forall x \in \mathbb{R}^{\rm n}, i = 1, \ldots, {\rm n},$$

where e_i denotes the usual Cartesian basis. We set $\Omega^- = \Omega / \overline{\Omega^+}^c$.

We establish elliptic regularity for the following two-phase vector valued elliptic problem:

$$-\frac{\partial}{\partial x_j}\left(a^{jk}_\pm \frac{\partial \boldsymbol{u}_\pm}{\partial x_k}\right) = \boldsymbol{f}_\pm \qquad \text{in} \quad \Omega^\pm, \tag{6.1a}$$

$$[\![\boldsymbol{u}]\!] = 0 \qquad \text{on} \quad \Gamma, \tag{6.1b}$$

$$\left[a^{jk} \frac{\partial \boldsymbol{u}}{\partial x_k} N_j\right] = \boldsymbol{g} \qquad \text{on} \quad \Gamma \tag{6.1c}$$

$$\boldsymbol{u}_- \text{ is periodic} \qquad \text{on} \quad \partial\Omega\,, \tag{6.1d}$$

where $\boldsymbol{u}_\pm = (\boldsymbol{u}^1_\pm, \cdots, \boldsymbol{u}^{\rm n}_\pm)$ and $\boldsymbol{f}_\pm = (\boldsymbol{f}^1, \cdots, \boldsymbol{f}^{\rm n})$, $\boldsymbol{g} = (\boldsymbol{g}^1, \cdots, \boldsymbol{g}^{\rm n})$ are vector valued functions, and $a^{jk}_\pm$ are two-tensors which satisfy the positivity condition

$$a^{jk}_\pm \xi_j \xi_k \geq \lambda |\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^{\rm n} \tag{6.2}$$

for some $\lambda > 0$. We use the notation $[\![\boldsymbol{w}]\!] = \boldsymbol{w}_+ - \boldsymbol{w}_-$ for vector fields $\boldsymbol{w}$ on Γ, and we let N denote the outward unit normal to $\partial\Omega^+$. The system (6.1) has a unique solution in $H^1(\Omega)$ when we additionally assume that $\int_\Omega u(x)dx = 0$.

Let $\mathcal{V} = H^1(\Omega)$, the space of H^1 functions on $[-\mathcal{L}, \mathcal{L}]^{\rm n}$ which are $2\mathcal{L}$-periodic. Let $\boldsymbol{u} = \boldsymbol{u}_+ \mathbf{1}_{\overline{\Omega^+}} + \boldsymbol{u}_- \mathbf{1}_{\Omega^-}$, $\boldsymbol{f} = \boldsymbol{f}_+ \mathbf{1}_{\overline{\Omega^+}} + \boldsymbol{f}_- \mathbf{1}_{\Omega^-}$. The variational (or weak) form of (6.1) is given by

$$\int_{\Omega^\pm} a^{jk} \frac{\partial \boldsymbol{u}^i}{\partial x_k} \frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\, dx = \int_{\Omega^\pm} \boldsymbol{f}\, \boldsymbol{\varphi}\, dx + \int_\Gamma \boldsymbol{g}\, \boldsymbol{\varphi}\, dS \quad \forall\, \boldsymbol{\varphi} \in \mathcal{V}, \tag{6.3}$$

where we use the following integral notation:

$$\int_{\Omega^\pm} a^{jk} \frac{\partial \boldsymbol{u}^i}{\partial x_k} \frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\, dx = \int_{\Omega^+} a^{jk}_+ \frac{\partial \boldsymbol{u}^i_+}{\partial x_k} \frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\, dx + \int_{\Omega^-} a^{jk}_- \frac{\partial \boldsymbol{u}^i_-}{\partial x_k} \frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\, dx$$

and

$$\int_{\Omega^\pm} \boldsymbol{f}\, \boldsymbol{\varphi}\, dx = \int_{\Omega^+} \boldsymbol{f}_+\, \boldsymbol{\varphi}\, dx + \int_{\Omega^-} \boldsymbol{f}_-\, \boldsymbol{\varphi}\, dx\,.$$

The regularity theory for solutions $\boldsymbol{u}$ of (6.3) is classical when the coefficient matrix a^{jk} is in $\mathcal{C}^k$, and can be summarized by the following

Theorem 3. *Suppose that for some* ${\rm k} \in \mathbb{N}$, $a^{jk}_\pm \in \mathcal{C}^{\rm k}(\overline{\Omega^\pm})$ *satisfies* (6.2). *Then for all* $\boldsymbol{f}_\pm \in H^{{\rm k}-1}(\Omega^\pm)$ *and* $\boldsymbol{g} \in H^{{\rm k}-0.5}(\Gamma)$, *the solution* $\boldsymbol{u}$ *to* (6.1) *is in* $H^{{\rm k}+1}(\Omega^\pm)$, *and satisfies*

$$\|\boldsymbol{u}_\pm\|_{H^{{\rm k}+1}(\Omega^\pm)} \leq C\Big[\|\boldsymbol{f}_\pm\|_{H^{{\rm k}-1}(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{{\rm k}-0.5}(\Gamma)}\Big] \tag{6.4}$$

for some constant C *depending on* $\|a_\pm\|_{\mathcal{C}^{\rm k}(\Omega^\pm)}$.

We use the following notation for norms:

$$\|(\cdot)_\pm\|_{H^{\mathrm{k}+1}(\Omega^\pm)} = \|(\cdot)_+\|_{H^{\mathrm{k}+1}(\Omega^+)} + \|(\cdot)_-\|_{H^{\mathrm{k}+1}(\Omega^-)} \,.$$

We shall need the corresponding result for the case that the coefficient matrix $a^{jk}_\pm$ has only Sobolev-class regularity:

Theorem 4. *Suppose that for some integer* $\mathrm{k} > \frac{\mathrm{n}}{2}$ *and* $1 \le \ell \le \mathrm{k}$, $a^{jk}_\pm \in H^{\mathrm{k}}(\Omega^\pm)$ *satisfies* (6.2). *Then if* $f \in H^{\ell-1}(\Omega^+)$ *and* $g \in H^{\ell-0.5}(\Gamma)$, *the weak solution* $\boldsymbol{u}_\pm$ *to* (6.1) *is in* $H^{\ell+1}(\Omega^\pm)$, *and satisfies*

$$(6.5)\quad \|\boldsymbol{u}_\pm\|_{H^{\ell+1}(\Omega^\pm)} \le C\Big[\|\boldsymbol{f}_\pm\|_{H^{\ell-1}(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)} + \mathcal{P}\big(\|a_\pm\|_{H^{\mathrm{k}}(\Omega^\pm)}\big)\Big(\|\boldsymbol{f}_\pm\|_{L^2(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{-0.5}(\Gamma)}\Big)\Big],$$

where $\mathcal{P}$ *is a polynomial function and the constant* C *depends on* $\Omega^\pm$.

We are using the notation

$$\begin{aligned}
&\mathcal{P}\big(\|a_\pm\|_{H^{\mathrm{k}}(\Omega^\pm)}\big)\Big(\|\boldsymbol{f}_\pm\|_{L^2(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{-0.5}(\Gamma)}\Big)\\
&\quad= \mathcal{P}\big(\|a_+\|_{H^{\mathrm{k}}(\Omega^+)}\big)\Big(\|\boldsymbol{f}_+\|_{L^2(\Omega^+)} + \|\boldsymbol{g}\|_{H^{-0.5}(\Gamma)}\Big)\\
&\qquad+ \mathcal{P}\big(\|a_-\|_{H^{\mathrm{k}}(\Omega^-)}\big)\Big(\|\boldsymbol{f}_-\|_{L^2(\Omega^-)} + \|\boldsymbol{g}\|_{H^{-0.5}(\Gamma)}\Big).
\end{aligned}$$

Proof. Let $\mathrm{E}^\pm : H^{\mathrm{k}+1}(\Omega^\pm) \to H^{\mathrm{k}+1}(\mathbb{R}^{\mathrm{n}})$ denote a Sobolev extension operator, and let $a_{\pm_\epsilon} = \eta_\epsilon \star (\mathrm{E}^\pm a_\pm)$ and $\boldsymbol{f}_\epsilon = \eta_\epsilon \star (\mathrm{E}^\pm f)$. Let $\{\mathcal{U}_m\}_{m=1}^K$ denote an open cover of Ω which intersects the interface Γ, and let $\{\theta_m\}_{m=1}^K$ denote a collection of charts such that

(1) $\theta_m : B(0, r_m) \to \mathcal{U}_m$ is an H^{k+1}-diffeomorphism,
(2) $\det(D\theta_m) > 0$,
(3) $\theta_m : B^0_m \equiv B(0, r_m) \cap \{x_n = 0\} \to \mathcal{U}_m \cap \Gamma$,
(4) $\theta_m : B^+_m \equiv B(0, r_m) \cap \{y_{\mathrm{n}} > 0\} \to \mathcal{U}_m \cap \Omega^+$,
(5) $\theta_m : B^-_m \equiv B(0, r_m) \cap \{y_{\mathrm{n}} < 0\} \to \mathcal{U}_m \cap \Omega^-$,
(6) $\|D\theta_m - \mathrm{Id}\|_{L^\infty(B(0,r_m))} \ll 1$.

Let $0 \le \zeta_m \le 1$ in $\mathcal{C}^\infty_c(\mathcal{U}_m)$ denote a partition of unity subordinate to the open covering $\mathcal{U}_m$; that is,

$$\sum_{m=0}^K \zeta_m = 1 \quad \text{and} \quad \operatorname{supp}(\zeta_m) \subseteq \mathcal{U}_m \quad \forall\, m\,.$$

Finally, let $\boldsymbol{g}_\epsilon$ denote a smooth regularization of $\boldsymbol{g}$ defined by

$$\boldsymbol{g}_\epsilon = \sum_{m=1}^K \sqrt{\zeta_m}\,\big[\Lambda_\epsilon\big((\sqrt{\zeta_m}\,\boldsymbol{g}) \circ \theta_m\big)\big] \circ \theta_m^{-1}\,.$$

It follows that for $\epsilon \ll 1$ sufficiently small,

$$(6.6)\qquad a^{jk}_{\pm\,\epsilon}(x)\xi_j\xi_k \ge \frac{\lambda}{2}\,|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^{\mathrm{n}}, x \in \Omega\,.$$

Hence, by Theorem 3, the solution $\boldsymbol{u}^\epsilon$ to the variational problem

$$\int_{\Omega^\pm} a^{jk}_\epsilon \frac{\partial \boldsymbol{u}^{\epsilon i}}{\partial x_k}\frac{\partial \boldsymbol{\varphi}^i}{\partial x_j}\,dx = \int_{\Omega^+} \boldsymbol{f}_\epsilon\,\boldsymbol{\varphi}\,dx + \int_\Gamma \boldsymbol{g}_\epsilon\,\boldsymbol{\varphi}\,dS \quad \forall\, \boldsymbol{\varphi} \in \mathcal{V},$$

satisfies $\boldsymbol{u}^\epsilon_\pm \in H^k(\Omega^\pm)$ for all $k \ge 1$; in particular, the vector fields $\boldsymbol{u}^\epsilon_\pm$ are smooth. We next establish an ϵ-independent upper bound for $\|\boldsymbol{u}^\epsilon_\pm\|_{H^{\ell+1}(\Omega^\pm)}$.

Step 1 (Regularity in horizontal directions near Γ). We fix $m \in \{1,\dots,K\}$ and set

$$U_\pm = \boldsymbol{u}^\epsilon_\pm \circ \theta_m,\ F = \boldsymbol{f}_\epsilon \circ \theta_m,\ G = \boldsymbol{g}_\epsilon \circ \theta_m,\ \xi = \zeta_m \circ \theta_m,\ \text{and } \Phi = \boldsymbol{\varphi} \circ \theta_m\,.$$

With $A = [D\theta_m]^{-1}$, we define $b^{rs} = (a^{jk} \circ \theta_m) A^s_k A^r_j$. Then, since $\|D\theta_m - \mathrm{Id}\|_{L^\infty(B^+_m)} \ll 1$, the matrix b is positive-definite:

$$b^{rs}\xi_r\xi_s = (a^{jk} \circ \theta_m) A^s_k A^r_j \xi_r \xi_s \ge \lambda |A^{\mathrm{T}}\xi|^2 \ge \frac{\lambda}{4}|\xi|^2 \quad \forall \xi \in \mathbb{R}^{\mathrm{n}}\,. \tag{6.7}$$

By the change-of-variables formula, the variational formulation is written as

$$\int_{B^\pm_m} b^{rs} \frac{\partial U^i}{\partial x_r}\frac{\partial \Phi^i}{\partial x_s}\, dx = \int_{B^\pm_m} F\,\Phi\, dx + \int_{B^0_m} G\,\Phi\, dS \quad \forall\, \Phi \in H^1_0(B_m),$$

where $\int_{B^\pm_m} b^{rs}\frac{\partial U^i}{\partial x_r}\frac{\partial \Phi^i}{\partial x_s}\, dx = \int_{B^+_m} b^{rs}_+ \frac{\partial U^i_+}{\partial x_r}\frac{\partial \Phi^i}{\partial x_s}\, dx + \int_{B^-_m} b^{rs}_- \frac{\partial U^i_-}{\partial x_r}\frac{\partial \Phi^i}{\partial x_s}\, dx$.

With $\Delta_0 = \sum_{\alpha=1}^{\mathrm{n}-1} \frac{\partial^2}{\partial x^2_\alpha}$ denoting the horizontal Laplace operator, we define the test function

$$\Phi = (-1)^\ell\big[\xi \Delta^\ell_0(\xi U)\big],$$

so that

$$\begin{aligned}\int_{B^\pm_m} & b^{rs}\frac{\partial U^i}{\partial x_r}\frac{\partial \Phi^i}{\partial x_s}\, dx \\ &\le C\Big[\|\boldsymbol{f}_+\|_{H^{\ell-1}(\Omega^+)} + \|\boldsymbol{f}_-\|_{H^{\ell-1}(\Omega^-)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)}\Big]\big\|\bar\partial^\ell(\xi U_\pm)\big\|_{H^1(B^\pm_m)}\,.\end{aligned} \tag{6.8}$$

We focus now on the left-hand side of (6.8). We let $\bar\partial = (\partial_1,\cdots,\partial_{n-1})$ denote the horizontal gradient, and write

$$\begin{aligned}\bar\partial^\ell V\,\bar\partial^\ell W &= \sum_{\alpha_1=1}^{\mathrm{n}-1}\cdots\sum_{\alpha_\ell=1}^{\mathrm{n}-1} \frac{\partial^\ell V}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_\ell}}\ \frac{\partial^\ell W}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_\ell}},\\ \bar\partial^{\ell-1} V\,\bar\partial^{\ell+1} W &= \sum_{\alpha_1=1}^{\mathrm{n}-1}\cdots\sum_{\alpha_{\ell-1}=1}^{\mathrm{n}-1} \frac{\partial^\ell V}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_{\ell-1}}}\ \frac{\partial^\ell \Delta_0 W}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_{\ell-1}}},\end{aligned}$$

and so forth. Then,

$$\begin{aligned}\int_{B^\pm_m} & b^{rs}\frac{\partial U^i}{\partial x_r}\frac{\partial \Phi^i}{\partial x_s}\, dx \\ &= \int_{B^\pm_m} \bar\partial^\ell\big[b^{rs}(\xi U),_r\big]\bar\partial^\ell(\xi U),_s\, dx - \int_{B^\pm_m} \bar\partial^\ell\big[b^{rs} U \xi,_r\big]\bar\partial^\ell(\xi U),_s\, dx \\ &\quad + \int_{B^\pm_m} \bar\partial^{\ell-1}\big[b^{rs} U,_r\,\xi,_s\big]\bar\partial^{\ell+1}(\xi U) dx\,.\end{aligned} \tag{6.9}$$

For the first term on the right-hand side of (6.9), we make use of (6.7) and Young's inequality to conclude that

$$\begin{aligned}\int_{B_m^\pm} \bar\partial^\ell \big[b^{rs}(\xi U),_r\big]\bar\partial^\ell(\xi U),_s\,dx &= \int_{B_m^\pm} b^{rs}\bar\partial^\ell(\xi U),_r\,\bar\partial^\ell(\xi U),_s\,dx \\ &\quad + \int_{B_m^\pm}\big[\{\bar\partial^\ell, b^{rs}\}(\xi U),_r\big]\bar\partial^\ell(\xi U),_s\,dx \\ &\geq \big(\frac{\lambda}{8}-\delta\big)\big\|\bar\partial^\ell D(\xi U_\pm)\big\|^2_{L^2(B_m^\pm)} - C_\delta\big\|\{\bar\partial^\ell, b\}D(\xi U_\pm)\big\|^2_{L^2(B_m^\pm)}\,.\end{aligned}$$

Then, Corollary 6 with $\epsilon = 1/8$ shows that

$$\begin{aligned}\int_{B_m^\pm} \bar\partial^\ell\big[b^{rs}(\xi U),_r\big]\bar\partial^\ell(\xi U),_s\,dy \geq \big(\frac{\lambda}{8}-\delta\big)\big\|\bar\partial^\ell D(\xi U_\pm)\big\|^2_{L^2(B_m^\pm)} \\ - C_\delta\|a_\pm\|^2_{H^{\rm k}(\Omega_\pm)}\|\boldsymbol{u}^\epsilon_\pm\|^2_{H^{\ell+\frac{7}{8}}(\Omega_\pm)}\,.\end{aligned} \tag{6.10}$$

By Lemma 5, for $0\leq \ell\leq {\rm k}+1$, $f_\pm\in H^{\max\{{\rm k},\ell\}}(\Omega_\pm)$ and $g_\pm\in H^\ell(\Omega_\pm)$, and for a generic C,

$$\begin{aligned}\|f_\pm\, g_\pm\|_{H^\ell(\Omega)} \leq C\|f_\pm\|_{H^{\max\{{\rm k},\ell\}}(\Omega^\pm)}\|g_\pm\|_{H^\ell(\Omega^\pm)} \\ \forall\, f_\pm\in H^{\max\{{\rm k},\ell\}}(\Omega^\pm), g_\pm\in H^\ell(\Omega^\pm)\,.\end{aligned} \tag{6.11}$$

For the second and third terms on the right-hand side of (6.9), we use the inequality (6.11), and find that

$$\begin{aligned}\left|\int_{B_m^\pm}\bar\partial^\ell\big[b^{rs}U\xi,_r\big]\bar\partial^\ell(\xi U),_s\,dx\right| + \left|\int_{B_m^\pm}\bar\partial^{\ell-1}\big[b^{rs}U,_r\xi,_s\big]\bar\partial^{\ell+1}(\xi U)dx\right| \\ \leq C_\delta\|a_\pm\|^2_{H^{\rm k}(\Omega)}\|\boldsymbol{u}^\epsilon_\pm\|^2_{H^\ell(\Omega^\pm)} + \delta\big\|\bar\partial^\ell D(\xi U_\pm)\big\|^2_{L^2(B_m^\pm)}\,.\end{aligned} \tag{6.12}$$

Choosing $\delta>0$ sufficiently small in (6.10) and (6.12), we conclude that

$$\big\|\xi\bar\partial^\ell DU_\pm\big\|_{L^2(B_m^\pm)} \leq C\Big[\|\boldsymbol{f}_\pm\|_{H^{\ell-1}(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)} + \|a\|_{H^{\rm k}(\Omega)}\|\boldsymbol{u}^\epsilon_\pm\|_{H^{\ell+\frac{7}{8}}(\Omega^\pm)}\Big]\,. \tag{6.13}$$

Step 2 (Regularity in the vertical direction near Γ). We write (6.1a) as

$$-\xi\big(b^{rs}U_\pm,_s\big),_r = \xi F_\pm \text{ in } B_m^\pm\,. \tag{6.14}$$

We analyze (6.14) in the $+$-phase and drop the $+$-subscript for notational clarity. With $U,_{\rm n}$ denoting $\partial U/\partial x_{\rm n}$, we have that

$$\begin{aligned}-\xi b^{\rm nn}U,_{\rm nn} \\ = \xi\Big[F - b^{\rm nn},_{\rm n}U,_{\rm n} - \sum_{(r,s)\neq({\rm n},{\rm n})} b^{rs},_r U,_s - \sum_{(r,s)\neq({\rm n},{\rm n})} b^{rs}U,_{sr}\Big] \text{ in } B_m^+\,.\end{aligned} \tag{6.15}$$

We analyze the terms on the right-hand side of (6.15). For any integer j such that $0\leq j\leq \ell-1$,

$$\|\bar\partial^{\ell-1-j}D^jF\|_{L^2(B_m^+)} \leq C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega^+)}\Big]\,.$$

Moreover, since $\ell \le \mathrm{k}$, by Lemma 5 with $\epsilon = 1/8$,

$$\|\overline{\partial}^{\ell-1-j} D^j(\xi b^{\mathrm{nn}},_{\mathrm{n}} U,_{\mathrm{n}})\|_{L^2(B_m^+)} + \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})} \|\overline{\partial}^{\ell-1-j} D^j b^{rs},_r U,_s \|_{L^2(B_m^+)}$$
$$\le C\sum_{r=0}^{\ell-1} \|D^{\ell-r} a D^{r+1}\boldsymbol{u}^\epsilon\|_{L^2(\Omega^+)}$$
$$\le C\sum_{r=1}^{\ell} \|D^{\ell+1-r} a D^{r}\boldsymbol{u}^\epsilon\|_{L^2(\Omega^+)} \le C_\epsilon \|a\|_{H^{\mathrm{k}}(\Omega^+)} \|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^+)} .$$

Finally, by Corollary 6 with $\epsilon = 1/8$,

$$\|\{\overline{\partial}^{\ell-1-j} D^j, \xi b^{\mathrm{nn}}\} U,_{\mathrm{nn}}\|_{L^2(B_m^+)} + \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})} \|\{\overline{\partial}^{\ell-1-j} D^j, \xi b^{rs}\} U,_{rs}\|_{L^2(B_m^+)}$$
$$\le C_\epsilon \|a\|_{H^{\mathrm{k}}(\Omega)} \|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^+)} .$$

Therefore, for $0 \le j \le \ell - 1$, letting $\overline{\partial}^{\ell-1-j} D^j$ act on (6.15),

$$\xi b^{\mathrm{nn}} \overline{\partial}^{\ell-1-j} D^j U,_{\mathrm{nn}} = \boldsymbol{G}_{(\ell,j)} - \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})} \xi b^{rs} \overline{\partial}^{\ell-1-j} D^j U,_{rs} \tag{6.16}$$

for a function $\boldsymbol{G}_{(\ell,j)}$ satisfying

$$\|\boldsymbol{G}_{(\ell,j)}\|_{L^2(B_m^+)} \le C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega^+)} + \|a\|_{H^{\mathrm{k}}(\Omega^+)} \|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^+)}\Big] .$$

Now we argue by induction on $0 \le j \le \ell-1$. By (6.7), $b^{\mathrm{nn}} \ge \frac{\lambda}{2}$ so that when $j = 0$, the inequalities (6.13) and (6.16) show that

$$\|\xi \overline{\partial}^{\ell-1} U,_{\mathrm{nn}}\|_{L^2(B_m^+)} \le \|\boldsymbol{G}_{(\ell,j)}\|_{L^2(B_m^+)} + \sum_{(r,s)\neq(\mathrm{n},\mathrm{n})} \|b^{rs}\|_{L^\infty(B_m^+)} \|\xi \partial^{\ell-1} U,_{rs}\|_{L^2(B_m^+)}$$
$$\le C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega^+)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)} + \|a\|_{H^{\mathrm{k}}(\Omega^+)} \|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^+)}\Big]$$

which, combined with (6.13), provides the estimate

$$\|\xi \overline{\partial}^{\ell-1} D^2 U\|_{L^2(B_m^+)} \le C\Big[\|\boldsymbol{f}\|_{H^{\ell-1}(\Omega^+)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)} + \|a\|_{H^{\mathrm{k}}(\Omega^+)} \|\boldsymbol{u}^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^+)}\Big].$$

Repeating this process for $j = 1, \cdots, \ell$ and including the analysis in the $-$-phase, we conclude that

$$\|\xi D^{\ell+1} U_\pm\|_{L^2(B_m^\pm)} \le C\Big[\|\boldsymbol{f}_\pm\|_{H^{\ell-1}(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)} + \|a_\pm\|_{H^{\mathrm{k}}(\Omega^\pm)} \|\boldsymbol{u}_\pm^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^\pm)}\Big] . \tag{6.17}$$

Step 3 (Completing the regularity theory). Let $\chi_\pm \ge 0$ be in $\mathcal{C}_c^\infty(\Omega^\pm)$ so that $\mathrm{supp}(\chi_\pm) \subset\subset \Omega^\pm$. Repeating the computations above, we find that

$$\|\chi_\pm D^{\ell+1} \boldsymbol{u}_\pm^\epsilon\|_{L^2(\Omega^\pm)} \le C\Big[\|\boldsymbol{f}_\pm\|_{H^{\ell-1}(\Omega^\pm)} + \|a_\pm\|_{H^{\mathrm{k}}(\Omega^\pm)} \|\boldsymbol{u}_\pm^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^\pm)}\Big] . \tag{6.18}$$

The inequalities (6.17) and (6.18) establish the inequality

$$\|\boldsymbol{u}_\pm^\epsilon\|_{H^{\ell+1}(\Omega^\pm)} \le C\Big[\|\boldsymbol{f}_\pm\|_{H^{\ell-1}(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)} + \|a_\pm\|_{H^{\mathrm{k}}(\Omega^\pm)} \|\boldsymbol{u}_\pm^\epsilon\|_{H^{\ell+\frac{7}{8}}(\Omega^\pm)}\Big] . \tag{6.19}$$

Since

$$\|\boldsymbol{u}^\epsilon_\pm\|_{H^{\ell+\frac{7}{8}}(\Omega^\pm)} \le C\|\boldsymbol{u}^\epsilon_\pm\|^{1-\frac{1}{8\ell}}_{H^{\ell+1}(\Omega^\pm)}\|\boldsymbol{u}^\epsilon_\pm\|^{\frac{1}{8\ell}}_{H^1(\Omega^\pm)},$$

Young's inequality shows that

$$\begin{aligned}
&\|\boldsymbol{u}^\epsilon_\pm\|_{H^{\ell+1}(\Omega^\pm)} \\
&\qquad \le C_\delta\Big[\|\boldsymbol{f}_\pm\|_{H^{\ell-1}(\Omega^\pm)} + \|\boldsymbol{g}\|_{H^{\ell-0.5}(\Gamma)} + \mathcal{P}\big(\|a_\pm\|_{H^{\mathrm{k}}(\Omega^\pm)}\big)\|\boldsymbol{u}^\epsilon_\pm\|_{H^1(\Omega^\pm)}\Big] \\
&\qquad\quad + \delta\|\boldsymbol{u}^\epsilon_\pm\|_{H^{\ell+1}(\Omega^\pm)}
\end{aligned}$$

for some polynomial function $\mathcal{P}$. Finally, the inequality (6.4) is established by choosing $\delta > 0$ sufficiently small, letting $\epsilon \to 0$, and using the a priori H^1-estimate. ☐

Appendix A. Some basic inequalities

Lemma 5. *For* $\mathrm{k} > \frac{\mathrm{n}}{2}$ *and* $0 \le \ell \le \mathrm{k}$, *let* $\mathrm{O} \subseteq \mathbb{R}^{\mathrm{n}}$ *be a bounded smooth domain. Then for all* $\epsilon \in \big(0, \frac{1}{4}\big)$, *there exists a constant* C_ϵ *depending on* ϵ *such that for all* $f \in H^{\mathrm{k}}(\mathrm{O})$ *and* $g \in H^{\ell-\epsilon}(\mathrm{O})$,

$$\sum_{j=1}^{\ell}\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \le C_\epsilon\|f\|_{H^{\mathrm{k}}(\mathrm{O})}\|g\|_{H^{\ell-\epsilon}(\mathrm{O})}\,. \tag{A.1}$$

Proof. We estimate $D^j f D^{\ell-j} g$ for $j = 1, \cdots, \ell$ as follows:

Step 1. If $1 \le j \le \frac{\mathrm{n}}{2}$, by the Sobolev inequalities

$$\begin{aligned}
\|w\|_{L^{\frac{\mathrm{n}}{j-\epsilon}}(\mathrm{O})} &\le C_\epsilon\|w\|_{H^{\frac{\mathrm{n}}{2}-j+\epsilon}(\mathrm{O})} \quad (\text{if } 0<\epsilon<1),\\
\|w\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(j-\epsilon)}}(\mathrm{O})} &\le C\|w\|_{H^{j-\epsilon}(\mathrm{O})},
\end{aligned}$$

we find that

$$\begin{aligned}
\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} &\le \|D^j f\|_{L^{\frac{\mathrm{n}}{j-\epsilon}}(\mathrm{O})}\|D^{\ell-j} g\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(j-\epsilon)}}(\mathrm{O})}\\
&\le C_\epsilon\|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\|g\|_{H^{\ell-\epsilon}(\mathrm{O})}\,.
\end{aligned}$$

Step 2. If $j = \ell$, by the Sobolev inequality

$$\|w\|_{L^\infty(\mathrm{O})} \le C_\epsilon\|w\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})},$$

we find that

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \le C_\epsilon\|f\|_{H^\ell(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\,.$$

Step 3. If $\frac{\mathrm{n}}{2} < j < \ell$ (this happens only when $\frac{\mathrm{n}}{2} < \ell \le \mathrm{k}$), we consider the following two sub-cases:

Case A ($\ell \le \mathrm{n}$). Similar to the previous case, by the Sobolev inequalities

$$\|w\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(\ell-j)}}(\mathrm{O})} \le C\|w\|_{H^{\ell-j}(\mathrm{O})} \quad\text{and}\quad \|w\|_{L^{\frac{\mathrm{n}}{\ell-j}}(\mathrm{O})} \le C\|w\|_{H^{\frac{\mathrm{n}}{2}-\ell+j}(\mathrm{O})},$$

and hence, we obtain that

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \le \|D^j f\|_{L^{\frac{2\mathrm{n}}{n-2(\ell-j)}}(\mathrm{O})}\|D^{\ell-j} g\|_{L^{\frac{\mathrm{n}}{\ell-j}}(\mathrm{O})} \le C\|f\|_{H^\ell(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}}(\mathrm{O})}\,.$$

Case B $(\mathrm{n} < \ell \leq \mathrm{k})$. If $j > \mathrm{k} - \frac{\mathrm{n}}{2}$, by the Sobolev inequalities

$$\|w\|_{L^{\frac{2\mathrm{n}}{\mathrm{n}-2(\mathrm{k}-j)}}(\mathrm{O})} \leq C\|w\|_{H^{\mathrm{k}-j}(\mathrm{O})} \quad \text{and} \quad \|w\|_{L^{\frac{\mathrm{n}}{\mathrm{k}-j}}(\mathrm{O})} \leq C\|w\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+j}(\mathrm{O})},$$

we obtain that

$$\begin{aligned}\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} &\leq \|D^j f\|_{L^{\frac{2\mathrm{n}}{n-2(\mathrm{k}-j)}}(\mathrm{O})} \|D^{\ell-j} g\|_{L^{\frac{\mathrm{n}}{\mathrm{k}-j}}(\mathrm{O})} \\ &\leq C\|f\|_{H^{\mathrm{k}}(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})}.\end{aligned}$$

Now suppose that $\frac{\mathrm{n}}{2} < j \leq \mathrm{k} - \frac{\mathrm{n}}{2}$. Note that if $0 < \epsilon < \frac{1}{2}$,

$$\begin{aligned}\|w\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})} &\leq C_\epsilon \|w\|_{W^{j,\infty}(\mathrm{O})} \leq C_\epsilon\|w\|_{H^{\mathrm{k}}(\mathrm{O})}, \\ \|w\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})} &\leq C\|w\|_{H^{\ell-j}(\mathrm{O})} \leq C\|w\|_{H^{\ell-\epsilon}(\mathrm{O})}.\end{aligned}$$

Therefore, by the Gagliardo-Nirenberg-Sobolev interpolation inequality, we obtain that

$$\begin{aligned}\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} &\leq \|f\|_{W^{j,\infty}(\mathrm{O})}\|g\|_{H^{\ell-j}(\mathrm{O})} \\ &\leq C_\epsilon \|f\|^{1-\alpha_j}_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\|f\|^{\alpha_j}_{H^{\mathrm{k}}(\mathrm{O})}\|g\|^{\alpha_j}_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})}\|g\|^{1-\alpha_j}_{H^{\ell-\epsilon}(\mathrm{O})}\end{aligned}$$

for some $\alpha_j \in (0,1)$; hence, by Young's inequality,

$$\|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})} \leq C_\epsilon\Big[\|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\|g\|_{H^{\ell-\epsilon}(\mathrm{O})} + \|f\|_{H^{\mathrm{k}}(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}-\mathrm{k}+\ell}(\mathrm{O})}\Big].$$

Summing over ℓ, we conclude that for $0 < \epsilon < \frac{1}{2}$,

$$\sum_{j=1}^{\ell} \|D^j f D^{\ell-j} g\|_{L^2(\mathrm{O})}$$
$$\leq \begin{cases} C_\epsilon \|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\|g\|_{H^{\ell-\epsilon}(\mathrm{O})} & \text{if } \ell \leq \frac{\mathrm{n}}{2}, \\ C_\epsilon\Big[\|f\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\|g\|_{H^{\ell-\epsilon}(\mathrm{O})} + \|f\|_{H^{\mathrm{k}}(\mathrm{O})}\|g\|_{H^{\frac{\mathrm{n}}{2}+\epsilon}(\mathrm{O})}\Big] & \text{otherwise}. \end{cases}$$

Estimate (A.1) is then obtained from the fact that for all $\epsilon \in (0, \frac{1}{4})$,

$$\frac{\mathrm{n}}{2} + \epsilon \leq \mathrm{k} \quad \text{and} \quad \frac{\mathrm{n}}{2} + \epsilon \leq \ell - \epsilon \ \text{ if (in addition) } \ell > \frac{\mathrm{n}}{2}.$$

□

Corollary 6. *For any $m \in \{1, \dots, K\}$, and for $F \in H^{\mathrm{k}}(B_m^\pm)$ and $G = H^{\ell-\epsilon}(B_m^\pm)$ with $0 < \epsilon < 1/4$ and $1 \leq \ell \leq \mathrm{k}$,*

$$\big\|\{\overline{\partial}^\ell, F\}G\big\|_{L^2(B_m^\pm)} \leq C_\epsilon \|F\|_{H^{\mathrm{k}}(B_m^\pm)}\|G\|_{H^{\ell-\epsilon}(B_m^\pm)}, \tag{A.2}$$

where $\{\overline{\partial}^\ell, F\}G = \overline{\partial}^\ell(FG) - F\overline{\partial}^\ell G$.

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