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On the splash singularity for the free-surface of a Navier–Stokes fluid

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Abstract

In fluid dynamics, an interface *splash* singularity occurs when a locally smooth interface self-intersects in finite time. We prove that for d-dimensional flows, $d = 2$ or 3, the free-surface of a viscous water wave, modeled by the incompressible Navier–Stokes equations with moving free-boundary, has a finite-time splash singularity for a large class of specially prepared initial data. In particular, we prove that given a sufficiently smooth initial boundary (which is close to self-intersection) and a divergence-free velocity field designed to push the boundary towards self-intersection, the interface will indeed self-intersect in finite time.

Keywords: Navier–Stokes; Free-boundary problem; Finite-time singularity; Splash singularity; Interface singularity

1. Introduction

1.1. The interface splash singularity

The fluid interface *splash singularity* was introduced by Castro, Córdoba, Fefferman, Gancedo, & Gómez-Serrano in [8] in the context of the one-phase water waves problem. As shown in Fig. 1, a *splash singularity* occurs when a fluid interface remains locally smooth but self-intersects in finite time. Using methods from complex analysis together with a conformal transformation of the equations, Castro, Córdoba, Fefferman, Gancedo, & Gómez-Serrano [8] showed that a splash singularity occurs in finite time for the 2-d water waves equations. In Coutand & Shkoller [16], we showed the existence of a finite-time splash singularity for the one-phase incompressible Euler equations with free-boundary in 3-d, using a very different approach, founded upon an approximation of the self-intersecting fluid domain by a sequence of smooth fluid domains, each with non self-intersecting boundary. For one-phase flow, it is the vacuum state on one side of the interface which permits this finite-time interface self-intersection, and neither surface tension

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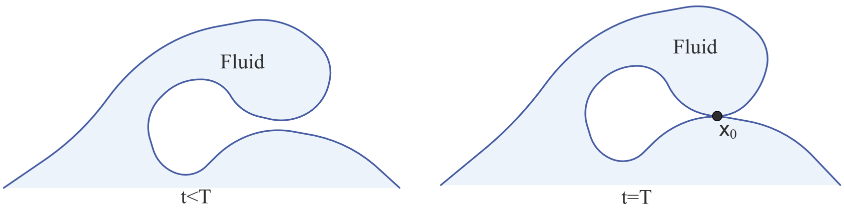

Fig. 1. The splash singularity at a point x_0 occurs when a locally smooth interface self-intersects in finite time $t = T$.

nor magnetic fields nor other inviscid regularizations of the interface change this fact [7,16], and even stationary solutions, having a splash singularity, have been shown to exist (see Córdoba, Enciso, & Grubic [10]).

On the other hand, for the two-phase incompressible Euler equations, wherein the moving interface is a vortex sheet,[1] it was proven by Fefferman, Ionescu, & Lie [20] and Coutand & Shkoller [17] that a splash singularity cannot occur in finite-time while the interface remains locally smooth. In particular, there is a fundamental difference in the behavior of the fluid interface when vacuum is replaced with fluid in the mathematical model.

Since these results have been established for inviscid flows, it is natural to ask if splash singularities can occur for viscous flows modeled by the incompressible Navier–Stokes equations with a moving free-surface. Specifically, given well-prepared initial data, in which the initial boundary is smooth but close to self-intersection, and the initial velocity[2] is chosen so as to move the boundary towards self-intersection, does the boundary in fact self-intersect in a finite amount of time?

Because the methods of constructing splash singularities for inviscid flows have relied on the ability to flow backward-in-time, a new strategy must be devised to study the parabolic Navier–Stokes equations. By using the change-of-variables employed in [8] together with stability estimates, Castro, Córdoba, Fefferman, Gancedo, & Gómez-Serrano in [9] have shown the existence of finite-time splash singularities for the Navier–Stokes equations. Herein, we give a different proof which is amenable to any dimension of space $d \geqslant 2$. Our idea is to prove that the time-of-existence as well as Sobolev estimates for solutions to the free-surface Navier–Stokes equations can be made independent of the distance ϵ between two nearby portions of the free-surface. In particular, we prove that there exists initial data, allowing us to obtain a smooth self-intersecting geometry which is arbitrarily close to *any* given domain with a splash singularity.

Herein, we present a rather simple proof of finite-time self-intersection, based on the construction of fluid domains whose boundary curvature does not change very much (or does not change at all) during the deformation of the domain as it moves closer toward self-intersection. Our stability estimates fundamentally rely upon Sobolev inequalities and elliptic estimates whose constants depend crucially on the curvature of the domain boundary, and hence our constructed geometries provide a simple strategy for keeping such constants uniform. Our method not only works for the Navier–Stokes equations, but also provides a simpler proof of self-intersection for the Euler problems previously considered in [8,16], whose methods relied upon rather technical constructions.

1.2. The Eulerian description of the Navier–Stokes free-boundary problem

For $0 \leqslant t \leqslant T$, the evolution of a d-dimensional ($d = 2$ or 3) one-phase, incompressible, viscous fluid with a moving free boundary is modeled by the incompressible Navier–Stokes equations:

$$u_t + u \cdot \nabla u + \nabla p = \nu \Delta u \quad \text{in } \Omega(t)\,, \tag{1a}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega(t)\,, \tag{1b}$$

$$\nu \operatorname{Def} u \cdot n - p\, n = 0 \quad \text{on } \Gamma(t)\,, \tag{1c}$$

$$\mathcal{V}(\Gamma(t)) = u \cdot n \tag{1d}$$

[1] For the vortex sheet problem, it is necessary to have surface tension in order to ensure well-posedness in Sobolev spaces.

[2] For both the Navier–Stokes and Euler equations, an initial velocity field must be prescribed at time $t = 0$; this is in sharp contrast to Muskat-type problems, wherein the velocity field at time $t = 0$ is determined by the initial geometry of the domain.

$$u = u_0 \quad \text{on } \Omega(0)\,, \tag{1e}$$
$$\Omega(0) = \Omega_0\,. \tag{1f}$$

The open subset $\Omega(t) \subset \mathbb{R}^d$, $d = 2$ or 3, denotes the time-dependent volume occupied by the fluid, $\Gamma(t) := \partial\Omega(t)$ denotes the moving free-surface, $\mathcal{V}(\Gamma(t))$ denotes the normal velocity of $\Gamma(t)$, and $n(t)$ denotes the exterior unit normal vector to the free-surface $\Gamma(t)$. The vector-field $u = (u_1, ..., u_d)$ denotes the Eulerian velocity field, and p denotes the pressure function. We use the notation $\nabla = (\partial_1, ..., \partial_d)$ to denote the gradient operator, and set $\operatorname{Def} u = \nabla u + \nabla u^T$, twice the symmetric part of the gradient of velocity. We have normalized the equations to have all physical constants equal to 1.

The pressure p is a solution to the following Dirichlet problem:

$$-\Delta p = u^i{}_{,j}\, u^j{}_{,i} \quad \text{in } \Omega(t)\,, \tag{2a}$$
$$p = n \cdot [\nu \operatorname{Def} u \cdot n] \quad \text{on } \Gamma(t)\,, \tag{2b}$$

so that given an initial domain Ω and an initial velocity field u_0, the initial pressure is obtained as the solution of (2) at $t = 0$.

Definition 1. Given a locally smooth, time-dependent fluid interface or free-boundary, if there exists a time $T < \infty$ such that the interface $\Gamma(T)$ self-intersects at a point while remaining locally smooth, we call this point of self-intersection at time T a "splash" singularity.

We prove that there exist smooth initial data for the Navier–Stokes equations (1) for which such a splash singularity occurs in finite time.

1.3. Statement of the main theorem

Theorem 1 *(Finite-time splash singularity). There exist*

1. *open bounded C^∞-class initial domains $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with N denoting the unit normal vector field on $\partial\Omega$, and*
2. *smooth divergence-free velocity fields u_0 satisfying the compatibility condition*

$$[\operatorname{Def} u_0 \cdot N] \times N = 0 \text{ on } \partial\Omega\,,$$

such that after a finite time $T^* > 0$, the solution to the Navier–Stokes equations (1) has a splash singularity; that is, the interface $\Gamma(T^*)$ self-intersects.

In Theorem 8, we show that the geometry of such a splash singularity can be prescribed arbitrarily close (in the H^3 norm) to any sufficiently smooth and prescribed self-intersecting domain.

1.4. Prior results for the incompressible Navier–Stokes equations with moving free-surface

Local-in-time well-posedness of solutions to (1) have been known since the pioneering work of Solonnikov [28–30]; his proof did not rely on energy estimates, but rather on Fourier–Laplace transform techniques, which required the use of exponentially weighted anisotropic Sobolev–Slobodeskii spaces with only fractional-order spatial derivatives for the analysis. Beale [5] proved local well-posedness in a similar functional framework, and Abels [1] established the existence theory in the L^p Sobolev space framework. Well-posedness in energy spaces was established by Coutand & Shkoller in [12] for the case of surface tension on the free-boundary, and for Navier–Stokes fluid-structure interaction problems wherein a viscous fluid is coupled to an elastic solid, in [13,14]. Guo & Tice [24] also used energy spaces for local well-posedness for the case of zero surface tension.

Beale [6] established global existence of solutions to (1) for small perturbations of equilibrium. More recent small-data global existence and decay results (both with and without surface tension) can be found in [32], [27], [26], [21],

[4], and [22,23]. Recent results on the limit of zero viscosity and the limit of zero surface tension can be found in [25], [19], and [33].

For the history of the well-posedness and singularity theory for the inviscid problem, we refer the reader to the introduction in [15] and [17].

1.5. Outline of the paper

In Section 2, we define our notation. In Section 3, we define a sequence of domains Ω^ϵ that we use as the initial data for the splash singularity, wherein the boundary Γ^ϵ of these domains is close to self-intersection with a distance ϵ between two approaching portions of Γ^ϵ. We convert the Navier–Stokes equations to Lagrangian coordinates in Section 4, thus fixing the domain. In Section 5, we present some preliminary lemmas which show that the constant appearing in elliptic estimates and the Sobolev embedding theorem is independent of ϵ. In Section 6, we define the sequence of initial divergence-free velocity fields that are guaranteed to satisfy the single compatibility condition that we require, and whose norm is independent of ϵ. Section 7 is devoted to the basic a priori estimates for the Navier–Stokes equations in Lagrangian coordinates; following our approach in [12], we establish estimates for velocity $v \in L^2(0,T;H^3(\Omega^\epsilon)) \cap C^0([0,T];H^2(\Omega^\epsilon))$ which are independent of ϵ. We then prove that the vertical component of velocity $v(\cdot,t)$ at time t remains in an $O(t^{\frac{1}{4}})$ neighborhood of the vertical component of the initial velocity field. Using this fact, we prove the main theorem in Section 8; we show that by choosing ϵ appropriately, a finite-time splash singularity must occur at some time $T^* \in (0, 10\epsilon)$. We consider a completely arbitrary geometry for a splash singularity in Section 9, by following our definition of a generalized splash domain from our previous work in [16]. This, then, allows us to show in Section 10, that we can construct a splash singularity for a geometry which is arbitrarily close in H^3 to *any* prescribed H^3 splash domain.

2. Notation, local coordinates, and some preliminary results

2.1. Notation for the gradient vector

Throughout the paper the symbol ∇ will be used to denote the d-dimensional gradient vector $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d}\right)$.

2.2. Notation for partial differentiation and the Einstein summation convention

The kth partial derivative of F will be denoted by $F_{,k} = \frac{\partial F}{\partial x_k}$. Repeated Latin indices i, j, k, etc., are summed from 1 to d, and repeated Greek indices α, β, γ, etc., are summed from 1 to $d-1$. For example, $F_{,ii} = \sum_{i=1}^d \frac{\partial^2 F}{\partial x_i \partial x_i}$, and $F^i{}_{,\alpha} I^{\alpha\beta} G^i{}_{,\beta} = \sum_{i=1}^d \sum_{\alpha=1}^{d-1} \sum_{\beta=1}^2 \frac{\partial F^i}{\partial x_\alpha} I^{\alpha\beta} \frac{\partial G^i}{\partial x_\beta}$.

2.3. Tangential (or horizontal) derivatives

On each boundary chart $U_l \cap \Omega$, for $1 \leqslant l \leqslant K$, we let $\bar\partial$ denote the *tangential derivative* whose αth-component given by

$$\bar\partial_\alpha f = \left(\frac{\partial}{\partial x_\alpha}[f \circ \theta_l]\right) \circ \theta_l^{-1} = \left((\nabla f \circ \theta_l)\frac{\partial \theta_l}{\partial x_\alpha}\right) \circ \theta_l^{-1}.$$

For functions defined directly on $B^+ = B(0,1) \cap \{x_d > 0\}$, $\bar\partial$ is simply the horizontal derivative $\bar\partial = (\partial_{x_1}, \ldots, \partial_{x_{d-1}})$.

2.4. Sobolev spaces

For integers $k \geqslant 0$ and a bounded domain U of $\mathbb{R}^d$, we define the Sobolev space $H^k(U)$ ($H^k(U;\mathbb{R}^d)$) to be the completion of $C^\infty(\bar U)$ ($C^\infty(\bar U;\mathbb{R}^d)$) in the norm

$$\|u\|_{k,U}^2 = \sum_{|a|\leqslant k} \int_U \left|\nabla^a u(x)\right|^2,$$

for a multi-index $a \in \mathbb{Z}_+^d$, with the convention that $|a| = a_1 + a_2 + \cdots + a_d$. When there is no possibility for confusion, we write $\|\cdot\|_k$ for $\|\cdot\|_{k,U}$. For real numbers $s \geqslant 0$, the Sobolev spaces $H^s(U)$ and the norms $\|\cdot\|_{s,U}$ are defined by interpolation. We will write $H^s(U)$ instead of $H^s(U;\mathbb{R}^d)$ for vector-valued functions.

2.5. *Sobolev spaces on a surface* Γ

For functions $u \in H^k(\Gamma)$, $k \geqslant 0$, we set

$$\|u\|_{k,\Gamma}^2 = \sum_{|a|\leqslant k} \int_\Gamma \left|\bar{\partial}^a u(x)\right|^2,$$

for a multi-index $a \in \mathbb{Z}_+^{d-1}$. For real $s \geqslant 0$, the Hilbert space $H^s(\Gamma)$ and the boundary norm $|\cdot|_s$ is defined by interpolation. The negative-order Sobolev spaces $H^{-s}(\Gamma)$ are defined via duality. That is, for real $s \geqslant 0$, $H^{-s}(\Gamma) = H^s(\Gamma)'$.

2.6. *The unit normal and tangent vectors*

We let $n(\cdot,t)$ denote the outward unit normal vector to the moving boundary $\Gamma(t)$. When $t = 0$, we let N_ϵ denote the outward unit normal to Γ^ϵ. For each $\alpha = 1, ..., d-1$ and $x \in \Gamma^\epsilon$, $\tau_\alpha(x)$ denotes an orthonormal basis of the $(d-1)$-dimensional tangent space to Γ^ϵ at the point x.

3. The sequence of initial domains Ω^ϵ

We shall use, as initial data, a sequence of domains, whose two-dimensional cross-section resembles a dinosaur neck arching over its body.

3.1. *The "dinosaur wave" domains*

Definition 2 *(The domain* Ω*).* Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a smooth bounded domain (as shown on the left of Fig. 2) with boundary Γ. We assume that there are three particular open subsets of Ω as follows:

1. There exists an open subset $\omega \subset \Omega$ such that its boundary $\partial\omega$ is a vertical circular cylinder of radius 1 and of length h.
2. There exists an open subset $\omega_+ \subset \Omega$ which is the lower-half of an open ball of radius 1, located directly below the cylindrical region ω, and in contact with the cylindrical region $\overline{\omega}$. The "south pole" of ω_+ is the point X_+ (see Fig. 3).
3. There exists an open subset $\omega_- \subset \Omega$ directly below, at a distance 1, from the "south pole" X_+ of ω_+, such that the points with maximal vertical coordinate in $\partial\omega_- \cap \Gamma$ form a subset of the horizontal plane $x_d = 0$.
4. Coordinates are assigned to subsets of Ω as follows:
   (a) The origin of $\mathbb{R}^d$ is contained in $\partial\omega_- \cap \Gamma \subset \{x_d = 0\}$.
   (b) The point X_+, the "south pole" of ω_+, has the coordinates $X_+^\alpha = 0$ for $\alpha = 1, ..., d-1$ and $X_+^d = 1$.
   (c) The top boundary of the hemisphere ω_+ is the set $\{(x_h, x_d) \in \mathbb{R}^d \ : \ x_d = 2, \ |x_h| < 1\}$.
   (d) The cylindrical region ω is given by $\{(x_h, x_d) \in \mathbb{R}^d \ : \ 2 < x_d < 2 + h, \ |x_h| < 1\}$.

Definition 3 *(The initial domains* Ω^ϵ*).* For $0 < \epsilon \ll 1$, let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a smooth bounded domain (as shown on the right of Fig. 2) with boundary Γ^ϵ. We define the domain Ω^ϵ to be the following modification of the domain Ω:

1. There exists an open subset $\omega^\epsilon \subset \Omega^\epsilon$, which is a vertical dilation of the domain ω, such that its boundary $\partial\omega^\epsilon \cap \Gamma^\epsilon$ is a vertical circular cylinder of radius r and of length $h + 1 - \epsilon$.

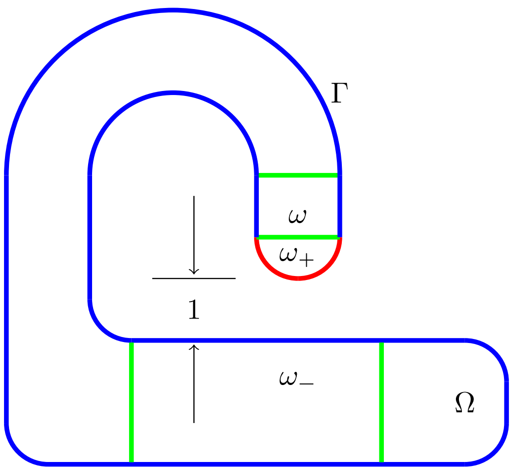

Fig. 2. Left: The "dinosaur wave" domain Ω with boundary Γ. Right: The sequence of "dinosaur waves" Ω^ϵ with boundary Γ^ϵ, $\epsilon > 0$, used as initial data for the Navier–Stokes splash singularity. In order to ensure that a splash occurs, the "dinosaur neck" ω^ϵ stretches downward so that there is a distance ϵ between the two portions. The domains Ω^ϵ simply stretch the neck of the dinosaur, and are identical to Ω away from the neck.

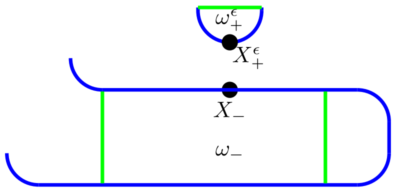

Fig. 3. In a neighborhood of the intended splash point, we suppose that Ω^ϵ consists of two sets: the upper set ω_+^ϵ and the lower set ω_- containing the horizontally flat "dinosaur belly." The point X_+^ϵ is at a distance ϵ from the set ω_- and the point X_- is assumed to be the origin in $\mathbb{R}^d$.

2. There exists an open subset $\omega_+^\epsilon \subset \Omega^\epsilon$ which is the set ω_+ translated vertically downward a distance $1-\epsilon$; hence, ω_+^ϵ is the lower-half of an open ball of radius 1, located directly below the cylindrical region ω^ϵ, and in contact with the cylindrical region $\overline{\omega^\epsilon}$. The "south pole" of ω_+^ϵ is the point X_+^ϵ.
3. There exists an open subset $\omega_- \subset \Omega^\epsilon$ directly below, and a distance ϵ, from the "south pole" X_+^ϵ of ω_+^ϵ, such that the points with maximal vertical coordinate in $\partial\omega_- \cap \Gamma$ form a subset of the horizontal plane $x_d = 0$. We assume that $\partial\omega_- \cap \Gamma$ contains a $d-1$-dimensional ball of radius $\sqrt{\epsilon}$.
4. Coordinates are assigned to subsets of Ω^ϵ as follows:
   (a) The origin of $\mathbb{R}^d$ is contained in $\partial\omega_- \cap \Gamma \subset \{x_d = 0\}$.
   (b) The point X_+^ϵ, the "south pole" of ω_+^ϵ, has the coordinates $X_+^\alpha = 0$ for $\alpha = 1, ..., d-1$ and $X_+^d = \epsilon$.
   (c) The top boundary of the hemisphere ω_+^ϵ is the set $\{(x_h, x_d) \in \mathbb{R}^d \ : \ x_d = \epsilon + 1, \ |x_h| < 1\}$.
   (d) The cylindrical region ω^ϵ is given by $\{(x_h, x_d) \in \mathbb{R}^d \ : \ \epsilon + 1 < x_d < \epsilon + 1 + h, \ |x_h| < 1\}$.

3.2. *Local coordinate charts for Ω and Ω^ϵ*

3.2.1. *Local charts for Ω*

We let $s \geqslant 3$ and $0 < \epsilon \ll 1$. Let $\Omega \subset \mathbb{R}^d$ denote a smooth open set, and let $\{U_l\}_{l=1}^K$ denote an open covering of $\Gamma = \partial\Omega$, such that for each $l \in \{1, 2, \ldots, K\}$, with

$$B = B(0,1), \text{ denoting the open ball of radius 1 centered at the origin and,}$$
$$B^+ = B \cap \{x_d > 0\},$$
$$B^0 = \overline{B} \cap \{x_d = 0\},$$

there exist C^∞ charts θ_l which satisfy

$$\theta_l : B \to U_l \text{ is an } C^\infty \text{ diffeomorphism,} \tag{3a}$$

$$\theta_l(B^+) = U_l \cap \Omega, \quad \theta_l(B^0) = U_l \cap \Gamma\,, \tag{3b}$$

and $\det \nabla\theta_l = C_l$ for a constant $C_l > 0$. We assume these boundary charts can be split into three nonempty categories; to do so, we introduce two additional length scales for the dinosaur neck. We set

$$\delta_1 = \frac{h}{15}\frac{h}{h+3} \text{ and } \delta_2 = \left(\frac{15+4h}{h+3}\right)\frac{h}{15} < \frac{5h}{15}$$

these number being chosen so that,

$$0 < \delta_1 < \delta_2 < \frac{h}{3}\,.$$

The set $\omega = \{(x_h, x_d) \in \mathbb{R}^d \ : \ 2 < x_d < 2+h,\ |x_h| < 1\}$ of the dinosaur neck will be split into three sets:

$$2 \leqslant x_d \leqslant 2 + \frac{h}{3}\,, \quad 2 + \frac{h}{3} \leqslant x_d \leqslant 2 + \frac{2h}{3}\,, \quad 2 + \frac{2h}{3} \leqslant x_d \leqslant 2 + h \tag{4}$$

and the "middle" cylinder $2 + \frac{h}{3} \leqslant x_d \leqslant 2 + \frac{2h}{3}$ will be further refined using the smaller cylinder

$$\{2 + \frac{h}{3} + \delta_1 \leqslant x_d \leqslant 2 + \frac{h}{3} + \delta_2\} \subset \{2 + \frac{h}{3} \leqslant x_d \leqslant 2 + \frac{2h}{3}\}\,. \tag{5}$$

We define three distinct sets of indices l for our boundary charts θ_l, which depend on the location of $\theta_l(B^+)$ with respect to the vertical interval (5) as follows:

- We choose the first K_1 charts such that

$$\omega \cap \{2 + \frac{h}{3} + \delta_1 < x_d < 2 + \frac{h}{3} + \delta_2\} \subset \bigcup_{l=1}^{K_1} \theta_l(B^+) \subset \omega \cap \{2 + \frac{h}{3} < x_d < 2 + \frac{2h}{3}\}\,. \tag{6}$$

- For $K_1 + 1 \leqslant l \leqslant K_2$, $\theta_l(B^+) \not\subset \omega$ and $\theta_l(B^+) \cap \omega_+ = \emptyset$ and

$$\theta_l(B^+) \cap \omega \cap \{2 + \frac{h}{3} + \delta_1 < x_d < 2 + \frac{h}{3} + \delta_2\} = \emptyset\,. \tag{7}$$

- For $K_2 + 1 \leqslant l \leqslant K$, $\theta_l(B^+) \not\subset \omega$ and $\theta_l(B^+) \cap \omega_+ \neq \emptyset$ and

$$\theta_l(B^+) \cap \omega \cap \{2 + \frac{h}{3} + \delta_1 < x_d < 2 + \frac{h}{3} + \delta_2\} = \emptyset\,. \tag{8}$$

We also have that the images of any charts θ_l for $K_1 + 1 \leqslant l \leqslant K_2$ does not intersect any of the images of the charts θ_l for $K_2 + 1 \leqslant l \leqslant K$.

We now repeat this indexing construction for the interior charts. For $L > K$, we let $\{U_l\}_{l=K+1}^L$ denote a family of open sets contained in Ω such that $\{U_l\}_{l=1}^L$ is an open cover of Ω and there exist smooth diffeomorphisms $\theta_l : B \to U_l$ with $\det \nabla\theta_l$ equal to a constant $C_l > 0$ (which is always possible by the construction of [18]).

Just as for the case of the boundary charts and repeating our construction in (6)–(8), we split the index l on the interior charts into three nonempty categories:

- We choose our charts θ_l for $K + 1 \leqslant l \leqslant L_1$ such that

$$\omega \cap \{2 + \frac{h}{3} + \delta_1 < x_d < 2 + \frac{h}{3} + \delta_2\} \subset \cup_{l=K+1}^{L_1} \theta_l(B) \subset \omega \cap \{2 + \frac{h}{3} < x_d < 2 + \frac{2h}{3}\}\,. \tag{9}$$

- For $L_1 + 1 \leqslant l \leqslant L_2$, $\theta_l(B) \not\subset \omega$ and $\theta_l(B) \cap \omega_+ = \emptyset$ and

$$\theta_l(B) \cap \omega \cap \{2 + \frac{h}{3} + \delta_1 < x_d < 2 + \frac{h}{3} + \delta_2\} = \emptyset\,. \tag{10}$$

- For $L_2+1 \leqslant l \leqslant L$, $\theta_l(B) \not\subset \omega$ and $\theta_l(B) \cap \omega_+ \neq \emptyset$ and

$$\theta_l(B) \cap \omega \cap \{2+\frac{h}{3}+\delta_1 < x_d < 2+\frac{h}{3}+\delta_2\} = \emptyset\,. \tag{11}$$

Furthermore, we have that the images of any of the charts θ_l for $L_1+1 \leqslant l \leqslant L_2$ do not intersect any of the images of the charts θ_l for $L_2+1 \leqslant l \leqslant L$.

Definition 4. We set

$$\mathcal{B}_l = B^+ \text{ (upper half-ball) for } l=1,\dots,K \text{ and, } \mathcal{B}_l = B \text{ (ball) for } l=K+1,\dots,L\,. \tag{12}$$

We introduce the sets of indices I_1, I_2, and I_3 as follows:

$$\begin{aligned} I_1 &= \{1 \leqslant l \leqslant K_1\} \cup \{K+1 \leqslant l \leqslant L_1\}\,,\\ I_2 &= \{K_1+1 \leqslant l \leqslant K_2\} \cup \{L_1+1 \leqslant l \leqslant L_2\}\,,\\ I_3 &= \{K_2+1 \leqslant l \leqslant K\} \cup \{L_2+1 \leqslant l \leqslant L\}\,. \end{aligned} \tag{13}$$

These indices correspond to the following regions in Ω:

I_1: *Middle region of the "dinosaur neck" ω.*

$$\omega \cap \{2+\frac{h}{3}+\delta_1 < x_d < 2+\frac{h}{3}+\delta_2\} \subset \bigcup_{l\in I_1} \theta_l(\mathcal{B}_l) \subset \omega \cap \{2+\frac{h}{3} < x_d < 2+\frac{2h}{3}\}\,.$$

I_2: *Above the middle region.*

$$\theta_l(\mathcal{B}_l) \subset \omega \cap \{x_d > 2+\frac{h}{3}+\delta_2\}\,.$$

I_3: *Below the middle region.*

$$\theta_l(\mathcal{B}_l) \subset \omega \cap \{x_d < 2+\frac{h}{3}+\delta_1\}\,.$$

We also assume that

$$\omega \cap \check{\omega}^c \subset \bigcup_{l\in I_2} \theta_l(\mathcal{B}_l)$$

where $\check{\omega}$ denotes the (bottom third) shortened cylindrical region

$$\check{\omega} = \{(x_h, x_d) \in \mathbb{R}^d \;:\; 2 < x_d < 2+\frac{2h}{3},\ |x_h| < 1\}$$

of length $\frac{2h}{3}$, so that the vertical length of $\omega \cap \check{\omega}^c$ is $\frac{h}{3}$.

We finally assume that

$$\omega \cap \tilde{\omega}^c \subset \bigcup_{l\in I_3} \theta_l(\mathcal{B}_l)$$

where $\tilde{\omega}$ denotes the (top third) shortened cylindrical region

$$\tilde{\omega} = \{(x_h, x_d) \in \mathbb{R}^d \;:\; 2+\frac{h}{3} < x_d < 2+h,\ |x_h| < 1\}$$

of length $\frac{2h}{3}$, so that the vertical length of $\omega \cap \tilde{\omega}^c$ is $\frac{h}{3}$.

3.2.2. Local charts for Ω^ϵ

We next explain how the system of coordinate charts $\{\theta_l\}_{l=1}^L$ can be modified to be a system of coordinate charts on the domains Ω^ϵ; for $\epsilon > 0$ sufficiently small, we use the following three steps to define the new charts θ_l^ϵ:

1. For $l \in I_1$, we define the vertically dilated charts (which cover a middle cylinder $\overset{\circ}{\omega}$ with length dilated from $\frac{h}{3}$ to $\frac{h}{3} + 1 - \epsilon$)
$$\theta_l^\epsilon = F^\epsilon(\theta_l)\,,$$
with
$$F^\epsilon(x_1, ..., x_d) = \left(x_1, ..., \frac{h+3+3\epsilon}{h}(x_d - 2 - \frac{h}{3}) + \frac{h}{3} + 1 - \epsilon\right). \tag{14}$$
Note that F^ϵ sends any point with $x_d = 2 + \frac{h}{3}$ in $\overline{\omega}$ (respectively $x_d = 2 + \frac{2h}{3}$) to a point with $x_d = 1 - \epsilon + \frac{h}{3}$ (respectively $x_d = 2 + \frac{2h}{3}$) in Ω^ϵ.
2. For $l \in I_2$, we set $\theta_l^\epsilon = \theta_l$.
3. For $l \in I_3$, we define the vertically-translated charts $\theta_l^\epsilon = \theta_l - (1-\epsilon)e_d$.

Note that
$$\det \nabla\theta_l^\epsilon = \begin{cases} \frac{h+3+3\epsilon}{h} C_l\,, & l \in I_1 \\ C_l\,, & l \in I_2 \cup I_3 \end{cases},$$
where we recall that the charts θ_l were chosen such that $\det \nabla\theta_l = C_l$ for a constant $C_l > 0$. In summary, for $l \in I_1$, the charts θ_l^ϵ are dilated using (14), for $l \in I_2$ the charts $\theta_l^\epsilon = \theta_l$ and are not changed, while for $l \in I_3$ the charts $\theta_l^\epsilon = \theta_l - (1-\epsilon)e_d$ are merely translated in the vertical direction.

3.2.3. Cut-off functions on charts covering Ω

Let $\{\xi_l\}_{l=1}^L$ denote a smooth partition of unity, subordinate to the covering $\{U_l\}_{l=1}^L$; i.e., $\xi_l \in C_c^\infty(U_l)$, $0 \leqslant \xi_l \leqslant 1$, and $\sum_{l=1}^L \xi_l = 1$. With $\mathcal{B}_l$ defined in (12), for each $l = 1, ..., L$, we set $\zeta_l = \xi_l \circ \theta_l$, so that $\zeta_l \in C_c^\infty(\mathcal{B}_l)$ whenever the charts θ_l are smooth.

3.2.4. Cut-off functions supported on the charts covering Ω^ϵ

We next define cut-off functions ξ_l^ϵ which are supported on the image of the charts θ_l^ϵ as follows:
$$\xi_l^\epsilon \circ \theta_l^\epsilon = \xi_l \circ \theta_l\,.$$
With the set $\mathcal{B}_l$ defined in (12), and setting $\zeta_l = \xi_l^\epsilon \circ \theta_l^\epsilon$, we see that (by definition) $\|\zeta_l\|_{k,\mathcal{B}_l}$ is bounded by a constant which is independent of ϵ.

With the set of indices I_1, I_2, and I_3 defined in (13), given our expressions for θ_l^ϵ, we have that for any $x \in \Omega^\epsilon$,
$$\sum_{l=1}^L \xi_l^\epsilon(x) = \sum_{l\in I_2} \xi_l(x) + \sum_{l\in I_3} \xi_l(x + (1-\epsilon)e_d) + \sum_{l\in I_1} \xi_l(x_1, x_2, g^\epsilon(x_d))\,, \tag{15}$$
where g^ϵ is the inverse of F_d^ϵ (defined in (14)) with
$$g^\epsilon(x_d) = \frac{h}{h+3+3\epsilon}(x_d - 1 - \frac{h}{3} + \epsilon) + 2 + \frac{h}{3}\,. \tag{16}$$
The following three possibilities exist for a lower-bound of the sum $\sum_{l=1}^L \xi_l^\epsilon(x)$:

i) If $x \in (\omega^\epsilon \cup \omega_+^\epsilon)^c \cap \Omega^\epsilon$ or $x \in \omega^\epsilon \cap \{x_d \geqslant 2 + \frac{2h}{3}\}$, then,
$$\sum_{l=1}^L \xi_l^\epsilon(x) = \sum_{l\in I_2} \xi_l(x) = 1\,. \tag{17}$$

ii) If $x \in \omega_+^\epsilon$ or $x \in \omega^\epsilon \cap \{x_d \leqslant 1-\epsilon+\frac{h}{3}\}$, then,

$$\sum_{l=1}^{L} \xi_l^\epsilon(x) = \sum_{l\in I_3} \xi_l(x+(1-\epsilon)e_d) = 1\,. \tag{18}$$

iii) If $x \in \omega^\epsilon \cap \{1-\epsilon+\frac{h}{3} \leqslant x_d \leqslant 2+\frac{2h}{3}\}$, then x is in the middle cylindrical region of the dinosaur neck $\overset{\circ}{\omega}$ whose length is stretched from $\frac{h}{3}$ to $\frac{h}{3}+1-\epsilon$, which means that the vertical derivative $\partial_{x_d}\xi_l^\epsilon(x)$ can change with ϵ, and in turn, the sum $\sum_{l=1}^{L}\xi_l^\epsilon(x)$ may drop below the value of 1. As we do not a priori know what this lower-bound will be, we add more charts into this region (with corresponding cut-off functions) in such a way as to ensure that we indeed have a lower-bound of 1 on the sum of the $\xi_l^\epsilon(x)$ for each x in this region.

Specifically, we add an additional $\tilde L$ local charts θ_l to our domain Ω; of these additional $\tilde L$ charts, we add $\mathcal{K}$ additional boundary charts and $\mathcal{L}$ additional interior charts so that $\tilde L = \mathcal{K}+\mathcal{L}$. We then choose the positive integers $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{L}_1$ and $\mathcal{L}_2$, such that the indices $L+1 \leqslant l \leqslant \tilde L$ are split into

$$L+1 \leqslant l \leqslant L+\mathcal{K}_1\,,\ L+\mathcal{K}_1+1 \leqslant l \leqslant L+\mathcal{K}_2\,,\ L+\mathcal{K}_2+1 \leqslant l \leqslant L+\mathcal{K}\,,$$
$$L+\mathcal{K}+1 \leqslant l \leqslant L+\mathcal{K}+\mathcal{L}_1\,,\ L+\mathcal{K}+\mathcal{L}_1+1 \leqslant l \leqslant L+\mathcal{K}+\mathcal{L}_2\,,\ L+\mathcal{K}+\mathcal{L}_2+1 \leqslant l \leqslant L+\tilde L\,.$$

The integers $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{L}_1$, and $\mathcal{L}_2$ are chosen by repeating the index construction for the integers K_1, K_2, L_1, and L_2 in (6)–(11), but with a modification in the vertical splitting of the dinosaur neck ω in (4), in which

$$\frac{h}{3} \text{ is replaced by } \frac{2h}{5}\,,$$
$$\frac{2h}{3} \text{ is replaced by } \frac{3h}{5}\,.$$

Following Definition 4, we introduce the sets of indices $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ as follows:

$$\mathcal{I}_1 = \{L+1 \leqslant l \leqslant L+\mathcal{K}_1\} \cup \{L+\mathcal{K}+1 \leqslant l \leqslant L+\mathcal{K}+\mathcal{L}_1\}\,,$$
$$\mathcal{I}_2 = \{L+\mathcal{K}_1+1 \leqslant l \leqslant L+\mathcal{K}_2\} \cup \{L+\mathcal{K}+\mathcal{L}_1+1 \leqslant l \leqslant L+\mathcal{K}+\mathcal{L}_2\}\,,$$
$$\mathcal{I}_3 = \{L+\mathcal{K}_2+1 \leqslant l \leqslant L+\mathcal{K}\} \cup \{L+\mathcal{K}+\mathcal{L}_2+1 \leqslant l \leqslant L+\tilde L\}\,.$$

We define

$$\tilde\delta_1 = \frac{\delta_1}{10} \ \text{ and } \ \tilde\delta_2 = \frac{\delta_2}{10}$$

these number being chosen so that,

$$0 < \tilde\delta_1 < \tilde\delta_2 < \frac{h}{30}\,.$$

The indices $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ correspond to the following regions in Ω:

$\mathcal{I}_1$: *Middle region of the "dinosaur neck."*

$$\omega \cap \{2+\frac{2h}{5}+\tilde\delta_1 < x_d < 2+\frac{2h}{5}+\tilde\delta_2\} \subset \bigcup_{l\in\mathcal{I}_1}\theta_l(\mathcal{B}_l) \subset \omega\cap\{2+\frac{2h}{5} < x_d < 2+\frac{3h}{5}\}\,.$$

$\mathcal{I}_2$: *Above the middle region.*

$$\text{Each chart } \theta_l(\mathcal{B}_l) \subset \omega \cap \{x_d > 2+\frac{2h}{5}+\tilde\delta_2\}\,.$$

$\mathcal{I}_3$: *Below the middle region.*

$$\text{Each chart } \theta_l(\mathcal{B}_l) \subset \omega \cap \{x_d < 2+\frac{2h}{5}+\tilde\delta_1\}\,.$$

We define the additional charts for the dilated region Ω^ϵ as follows:

1. For $l \in \mathcal{I}_1$, we define the vertically dilated charts

$$\theta_l^\epsilon = \mathcal{F}^\epsilon(\theta_l)\,,$$

with

$$\mathcal{F}^\epsilon(x_1, ..., x_d) = \left(x_1, ..., \frac{h+5+5\epsilon}{h}(x_d - 2 - \frac{2h}{5}) + \frac{2h}{5} + 1 - \epsilon\right).$$

Note that $\mathcal{F}^\epsilon$ sends any point with $x_d = 2 + \frac{2h}{5}$ in $\overline{\omega}$ (respectively $x_d = 2 + \frac{3h}{5}$) to a point with $x_d = 1 - \epsilon + \frac{2h}{5}$ (respectively $x_d = 2 + \frac{3h}{5}$) in Ω^ϵ, and the inverse function $\mathcal{G}^\epsilon$ is given by

$$\mathcal{G}^\epsilon(x_d) = \frac{h}{h+5+5\epsilon}(x_d - 1 - \frac{2h}{5} + \epsilon) + 2 + \frac{2h}{5}\,.$$

The set $\omega = \{(x_h, x_d) \in \mathbb{R}^d \ : \ 2 < x_d < 2+h,\ |x_h| < 1\}$ of the dinosaur neck is split into three sets:

$$2 \leqslant x_d \leqslant 2 + \frac{2h}{5}\,, \quad 2 + \frac{2h}{5} \leqslant x_d \leqslant 2 + \frac{3h}{5}\,, \quad 2 + \frac{3h}{5} \leqslant x_d \leqslant 2 + h$$

and the "middle" cylinder $2 + \frac{2h}{5} \leqslant x_d \leqslant 2 + \frac{3h}{5}$ will be further refined using the smaller cylinder

$$\{2 + \frac{2h}{5} + \tilde{\delta}_1 \leqslant x_d \leqslant 2 + \frac{2h}{5} + \tilde{\delta}_2\} \subset \{2 + \frac{2h}{5} \leqslant x_d \leqslant 2 + \frac{3h}{5}\}\,.$$

2. For $l \in \mathcal{I}_2$, we set $\theta_l^\epsilon = \theta_l$.
3. For $l \in \mathcal{I}_3$, we define the vertically-translated charts $\theta_l^\epsilon = \theta_l - (1-\epsilon)e_d$.

Note that

$$\det \nabla \theta_l^\epsilon = \begin{cases} \frac{h+5+5\epsilon}{h} C_l\,, & l \in \mathcal{I}_1 \\ C_l\,, & l \in \mathcal{I}_2 \cup \mathcal{I}_3 \end{cases}\,,$$

where we recall that the charts θ_l were chosen such that $\det \nabla \theta_l = C_l$ for a constant $C_l > 0$.
We denote by $\{\xi_l\}_{l=L+1}^{L+\tilde{L}}$ a smooth partition of unity associated to the covering $\{\theta_l(\mathcal{B}_l)\}_{l=L+1}^{L+\tilde{L}}$. We then repeat our previous construction of the functions ξ_l^ϵ and just as in (17)–(18), we have the following two analogous cases:

a. If $x \in (\omega^\epsilon \cup \omega_+^\epsilon)^c \cap \Omega^\epsilon$ or $x \in \omega^\epsilon \cap \{x_d \geqslant 2 + \frac{3h}{5}\}$, and therefore if

$$x \in \omega^\epsilon \cap \{2 + \frac{3h}{5} \leqslant x_d \leqslant 2 + \frac{2h}{3}\}\,,$$

then

$$\sum_{l=L+1}^{L+\tilde{L}} \xi_l^\epsilon(x) = \sum_{l \in \mathcal{I}_2} \xi_l^\epsilon(x) = 1\,. \tag{19}$$

b. If $x \in \omega_+^\epsilon$ or $x \in \omega^\epsilon \cap \{x_d \leqslant 1 - \epsilon + \frac{2h}{5}\}$, and therefore if

$$x \in \omega^\epsilon \cap \{1 - \epsilon + \frac{h}{3} \leqslant x_d \leqslant 1 - \epsilon + \frac{2h}{5}\}\,,$$

then

$$\sum_{l=L+1}^{L+\tilde{L}} \xi_l^\epsilon(x) = \sum_{l \in \mathcal{I}_3} \xi_l^\epsilon(x) = 1\,. \tag{20}$$

In equations (19) and (20), we have shown that the additional $l = L+1, ..., L+\tilde{L}$ partition functions sum to 1, while the original $l = 1, ..., L$ partition functions sum to a number greater than zero.
The remaining possibility is that

$$x \in \omega^\epsilon \cap \{1 - \epsilon + \frac{2h}{5} \leqslant x_d \leqslant 2 + \frac{3h}{5}\},$$

in which case,

$$2+\frac{h}{3}+\frac{h}{15}\frac{h}{h+3+3\epsilon}\leqslant g^\epsilon(x)\leqslant 2+\frac{h}{3}+h\frac{1+\frac{4h}{15}+\epsilon}{3+h+3\epsilon}\,,$$

and thus since

$$\lim_{\epsilon\to 0} 2+\frac{h}{3}+\frac{h}{15}\frac{h}{h+3+3\epsilon}=2+\frac{h}{3}+\delta_1\,,\ \text{and}\ \lim_{\epsilon\to 0} 2+\frac{h}{3}+h\frac{1+\frac{4h}{15}+\epsilon}{3+h+3\epsilon}=2+\frac{h}{3}+\delta_2\,,$$

we have from the assumptions (7), (8), (10) and (11) that (for $\epsilon>0$ small enough)

$$\sum_{l\in I_2}\xi_l(x_1,\dots,g^\epsilon(x_d))=0=\sum_{l\in I_3}\xi_l(x_1,\dots,g^\epsilon(x_d))\,,$$

and so

$$\sum_{l\in I_1}\xi_l(x_1,\dots,g^\epsilon(x_d))=1\,.$$

Together with (15), we have established that

$$\sum_{l=1}^{L}\xi_l^\epsilon(x)=1\,. \tag{21}$$

In this remaining case, we have shown that the original $l=1,\dots,L$ partition functions sum to 1, while the additional $l=L+1,\dots,L+\tilde{L}$ partition functions some to a number greater than zero.

We then use the open covering $\{\theta_l^\epsilon(B)\}_{l=1}^{L+\tilde{L}}$ of Ω^ϵ, with the associated compactly supported functions $\{\xi_l^\epsilon\}_{l=1}^{L+\tilde{L}}$. Using (17), (18), (19), (20) and (21), it follows that the functions $\{\xi_l^\epsilon\}_{l=1}^{L+\tilde{L}}$ satisfy

$$\sum_{l=1}^{L+\tilde{L}}\xi_l^\epsilon(x)\geqslant 1\ \ \forall x\in\Omega^\epsilon\,, \tag{22}$$

and we have therefore established the strictly positive uniform-in-ϵ lower-bound for the functions $\{\xi_l^\epsilon\}_{l=1}^{L+\tilde{L}}$.

4. The Lagrangian description of the Navier–Stokes free-boundary problem

For $\epsilon>0$, we let Ω^ϵ with boundary Γ^ϵ be given by Definition 3, and we transform the system (1) into a system of equations set on this reference domain. To do so, we shall employ the Lagrangian coordinates.

The Lagrangian flow map $\eta(\cdot,t)$ is the solution of the $\eta_t(x,t)=u(\eta(x,t),t)$ for $t>0$ with initial condition $\eta(x,0)=0$. Since $\operatorname{div} u=0$, it follows that $\det\nabla\eta=1$. For each instant of time t for which the flow is well-defined, we have

$$\eta(\cdot,t):\Omega^\epsilon\to\Omega(t)\text{ is a diffeomorphism;}$$

furthermore, thanks to (1d),

$$\Gamma(t)=\eta(\Gamma^\epsilon,t)\,.$$

Notationally, we keep the dependence on $\epsilon>0$ implicit, except for the initial domain and boundary.

Next, we define

$$\begin{aligned}
v&=u\circ\eta\text{ (Lagrangian velocity)},\\
q&=p\circ\eta\text{ (Lagrangian pressure)},\\
A&=[\nabla\eta]^{-1}\text{ (inverse of the deformation tensor)}\,,\\
g_{\alpha\beta}&=\eta,_\alpha\cdot\eta,_\beta\quad \alpha,\beta=1,\dots,d-1\text{ (induced metric on }\Gamma)\,,\\
\mathfrak{g}&=\det(g_{\alpha\beta})\,.
\end{aligned}$$

We also define the Lagrangian analogue of some of the fundamental differential operators present in this equation:

$$\begin{aligned}
\operatorname{div}_\eta v &= (\operatorname{div} u)\circ\eta = v^i,_j A^j_i\,,\\
\operatorname{curl}_\eta v &= (\operatorname{curl} u)\circ\eta \text{ or } [\operatorname{curl}_\eta v]_i = \varepsilon_{ijk} v^k,_r A^r_j\,,\\
\operatorname{Def}_\eta v &= (\operatorname{Def} u)\circ\eta \text{ or } [\operatorname{Def}_\eta v]^i_j = v^i,_r A^r_j + v^j,_r A^r_i\,,\\
\Delta_\eta v &= (\Delta u)\circ\eta = (A^j_r A^k_r v,_k),_j\ .
\end{aligned}$$

The Lagrangian version of equations (1) is given on the fixed reference domain Ω^ϵ by

$$\eta(\cdot,t) = e + \int_0^t v(\cdot,s)ds \quad \text{in } \Omega^\epsilon\times[0,T]\,, \tag{23a}$$

$$v_t + A^T\nabla q = \nu\Delta_\eta v \qquad \text{in } \Omega^\epsilon\times(0,T]\,, \tag{23b}$$

$$\operatorname{div}_\eta v = 0 \qquad \text{in } \Omega^\epsilon\times[0,T]\,, \tag{23c}$$

$$\nu\operatorname{Def}_\eta v\cdot n - qn = 0 \qquad \text{on } \Gamma^\epsilon\times[0,T]\,, \tag{23d}$$

$$(\eta,v) = (e,u_0) \qquad \text{in } \Omega^\epsilon\times\{t=0\}\,, \tag{23e}$$

where $e(x)=x$ denotes the identity map on Ω, and where we write n for $n(\eta)$ in the Lagrangian description; in particular, the unit normal vector n at the point $\eta(x,t)$ can be expressed in terms of the cofactor matrix A and the time $t=0$ normal vector N_ϵ as

$$n = A^T N_\epsilon/|A^T N_\epsilon|\,.$$

Due to (23c),

$$\Delta_\eta v = \operatorname{div}_\eta \operatorname{Def}_\eta v\,,$$

so that (23d) can be viewed as the natural boundary condition. The variables η, v, and q have an a priori dependence on $\epsilon>0$, but we do not explicitly write this.

Local-in-time existence and uniqueness of solutions to (23) have been known since the pioneering work of Solonnikov [28]. We shall establish a priori estimates for (23) with the initial domain Ω^ϵ and with divergence-free initial velocity fields satisfying the single compatibility condition

$$[\operatorname{Def} u_0^\epsilon\cdot N^\epsilon]\cdot\tau_\alpha^\epsilon = 0 \text{ on } \Gamma^\epsilon\,, \tag{24}$$

where N^ϵ denotes the outward unit normal to Γ^ϵ and τ^ϵ_α, $\alpha=1,...,d-1$, denotes the $d-1$ tangent vectors to Γ^ϵ.

We will show that both the a priori estimates and the time of existence for solutions are independent of the distance $\epsilon>0$ between the falling dinosaur head X^ϵ_+ and the flat trough $\partial\omega_-\cap\{x_d=0\}$ (see Fig. 2). To do so, we shall rely on some basic lemmas that provide us constants which are independent of ϵ.

5. The constants for elliptic estimates and Sobolev inequalities are independent of ϵ

We consider the following linear Stokes problem

$$-\Delta u + \nabla p = f \quad \text{in } \Omega^\epsilon\,, \tag{25a}$$

$$\operatorname{div} u = \phi \quad \text{in } \Omega^\epsilon\,, \tag{25b}$$

$$u = g \quad \text{on } \Gamma^\epsilon\,, \tag{25c}$$

Lemma 2 *(Estimates for the Stokes problem on Ω^ϵ). Suppose that for integers $k\geqslant 3$, $f\in H^{k-2}(\Omega^\epsilon)$, $\phi\in H^{k-1}(\Omega^\epsilon)$, and $g\in H^{k-1/2}(\Gamma^\epsilon)$, and $\int_{\Omega^\epsilon}\phi(x)dx = \int_{\Gamma^\epsilon} g\cdot N\,dS$. Then, there exists a unique solution $u\in H^k(\Omega^\epsilon)$ and $p\in H^{k-1}(\Omega^\epsilon)/\mathbb{R}$ to the Stokes problem (25). Moreover, there is a constant C depending only on Ω, but independent of $\epsilon>0$, such that*

$$\|u\|_{k,\Omega^\epsilon} + \|p\|_{k-1,\Omega^\epsilon} \leqslant C\left(\|f\|_{k-2,\Omega^\epsilon} + \|\phi\|_{k-1,\Omega^\epsilon} + |g|_{k-1/2,\Gamma^\epsilon}\right)\,. \tag{26}$$

Proof. The estimate (26) is well-known on the domain Ω; see, for example, [2]. The corresponding elliptic estimate on the sequence of domains Ω^ϵ follows by localization using the charts θ_l^ϵ, defined in Section 3.2. With the domains $\mathcal{B}_l$ defined by (12), following the elliptic estimates of [2] and using the Sobolev embedding theorem to bound the $H^{k-1}(\mathcal{B}_l)$-class coefficients arising from polynomial combinations of components of $\nabla\theta_l^\epsilon$, we have that

$$\|\zeta_l u\circ\theta_l^\epsilon\|_{k,\mathcal{B}_l}+\|\zeta_l p\circ\theta_l^\epsilon\|_{k-1,\mathcal{B}_l}\leqslant D_1(\|\nabla\theta_l^\epsilon\|_{k-1,\mathcal{B}_l})\left(\|f\|_{k-2,\Omega^\epsilon}+\|\phi\|_{k-1,\Omega^\epsilon}+|g|_{k-1/2,\Gamma^\epsilon}\right),\tag{27}$$

where D_1 is a polynomial function that does not depend on ϵ.

As we shall explain, since the charts θ_l^ϵ are modifications of the charts θ_l by vertical dilation with lower and upper bound that is uniform in ϵ, the constant for the elliptic estimate in each chart is independent of $\epsilon>0$. This follows from our explicit formulas for θ_l^ϵ in Section 3.2.2; for each ϵ and each θ_l^ϵ, using the definition of the dilation F^ϵ given by (14), we have that

$$\|\nabla\theta_l^\epsilon\|_{k-1,\mathcal{B}_l}\leqslant\frac{h+3+3\epsilon}{h}\|\nabla\theta_l\|_{k-1,\mathcal{B}_l}\leqslant(1+\frac{4}{h})\|\nabla\theta_l\|_{k-1,\mathcal{B}_l},\tag{28}$$

for $\epsilon>0$ small enough. Using the bound (28) in the elliptic estimate (27), there exists a constant $D_2>0$ independent of ϵ, such that

$$\|\zeta_l u\circ\theta_l^\epsilon\|_{k,\mathcal{B}_l}+\|\zeta_l p\circ\theta_l^\epsilon\|_{k-1,\mathcal{B}_l}\leqslant D_2\left(\|f\|_{k-2,\Omega^\epsilon}+\|\phi\|_{k-1,\Omega^\epsilon}+|g|_{k-1/2,\Gamma^\epsilon}\right).\tag{29}$$

Moreover, for a polynomial function $D_3>0$ which is independent of ϵ,

$$\|\nabla(\theta_l^\epsilon)^{-1}\|_{k-1,\theta_l(\mathcal{B}_l)}\leqslant D_3(\|\nabla\theta_l\|_{k-1,\mathcal{B}_l}).\tag{30}$$

To prove (30), we begin with the L^2 estimate. We define

$$\mathcal{A}_l^\epsilon(x)=[\nabla\theta_l^\epsilon(x)]^{-1},\quad\mathcal{J}_l^\epsilon=\det[\nabla\theta_l^\epsilon(x)],\quad\text{and }\mathscr{A}_l^\epsilon=\mathcal{A}_l^\epsilon\mathcal{J}_l^\epsilon,$$

with $\mathscr{A}_l^\epsilon$ denoting the cofactor matrix. Recall that $\mathcal{J}_l^\epsilon$ is equal to a constant given by either C_l or $\frac{h+3+3\epsilon}{h}C_l$, so that $1/\mathcal{J}_l^\epsilon\leqslant 1/C_l$. By the inverse function theorem, $\nabla_y(\theta_l^\epsilon)^{-1}(y)=\mathcal{A}_l^\epsilon(x)$ so that

$$\begin{aligned}\|\nabla(\theta_l^\epsilon)^{-1}\|^2_{0,\theta_l(\mathcal{B}_l)}&=\int\limits_{\theta_l(\mathcal{B}_l)}|\nabla_y(\theta_l^\epsilon)^{-1}(y)|^2dy\\&=\int\limits_{\mathcal{B}_l}|\mathcal{A}_l^\epsilon(x))|^2\mathcal{J}_l^\epsilon dx=\int\limits_{\mathcal{B}_l}|\mathscr{A}_l^\epsilon(x))|^2[\mathcal{J}_l^\epsilon]^{-1}dx\leqslant C_l^{-1}\int\limits_{\mathcal{B}_l}|\nabla\theta_l^\epsilon|^{2(d-1)}dx\end{aligned}$$

and hence, using (28), we see that

$$\|\nabla(\theta_l^\epsilon)^{-1}\|^2_{0,\theta_l(\mathcal{B}_l)}\leqslant D_3(\|\nabla\theta_l\|_{k-1,\mathcal{B}_l}).$$

Next, for the H^1 estimate, we use the chain-rule identity that $\frac{\partial}{\partial y_i}=[\mathcal{A}_l^\epsilon]_i^k\frac{\partial}{\partial x_k}$ and write

$$\begin{aligned}\|\nabla\nabla(\theta_l^\epsilon)^{-1}\|^2_{0,\theta_l(\mathcal{B}_l)}&=\int\limits_{\theta_l(\mathcal{B}_l)}\frac{\partial}{\partial y_i}\nabla_y(\theta_l^\epsilon)^{-1}(y)\ \frac{\partial}{\partial y_i}\nabla_y(\theta_l^\epsilon)^{-1}(y)\,dy\\&=\int\limits_{\mathcal{B}_l}[\mathcal{A}_l^\epsilon]_i^k\frac{\partial}{\partial x_k}\left([\mathcal{A}_l^\epsilon]_s^r\right)\ [\mathcal{A}_l^\epsilon]_i^j\frac{\partial}{\partial x_j}\left([\mathcal{A}_l^\epsilon]_s^r\right)\ \mathcal{J}_l^\epsilon dx.\end{aligned}$$

Using the identity

$$\frac{\partial}{\partial x_k}\left([\mathcal{A}_l^\epsilon]_s^r\right)=-[\mathcal{A}_l^\epsilon]_m^r\frac{\partial^2[\theta_l^\epsilon]^m}{\partial x_j\partial x_k}[\mathcal{A}_l^\epsilon]_s^j,$$

we have that

$$\|\nabla\nabla(\theta_l^\epsilon)^{-1}\|^2_{0,\theta_l(\mathcal{B}_l)}\leqslant C_l^{-5}\int\limits_{\mathcal{B}_l}|\nabla\theta_l^\epsilon|^{6(d-1)}\,|\nabla^2\theta_l^\epsilon|^2dx\leqslant D_3(\|\nabla\theta_l\|_{k-1,\mathcal{B}_l}),$$

the last inequality coming from the Sobolev embedding theorem and the fact that $k \geqslant 3$. The estimate for $\nabla^{k-1}\nabla(\theta_l^\epsilon)^{-1}$ follows in the same manner, and we obtain (30).

Since $\sum_{l=1}^{L+\tilde{L}} \xi_l^\epsilon \geqslant 1$ from (22) in Ω^ϵ this proves the lemma. □

Lemma 3 *(Sobolev constant on Ω^ϵ). Independent of ϵ, there exists a constant $C > 0$ which depends only on the domain Ω, such that*

$$\forall u \in H^s(\Omega^\epsilon)\,, \quad s > d/2\,, \quad \max_{x\in\Omega^\epsilon} |u(x)| \leqslant C\|u\|_{s,\Omega^\epsilon}\,.$$

Proof. By Morrey's inequality, for $1 \leqslant l \leqslant L$,

$$\forall u \in H^s(\Omega^\epsilon)\,, \quad s > d/2\,, \quad \max_{\mathcal{B}_l} |u \circ \theta_l^\epsilon| \leqslant C_1\|u \circ \theta_l^\epsilon\|_{s,\mathcal{B}_l}\,, \tag{31}$$

for some $C_1 > 0$ independent of ϵ. Now, depending on the index l, θ_l^ϵ is either equal to θ_l, a vertical translation of θ_l, or a vertical dilation of θ_l given by the map F^ϵ in (14) (see Section 3.2). Thus, as we proved in (28), for $\epsilon > 0$ small enough,

$$\|\nabla\theta_l^\epsilon\|_{s-1,\mathcal{B}_l} \leqslant (1 + \frac{4}{h})\|\nabla\theta_l\|_{s-1,\mathcal{B}_l}\,. \tag{32}$$

By the chain rule, using (32) in (31) shows that we have the existence of a constant $C_2 > 0$ (independent of $\epsilon > 0$ small enough) such that

$$\forall u \in H^s(\Omega^\epsilon)\,, \quad s > d/2\,, \quad \max_{\mathcal{B}_l} |u \circ \theta_l^\epsilon| \leqslant C_2\|u\|_{s,\theta_l^\epsilon(\mathcal{B}_l)}\,. \tag{33}$$

Given that the $\theta_l^\epsilon(\mathcal{B}_l)$ provide a cover of Ω^ϵ, we indeed have proved the lemma. □

The same argument also proves the following

Lemma 4 *(Sobolev constant on Γ^ϵ). Independent of ϵ, there exists a constant $C > 0$ which depends only on Γ, such that*

$$\forall u \in H^s(\Gamma^\epsilon)\,, \quad s > \frac{d}{2} - \frac{1}{2}\,, \quad \max_{x\in\Gamma^\epsilon} |u(x)| \leqslant C\|u\|_{s,\Gamma^\epsilon}\,.$$

Lemma 5 *(Trace theorem on Ω^ϵ). Independent of ϵ, there exists a constant $C > 0$ which depends only on the domain Ω, such that for $s \in (\frac{1}{2}, 3]$*

$$\|u\|_{s-\frac{1}{2},\Gamma^\epsilon} \leqslant C\|u\|_{s,\Omega^\epsilon} \quad \forall u \in H^s(\Omega^\epsilon)\,.$$

Proof. From the standard trace theorem in B^+, we have the existence of a constant $C_1 > 0$ (independent of $\epsilon > 0$ small enough) such that for any boundary chart,

$$\|u \circ \theta_l^\epsilon\|_{s-\frac{1}{2},B_0} \leqslant C\|u \circ \theta_l^\epsilon\|_{s,B^+} \quad \forall u \in H^s(\Omega^\epsilon)\,.$$

By differentiating the (inverse) dilation map g^ϵ in (16), we see that for $\epsilon > 0$ small enough,

$$\|\nabla\theta_l\|_{s-1,B^+} \leqslant \|\nabla\theta_l^\epsilon\|_{s-1,B^+} \leqslant (1 + \frac{4}{h})\|\nabla\theta_l\|_{s-1,B^+}\,. \tag{34}$$

This implies that by the chain rule, we have the existence of a constant $C_2 > 0$ (independent of $\epsilon > 0$ small enough) such that

$$\|u\|_{s-\frac{1}{2},\theta_l^\epsilon(B_0)} \leqslant C_2\|u\|_{s,\theta_l^\epsilon(B^+)} \quad \forall u \in H^s(\Omega^\epsilon)\,.$$

Since Γ^ϵ is the union of all $\theta_l^\epsilon(B_0)$, $1 \leqslant l \leqslant K$, the above inequality implies the result. □

6. The sequence of initial velocity fields u_0^ϵ

6.1. Constructing the sequence of initial velocity fields u_0^ϵ

As described in Definition 3, near the intended splash (or self-intersection) point, the open set Ω^ϵ consists of two sets: the upper set ω_+^ϵ and the lower set ω_- whose boundary contains the flat "dinosaur belly" at $x_d = 0$, as shown in Fig. 3. We let X_+^ϵ denote the point which has the smallest vertical coordinate in $\partial\omega_+^\epsilon$. Directly below, we let X_- be the point in $\partial\omega_- \cap \{x_d = 0\}$ with the same horizontal coordinate as X_+^ϵ. Without loss of generality, we set X_- to be the origin of $\mathbb{R}^d$.

We choose a smooth function $b_0^\epsilon \in C^\infty(\Gamma^\epsilon)$ such that $b_0^\epsilon = -1$ in a small neighborhood of X_+^ϵ on $\partial\omega_+^\epsilon$, $b_0^\epsilon = 0$ on $\partial\omega_-$, $b_0^\epsilon = 0$ on $\partial\omega^\epsilon \cap \Gamma^\epsilon$, $\int_{\Gamma^\epsilon} b_0^\epsilon \, dS = 0$, and satisfying the estimate

$$\|b_0^\epsilon\|_{2.5,\Gamma^\epsilon} \leqslant m_0 < \infty\,, \tag{35}$$

where m_0 does not depend on ϵ.

We define the initial velocity field u_0^ϵ at $t = 0$ as the solution to the following Stokes problem:

$$-\Delta u_0^\epsilon + \nabla r_0^\epsilon = 0 \quad \text{in } \Omega^\epsilon\,, \tag{36a}$$

$$\operatorname{div} u_0^\epsilon = 0 \quad \text{in } \Omega^\epsilon\,, \tag{36b}$$

$$[\operatorname{Def} u_0^\epsilon \cdot N^\epsilon] \cdot \tau_\alpha^\epsilon = 0 \quad \text{on } \Gamma^\epsilon\,, \tag{36c}$$

$$u_0^\epsilon \cdot N^\epsilon = b_0^\epsilon \quad \text{on } \Gamma^\epsilon\,, \tag{36d}$$

with N^ϵ denoting the outward unit normal to Γ^ϵ and τ_α^ϵ, $\alpha = 1, 2$ denoting an orthonormal basis of the tangent space to Γ^ϵ (if the dimension $d = 2$, then there is only one tangent vector). Using the regularity theory of this elliptic system (see, for example, [31] or [3] and references therein), together with the proof of Lemma 2, for a constant independent of $\epsilon > 0$,

$$\|u_0^\epsilon\|_{3,\Omega^\epsilon} \leqslant C\|b_0^\epsilon\|_{2.5,\Gamma^\epsilon} \leqslant C\, m_0\,. \tag{37}$$

The boundary condition (36c) ensures that u_0^ϵ satisfies (24).

6.2. The initial pressure function p_0^ϵ

The initial pressure function p_0^ϵ at $t = 0$ then satisfies

$$-\Delta p_0^\epsilon = (u_0^\epsilon)^i,_j\, (u_0^\epsilon)^j,_i \qquad \text{in } \Omega^\epsilon\,, \tag{38a}$$

$$p_0^\epsilon = N_0^\epsilon \cdot \left[\nu \operatorname{Def} u_0^\epsilon \cdot N_0^\epsilon\right] \text{ on } \Gamma^\epsilon\,, \tag{38b}$$

so that using the same proof as that of Lemma 2, we have the following ϵ-independent elliptic estimate:

$$\|p_0^\epsilon\|_{2,\Omega^\epsilon} \leqslant C\left[\|u_0^\epsilon\|^2_{3,\Omega^\epsilon} + \|u_0^\epsilon\|_{3,\Omega^\epsilon}\right], \tag{39}$$

where $C > 0$ does not depend on $\epsilon > 0$ small enough. Using (37) in (39) shows that

$$\|p_0^\epsilon\|_{2,\Omega^\epsilon} \leqslant C\left[Cm_0 + Cm_0^2\right] = \mathcal{P}(m_0)\,, \tag{40}$$

where we use $\mathcal{P}$ to denote a generic polynomial function that depends only on Ω (since the elliptic constant C depends on Ω).

7. A priori estimates

Let Ω^ϵ denote the dinosaur domain shown in Fig. 2, and let θ_l denote the system of local charts for Ω^ϵ as defined in (3). By denoting $\eta_l = \eta \circ \theta_l$ we see that

$$\eta_l(t) : B^+ \to \Omega(t) \ \text{ for } \ l = 1, ..., K\,.$$

We set $v_l = u \circ \eta_l$, $q_l = p \circ \eta_l$ and $A_l = [D\eta_l]^{-1}$, $J_l = C_l$ (where $C_l > 0$ is a constant), and $a_l = J_l A_l$. The unit normal n_l is defined as $\mathfrak{g}^{-\frac{1}{2}} \frac{\partial \eta_l}{\partial x_1} \times \frac{\partial \eta_l}{\partial x_2}$ if $d = 3$ and by $\mathfrak{g}^{-\frac{1}{2}} \frac{\partial \eta_l}{\partial x_1}^{\perp}$ if $d = 2$.

It follows that for $l = 1, \ldots, K$,

$$\eta_l(t) = \theta_l + \int_0^t v_l \qquad \text{in } B^+ \times [0, T], \tag{41a}$$

$$\partial_t v_l + A_l^T \nabla q_l = \Delta_{\eta_l} v_l \qquad \text{in } B^+ \times (0, T], \tag{41b}$$

$$\operatorname{div}_{\eta_l} v_l = 0 \qquad \text{in } B^+ \times [0, T], \tag{41c}$$

$$\nu \operatorname{Def}_{\eta_l} v_l \cdot n_l - q_l\, n_l = 0 \qquad \text{on } B^0 \times [0, T], \tag{41d}$$

$$(\eta_l, v_l) = (\theta_l, u_0 \circ \theta_l) \qquad \text{in } B^+ \times \{t = 0\}, \tag{41e}$$

where we have set $\nu = 1$.

Definition 5 *(Higher-order energy function).* For each $t \in [0, T]$, we define the higher-order energy function

$$E^\epsilon(t) = 1 + \|\eta(\cdot, t)\|^2_{3,\Omega^\epsilon} + \|v(\cdot, t)\|^2_{2,\Omega^\epsilon} + \int_0^t \|v(\cdot, s)\|^2_{3,\Omega^\epsilon} ds + \int_0^t \|q(\cdot, s)\|^2_{2,\Omega^\epsilon} ds$$
$$+ \|v_t(\cdot, t)\|^2_{0,\Omega^\epsilon} + \int_0^t \|v_t(\cdot, s)\|^2_{1,\Omega^\epsilon} ds$$

We then set $M_0 = \mathcal{P}(E^\epsilon(0))$ where $\mathcal{P}$ denotes a generic polynomial whose coefficients depend only on Ω. The constant M_0 is then equal to $\mathcal{P}(m_0)$, a polynomial function of the constant m_0 introduced in (37).

Remark 1. Given that $u_0 \in H^2(\Omega^\epsilon)$ satisfies the compatibility conditions:

$$\operatorname{div} u_0^\epsilon = 0 \quad \text{in } \Omega^\epsilon, \tag{42a}$$

$$[\operatorname{Def} u_0^\epsilon \cdot N^\epsilon] \cdot \tau_\alpha^\epsilon = 0 \quad \text{on } \Gamma^\epsilon, \tag{42b}$$

it follows from the energy estimates (that we next obtain) together with classical existence theorems for the free-boundary Navier–Stokes problem, that (1) admits a unique solution for some time $T^\epsilon > 0$, which has the regularity:

$$v \in L^\infty(0, T^\epsilon; H^2(\Omega^\epsilon)) \cap L^2(0, T^\epsilon; H^3(\Omega^\epsilon)),$$
$$v_t \in L^\infty(0, T^\epsilon; L^2(\Omega^\epsilon)) \cap L^2(0, T^\epsilon; H^1(\Omega^\epsilon)),$$
$$q \in L^2(0, T^\epsilon; H^2(\Omega^\epsilon)).$$

Our energy function E^ϵ contains all of these terms, and additionally, the term 1 to ensure that E^ϵ is smaller than its square; the term $\|\eta(\cdot, t)\|_{3,\Omega^\epsilon}$ is well-defined whenever $v \in L^2(0, T^\epsilon; H^3(\Omega^\epsilon))$.

So long as the solution has this regularity and the moving free surface does not self-intersect, the Eulerian formulation (1) and the Lagrangian formulation (written in each chart) (41) are equivalent, and we will work with the latter one.

We will first prove that the solution is defined over a time interval which is independent of $\epsilon > 0$.

Theorem 6. *Assuming that $\Gamma(t)$ does not self-intersect, independent of $\epsilon > 0$, there exists a time $T > 0$ and a constant $C > 0$ such that the solution*

$$v \in C([0, T], H^2(\Omega^\epsilon)) \cap L^2(0, T; H^3(\Omega^\epsilon)), \quad q \in L^2(0, T; H^2(\Omega^\epsilon))$$

to (23) satisfies the a priori estimate:

$$\max_{t \in [0,T]} E^\epsilon(t) \leqslant C\, M_0. \tag{43}$$

Proof. The proof will proceed in five steps.

Step 1. Estimates for $\nabla\eta$ and A. Using (41a), we see that

$$\|\nabla\eta(\cdot,t)-\mathrm{Id}\,\|_{2,\Omega^\epsilon} \leqslant \left\| \int_0^t \nabla v(\cdot,s)ds \right\|_{2,\Omega^\epsilon} \leqslant \sqrt{t}\sup_{s\in[0,t]}\sqrt{E^\epsilon(t)}\,. \tag{44}$$

Thanks to Lemma 3, there exists a constant $C>0$, independent of ϵ, such that

$$\|\nabla\eta(\cdot,t)-\mathrm{Id}\,\|_{L^\infty(\Omega^\epsilon)} \leqslant C\sqrt{t}\sup_{s\in[0,t]}\sqrt{E^\epsilon(t)}\,. \tag{45}$$

Since $\det\nabla\eta=1$, the matrix A is simply the cofactor matrix of $\nabla\eta$:

$$A=\begin{bmatrix} -\eta,_2^{\perp} \\ \eta,_1^{\perp}\end{bmatrix} \text{ for } d=2, \text{ and } A=\begin{bmatrix} \eta,_\mathbf{2}\times\eta,_\mathbf{3} \\ \eta,_\mathbf{3}\times\eta,_\mathbf{1} \\ \eta,_\mathbf{1}\times\eta,_\mathbf{2}\end{bmatrix} \text{ for } d=3\,, \tag{46}$$

where each row is a vector, and for a 2-vector $x=(x_1,x_2)$, $x^\perp=(-x_2,x_1)$.

We make the following basic assumption, that we shall verify below in Step 5: for a constant $0<\vartheta\ll 1$, we suppose that $t\in[0,T]$ and that T is chosen sufficiently small so that

$$\sup_{t\in[0,T]}\|\nabla\eta(\cdot,t)-\mathrm{Id}\,\|_{L^\infty(\Omega^\epsilon)}\leqslant\vartheta^{10}\,. \tag{47}$$

It follows from (46), that since $\|A(\cdot,t)-\mathrm{Id}\,\|_{L^\infty(\Omega^\epsilon)}\leqslant\int_0^t\|A_t(\cdot,s)\|_{L^\infty(\Omega^\epsilon)}ds$,

$$\sup_{t\in[0,T]}\|A(\cdot,t)-\mathrm{Id}\,\|_{L^\infty(\Omega^\epsilon)}+\|AA^T(\cdot,t)-\mathrm{Id}\,\|_{L^\infty(\Omega^\epsilon)}\leqslant\vartheta\,. \tag{48}$$

Step 2. Boundary regularity. We begin by considering a single boundary chart $\theta_l : B^+\to\Omega(t)$. Let ζ_l denote the smooth cut-off function defined in Section 3.2.4. Using equation (41b), we compute the following $L^2(B^+)$ inner-product:

$$\Big(\zeta_l\bar\partial^2[\partial_t v_l-\Delta_\eta v+A_l^T\,\nabla q_l]\,,\ \zeta_l\bar\partial^2 v_l\Big)_{L^2(B^+)}=0\,. \tag{49}$$

To simplify the notation, we fix $l\in\{1,\dots,K\}$ and drop the subscript. The chart θ_l was defined so that $\det\nabla\theta_l=C_l$ for a constant $C_l>0$. Then (49) can be written as be written as

$$\int_{B^+}\zeta^2\bar\partial^2 v_t^i\,\bar\partial^2 v^i\,dx-\int_{B^+}\zeta^2\bar\partial^2[A_s^kA_s^j v^i,_j],_k\,\bar\partial^2 v^i\,dx+\int_{B_+}\zeta^2\bar\partial^2[A_i^k q],_k\,\bar\partial^2 v^i\,dx=0\,. \tag{50}$$

Integration-by-parts with respect to x_k shows that

$$0=\frac{1}{2}\frac{d}{dt}\|\zeta\bar\partial^2 v(t)\|^2_{0,B^+}+\int_{B^+}\bar\partial^2[A_s^kA_s^j v^i,_j]\,\bar\partial^2[\zeta^2 v^i],_k\,dx+\int_{B^+}\bar\partial^2[A_i^k q]\,\bar\partial^2[\zeta^2 v^i],_k\,dx \tag{51}$$

where we have used the boundary condition (41d) to show that the boundary integral vanishes. Using δ^{jk} to denote the Kronecker delta function, we write (51) as

$$\frac{1}{2}\frac{d}{dt}\|\zeta\bar\partial^2 v(\cdot,t)\|^2_{0,B^+}+\|\zeta\bar\partial^2\nabla v(t)\|^2_{0,B^+}=-\int_{B^+}\bar\partial^2[A_i^k q]\,\bar\partial^2[\zeta^2 v^i],_k\,dx$$
$$-\int_{B^+}\bar\partial^2[(A_s^kA_s^j-\delta^{kj})v^i,_j]\,\bar\partial^2[\zeta^2 v^i],_k\,dx-\int_{B^+}\Big[\bar\partial^2 v^i,_k\,(\bar\partial^2\zeta^2 v^i+2\bar\partial\zeta^2\bar\partial v^i),_k+\xi,_k\,\bar\partial^2 v^i\Big]dx\,. \tag{52}$$

We integrate (52) over the time interval $[0,T]$:

$$\frac{1}{2}\|\zeta\bar\partial^2 v(\cdot,t)\|^2_{0,B^+}+\int_0^T\|\zeta\bar\partial^2 v(t)\|^2_{1,B^+}\leqslant M_0+\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3 \tag{53}$$

where

$$\mathcal{I}_1=\int_0^T\int_{B^+}\left|\bar\partial^2[A^k_i q]\,\bar\partial^2[\zeta^2 v^i],_k\right|dxdt\,,$$

$$\mathcal{I}_2=\int_0^T\int_{B^+}\left|\bar\partial^2[(A^k_s A^j_s-\delta^{kj})v^i,_j]\,\bar\partial^2[\zeta^2 v^i],_k\right|dxdt\,,$$

$$\mathcal{I}_3=\int_0^T\int_{B^+}\left|\bar\partial^2 v^i,_k\,[\bar\partial^2\zeta^2 v^i+2\bar\partial\zeta^2\bar\partial v^i],_k+\xi,_k\,\bar\partial^2 v^i\right|dxdt\,.$$

Using the Sobolev embedding theorem and Lemma 3, we estimate $\mathcal{I}_1$

$$\mathcal{I}_1\leqslant\underbrace{\int_0^T\int_{B^+}|\bar\partial^2 q|\,|A^k_i\bar\partial^2 v^i,_k|\,dxdt}_{\mathcal{I}_1^a}+\underbrace{\int_0^T\|q\|_{2,\epsilon}\|A\|_{2,\Omega^\epsilon}\|v\|_{2,\Omega^\epsilon}dt}_{\mathcal{I}_1^b}+\underbrace{\int_0^T\|q\|_{1.5,\epsilon}\|A\|_{2,\Omega^\epsilon}\|v\|_{3,\Omega^\epsilon}dt}_{\mathcal{I}_1^c}\,.$$

To estimate the integral $\mathcal{I}_1^a$, we use (41c) to write

$$v^i,_{k\alpha\beta}\,A^k_i=-A^k_i,_{\alpha\beta}\,v^i,_k-A^k_i,_\beta\,v^i,_{k\alpha}-A^k_i,_\alpha\,v^i,_{k\beta}\,,$$

so that the term with three derivatives on v is converted to a term with three derivatives on η plus lower-order terms. It follows that for $\delta>0$, and a constant C_δ (which blows-up as $\delta\to 0$),

$$\mathcal{I}_1^a\leqslant\delta\int_0^T\|q\|^2_{2,\Omega^\epsilon}dt+C_\delta T\,P(\sup_{t\in[0,T]}E^\epsilon(t))\,.$$

The integral $\mathcal{I}_1^b$ is estimated in the same way. For the integral $\mathcal{I}_1^c$ we use linear interpolation to estimate the norm $\int_0^T\|q\|_{1.5,\epsilon}$:

$$\mathcal{I}_1^c\leqslant\delta\int_0^T\|v\|^2_{3,\Omega^\epsilon}dt+\delta\int_0^T\|q\|^2_{2,\Omega^\epsilon}dt+C_\delta T\,P(\sup_{t\in[0,T]}E^\epsilon(t))\,.$$

It follows that

$$\mathcal{I}_1\leqslant M_0+C_\delta T\,P(\sup_{t\in[0,T]}E^\epsilon(t))+\delta\sup_{t\in[0,T]}E^\epsilon(t)\,. \tag{54}$$

Next, for the integral $\mathcal{I}_2$,

$$\mathcal{I}_2\leqslant\underbrace{\int_0^T\int_{B^+}\left|(A^k_sA^j_s-\delta^{kj})\bar\partial^2 v^i,_j\;\bar\partial^2[\zeta^2v^i],_k\right|dxdt}_{\mathcal{I}_2^a}+\underbrace{2\int_0^T\int_{B^+}\left|\bar\partial(A^k_sA^j_s-\delta^{kj})\bar\partial v^i,_j\;\bar\partial^2[\zeta^2v^i],_k\right|dxdt}_{\mathcal{I}_2^b}$$

$$+\underbrace{\int_0^T\int_{B^+}\left|\bar\partial^2(A^k_sA^j_s-\delta^{kj})v^i,_j\;\bar\partial^2[\zeta^2v^i],_k\right|dxdt}_{\mathcal{I}_2^c}\,.$$

Using (48) and choosing $\vartheta < \delta$,

$$\mathcal{I}_2^a \leqslant C_\delta T P(\sup_{t\in[0,T]} E^\epsilon(t)) + \delta \sup_{t\in[0,T]} E^\epsilon(t)\,.$$

In the same way as above, we again use Lemma 3, together with linear interpolation for term $\mathcal{I}_2^b$, to see that

$$\mathcal{I}_2 \leqslant M_0 + C_\delta T P(\sup_{t\in[0,T]} E^\epsilon(t)) + \delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{55}$$

The integral $\mathcal{I}_3$ is straightforward and also satisfies

$$\mathcal{I}_3 \leqslant M_0 + C_\delta T P(\sup_{t\in[0,T]} E^\epsilon(t)) + C\delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{56}$$

Summing over all of the boundary charts $l = 1, \ldots, K$ in (53), the inequalities (54)–(56) together with the trace theorem, Lemma 5, show that

$$\int_0^T \|v(\cdot,t)\|^2_{2.5,\Gamma^\epsilon} \leqslant M_0 + C_\delta T P(\sup_{t\in[0,T]} E^\epsilon(t)) + \delta \sup_{t\in[0,T]} E^\epsilon(t) \tag{57}$$

Step 3. Estimates for the time-differentiated problem. We consider the time-differentiated version of (23) which we write as the following system:

$$\eta_t = v \quad \text{in } \Omega^\epsilon \times [0,T]\,, \tag{58a}$$
$$v_{tt} - \Delta_\eta v_t + A^T \nabla q_t = -A_t^T \nabla q + [\partial_t(A_s^j A_s^k) v,_k],_j \quad \text{in } \Omega^\epsilon \times (0,T]\,, \tag{58b}$$
$$\operatorname{div}_\eta v_t = -v^i,_j \partial_t A_i^j \quad \text{in } \Omega^\epsilon \times [0,T]\,, \tag{58c}$$
$$\partial_t \left[\operatorname{Def}_\eta v \cdot n - qn\right] = 0 \quad \text{on } \Gamma^\epsilon \times [0,T]\,, \tag{58d}$$
$$(\eta, v, v_t) = (e, u_0^\epsilon, u_1^\epsilon) \quad \text{in } \Omega^\epsilon \times \{t=0\}\,, \tag{58e}$$

where $u_1^\epsilon = \Delta u_0^\epsilon - \nabla p_0^\epsilon$, with u_0^ϵ defined in (36) and p_0^ϵ defined in (38); therefore, independently of $\epsilon > 0$,

$$\|u_1^\epsilon\|_{0,\Omega^\epsilon} \leqslant \mathcal{P}(m_0)\,. \tag{59}$$

We define the space of div_η-free vectors fields on Ω^ϵ as

$$\mathcal{V}(t) = \{\phi \in H^1(\Omega^\epsilon; \mathbb{R}^d) \,:\, \operatorname{div}_{\eta(\cdot,t)} \phi = 0\}\,.$$

Taking the $L^2(\Omega^\epsilon)$ inner-product of equation (58b) with a test function $\phi \in \mathcal{V}(t)$, we have that

$$\int_{\Omega^\epsilon} v_{tt} \cdot \phi dx + \int_{\Omega^\epsilon} \partial_t [A_s^k A_s^j v^i,_j] \phi^i,_k \, dx = \int_\Omega q\, \partial_t A_i^k \phi^i,_k \, dx \quad \forall \phi \in \mathcal{V}(t)\,. \tag{60}$$

Next, we define a vector field w satisfying

$$\operatorname{div}_\eta w = -v^i,_j \partial_t A_i^j \quad \text{in } \Omega^\epsilon\,, \tag{61a}$$
$$w = \phi(t) n \quad \text{on } \Gamma^\epsilon\,, \tag{61b}$$

where $\phi(t) = -\int_{\Omega^\epsilon} v^i,_j \partial_t A_i^j dx / |\Gamma^\epsilon|$. A solution w can be found by solving a Stokes-type problem, and according to the proof of Lemma 3.2 in [11], for integers $k \geqslant 1$,

$$\|w(\cdot,t)\|_{k,\Omega^\epsilon} \leqslant C\left(\|v^i,_j(\cdot,t)\, \partial_t A_i^j(\cdot,t)\|_{k-1,\Omega^\epsilon} + \|\phi(t) n\|_{k-1/2,\Gamma^\epsilon}\right)\,, \tag{62}$$

where the constant C is independent of ϵ by Lemma 2. From (46), we see that $\partial_t A$ scales like ∇v in 2-D and like $\nabla v\, \nabla\eta$ in 3-D. Thus, the estimate (62) shows that

$$\sup_{t\in[0,T]} \|w(\cdot,t)\|^2_{0,\Omega^\epsilon} + \int_0^T \|w(\cdot,t)\|^2_{1,\Omega^\epsilon} \leqslant M_0 + T\,P(\sup_{t\in[0,T]} E^\epsilon(t))\,. \tag{63}$$

Similarly,

$$\operatorname{div}_\eta w_t = -\Big(w^i,_j\,\partial_t A^j_i + \partial_t(v^i,_j\,\partial_t A^j_i)\Big) \quad \text{in } \Omega^\epsilon\,, \tag{64a}$$

$$w_t = (\phi\, n)_t \qquad \text{on } \Gamma^\epsilon \tag{64b}$$

and

$$\|w_t\|_{1,\Omega^\epsilon} \leqslant C\left(\|w^i,_j\,\partial_t A^j_i + \partial_t(v^i,_j\,\partial_t A^j_i)\|_{0,\Omega^\epsilon} + \|(\phi_t n)_t\|_{1/2,\Gamma^\epsilon}\right),$$

so that

$$\int_0^T \|w_t\|^2_{1,\Omega^\epsilon} \leqslant P(\sup_{t\in[0,T]} E^\epsilon(t))\,. \tag{65}$$

Now, because of (61a), $v_t - w \in \mathcal{V}(t)$, and we are allowed to set $\phi = v_t - w$ in (60). We find that

$$\frac{1}{2}\frac{d}{dt}\|v_t(\cdot,t)\|^2_{0,\Omega^\epsilon} + \int_{\Omega^\epsilon} \partial_t[A^k_s A^j_s v^i,_j]\, v^i_t,_k\, dx = \int_{\Omega^\epsilon} v_{tt}\cdot w dx + \int_{\Omega^\epsilon} \partial_t(A^k_s A^j_s v^i,_j)\, w^i,_k\, dx$$
$$+ \int_\Omega q\,\partial_t A^k_i\left[v^i_t,_k + w^i,_k\right] dx$$

and hence for $t\in(0,T)$,

$$\frac{1}{2}\|v_t(\cdot,t)\|^2_{0,\Omega^\epsilon} + \int_0^t \|\nabla v_t\|^2_{0,\Omega^\epsilon} ds = \frac{1}{2}\|u_1\|^2_{0,\Omega^\epsilon} \overbrace{-\int_0^t\int_{\Omega^\epsilon} [A^k_s A^j_s - \delta^{kj}] v^i_t,_j\; v^i_t,_k\, dxds}^{\mathcal{J}_1}$$
$$\underbrace{-\int_0^t\int_{\Omega^\epsilon} \partial_t[A^k_s A^j_s] v^i,_j\; v^i_t,_k\, dxds}_{\mathcal{J}_2} + \underbrace{\int_0^t\int_{\Omega^\epsilon} v_{tt}\cdot w dxds}_{\mathcal{J}_3} + \underbrace{\int_0^t\int_{\Omega^\epsilon} \partial_t[A^k_s A^j_s v^i,_j]\, w^i,_k\, dxds}_{\mathcal{J}_4}$$
$$+ \underbrace{\int_0^t\int_\Omega q\,\partial_t A^k_i\left[v^i_t,_k + w^i,_k\right] dxds}_{\mathcal{J}_5}\,.$$

For $\delta > 0$ and using (48) with $\vartheta < \delta$, it follows from an L^∞-L^2-L^2 Hölder's inequality that

$$|\mathcal{J}_1| \leqslant \delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{66}$$

We next estimate $\mathcal{J}_2$. According to (46) the components of A are either linear ($d=2$) or quadratic ($d=3$) with respect to the components of $\nabla\eta$; hence, $\partial_t A$ behaves like ∇v for $d=2$ and like $\nabla\eta\,\nabla v$ for $d=3$. We consider the more difficult case that $d=3$ in which case $\partial_t(AA^T)$ behaves like $\nabla\eta\,\nabla\eta\,\nabla\eta\,\nabla v$, and

$$\int_{\Omega^\epsilon} |\nabla\eta|^3|\nabla v|^2\nabla v_t\, dxds \leqslant \|\nabla\eta\|^3_{L^\infty(\Omega^\epsilon)}\|\nabla v\|^2_{L^4(\Omega^\epsilon)}\|\nabla v_t\|_{L^2(\Omega^\epsilon)}$$
$$\leqslant \|\eta\|^3_{H^3(\Omega^\epsilon)}\|v\|^2_{H^2(\Omega^\epsilon)}\|v_t\|_{H^1(\Omega^\epsilon)}$$
$$\leqslant C_\delta\|\eta\|^6_{H^3(\Omega^\epsilon)}\|v\|^4_{H^2(\Omega^\epsilon)}| + \delta\|v_t\|^2_{H^1(\Omega^\epsilon)}\,,$$

where we have used Hölder's inequality for the first inequality, the Sobolev embedding theorem for the second inequality, and the Cauchy–Young inequality with $\delta > 0$ for the third inequality; the constant C_δ scales like $1/\delta$. It follows that

$$|\mathcal{J}_2| \leqslant M_0 + T\mathcal{P}(\sup_{t\in[0,T]} E^\epsilon(t)) + \delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{67}$$

To estimate $\mathcal{J}_3$, we integrate-by-parts in time:

$$\begin{aligned}
|\mathcal{J}_3| &\leqslant \int_0^t \int_{\Omega^\epsilon} |v_t \cdot w_t| dx ds + \left| \int_{\Omega^\epsilon} v_t \cdot w dx \Big|_0^t \right| \\
&\leqslant M_0 + \int_0^t \int_{\Omega^\epsilon} |v_t \cdot w_t| dx ds + \int_{\Omega^\epsilon} |v_t(\cdot,t) w(\cdot,0)| dx + \int_{\Omega^\epsilon} \left| v_t(\cdot,t) \int_0^t w_t(\cdot,s) ds \right| dx \\
&\leqslant M_0 + T^{\frac12}\mathcal{P}(\sup_{t\in[0,T]} E^\epsilon(t)) + \delta \|v_t(\cdot,t)\|^2_{0,\Omega^\epsilon} + C_\delta \left\| \int_0^t w_t(\cdot,s) ds \right\|^2_{0,\Omega^\epsilon}\,,
\end{aligned}$$

the last inequality following the estimates (62) and (65) and the Cauchy–Young inequality. Since

$$\left\| \int_0^t w_t(\cdot,s) ds \right\|^2_{0,\Omega^\epsilon} = \int_{\Omega^\epsilon} \left(\int_0^t w_t(x,s) ds \right)^2 dx \leqslant t \int_0^t \int_{\Omega^\epsilon} |w_t(x,s)|^2 dx dt\,,$$

and

$$\begin{aligned}
\int_0^t \int_{\Omega^\epsilon} |v_t \cdot w_t| dx ds &\leqslant \left(\int_0^t \|v_t(\cdot,s)\|^2_{0,\Omega^\epsilon} ds \right)^{\frac12} \left(\int_0^t \|w_t(\cdot,s)\|^2_{0,\Omega^\epsilon} ds \right)^{\frac12} \\
&\leqslant t^{\frac12} \sup_{s\in[0,t]} \|v_t(\cdot,s)\|_{0,\Omega^\epsilon} P(\sup_{s\in[0,t]} E^\epsilon(s)) \leqslant t^{\frac12} P(\sup_{s\in[0,t]} E^\epsilon(s))\,,
\end{aligned}$$

we see that

$$|\mathcal{J}_3| \leqslant M_0 + T^{\frac12}\mathcal{P}(\sup_{t\in[0,T]} E^\epsilon(t)) + \delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{68}$$

The integrals $\mathcal{J}_4$ and $\mathcal{J}_5$ (using (63) and (65)) are estimated in the same way as $\mathcal{J}_2$ so that

$$|\mathcal{J}_4| + |\mathcal{J}_5| \leqslant M_0 + T^{\frac12}\mathcal{P}(\sup_{t\in[0,T]} E^\epsilon(t)) + \delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{69}$$

Combining the estimates (66)–(69), we find that

$$\sup_{t\in[0,T]} \|v_t(\cdot,t)\|^2_{0,\Omega^\epsilon} + \int_0^T \|v_t\|^2_{1,\Omega^\epsilon} dt \leqslant M_0 + T^{\frac12}\mathcal{P}(\sup_{t\in[0,T]} E^\epsilon(t)) + C\delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{70}$$

Step 4. Regularity for the velocity and pressure. Next, we write equation (23b) as

$$-\Delta v + \nabla q = \operatorname{div}[(AA^T - \mathrm{Id})\nabla v] - (A^T - \mathrm{Id})\nabla q - v_t \quad \text{in } \Omega^\epsilon \times (0,T]\,, \tag{71a}$$

$$\operatorname{div} v = -(A^j_i - \delta^j_i) v^i,_j \quad \text{in } \Omega^\epsilon \times [0,T]\,, \tag{71b}$$

$$v \in L^2(0,T; H^{2.5}(\Gamma^\epsilon)) \tag{71c}$$

The two inequalities (57) and (70), together with the Stokes regularity given in Lemma 2, show that $v \in L^\infty([0,T]; H^2(\Omega^\epsilon)) \cap L^2(0,T; H^3(\Omega^\epsilon))$ and satisfies

$$\sup_{t\in[0,T]} \|v(\cdot,t)\|^2_{2,\Omega^\epsilon} + \int_0^T \|v\|^2_{3,\Omega^\epsilon} dt + \int_0^T \|q\|^2_{2,\Omega^\epsilon} dt \leqslant M_0 + T^{\frac{1}{2}}\mathcal{P}(\sup_{t\in[0,T]} E^\epsilon(t)) + C\delta \sup_{t\in[0,T]} E^\epsilon(t)\,. \tag{72}$$

By choosing $\delta > 0$ sufficiently small, we obtain that

$$\sup_{t\in[0,T]} E^\epsilon(t) \leqslant M_0 + T^{\frac{1}{2}}\mathcal{P}(\sup_{t\in[0,T]} E^\epsilon(t))\,, \tag{73}$$

for a constant M_0 and a polynomial function $\mathcal{P}$ which are both independent of ϵ.

From the estimate (72), $v \in L^2(0,T;H^3(\Omega^\epsilon))$, and the estimate (70), $v_t \in L^2(0,T;H^1(\Omega^\epsilon))$. Using the partition of unity functions ζ_l defined in Step 2 above, we then see that for each chart $\zeta_l v \in L^2(0,T;H^3(\mathcal{B}_l))$ where $\mathcal{B}_l = B^+$ for $l = 1,...,K$, and $\mathcal{B}_l = B$ for $l = K+1,...,L$. Similarly, $\zeta_l v_t \in L^2(0,T;H^1(\mathcal{B}_l))$. It is then standard that $\zeta_l v \in C^0([0,T];H^2(\mathcal{B}_l))$, and hence by summing over $l = 1,...,L$, $v \in C^0([0,T];H^2(\Omega^\epsilon))$.

Since the pressure satisfies the elliptic system:

$$-\Delta_\eta q = v^i,_r A^r_j v^j,_s A^s_i \quad \text{in } \Omega^\epsilon \times (0,T]\,,$$
$$q = n\cdot\left[\mathrm{Def}_\eta v\cdot n\right] \quad \text{on } \Gamma^\epsilon \times [0,T]\,,$$

we then infer that $q \in C^0([0,T];H^1(\Omega^\epsilon))$. Then, using the momentum equation (71a), it follows that $v_t \in C^0([0,T];L^2(\Omega^\epsilon))$.

This then shows that $E^\epsilon(t)$ is a continuous function of time. Following Section 9 in [14], from (73), we now may choose $T > 0$ sufficiently small and independent of ϵ, such that

$$\sup_{t\in[0,T]} E^\epsilon(t) \leqslant 2M_0\,. \tag{74}$$

Step 5. Verifying the basic assumption (47). Having established (74) on $[0,T]$ with T independent of ϵ, for any $\varepsilon > 0$, we may now use the formula (44) to choose T even smaller if necessary to ensure that (47) holds. This concludes the proof. □

We now establish a more quantitative estimate in order to assess the continuity of $\bar\partial^2 v(t,\cdot)$ in $L^2(\Omega^\epsilon)$.

Proposition 7. *For all $t \in [0,T]$,*

$$\max_{s\in[0,t]} \|\bar\partial^2(v^\epsilon(\cdot,s) - u_0^\epsilon)\|^2_{0,\Omega^\epsilon} + \int_0^t \|\bar\partial^2(v^\epsilon(\cdot,s) - u_0^\epsilon)\|^2_{1,\Omega^\epsilon} ds \lesssim t^{1/2}\mathcal{P}(M_0)\,. \tag{75}$$

Proof. We write $v(t) = v(\cdot,t)$ and again set viscosity $\nu = 1$. The difference $v(t) - u_0^\epsilon$ satisfies the equation

$$(v-u_0^\epsilon)_t - \Delta_\eta(v-u_0^\epsilon) + A^T\nabla q = \Delta_\eta u_0^\epsilon\,.$$

Following Step 2 in the proof of Theorem 6, and once again localize to a boundary chart θ_l, $l = 1,...,K$, with $\det\nabla\theta_l = C_l$ and with cut-off functions ζ_l, we obtain that

$$\begin{aligned} 0 = \frac{1}{2}\frac{d}{dt}\|\zeta\bar\partial^2[v(t) - u_0^\epsilon]\|^2_{0,B^+} &+ \int_{B^+} \bar\partial^2[A^k_s A^j_s (v-u_0^\epsilon),_j]\cdot\bar\partial^2[\zeta^2(v-u_0^\epsilon)],_k dx \\ &+ \int_{B^+} \bar\partial^2[A^k_i q]\,\bar\partial^2[\zeta^2 v^i],_k dx + \int_{B^+} \bar\partial^2[A^k_s A^j_s u^\epsilon_0,_j]\cdot\bar\partial^2[\zeta^2(v-u_0^\epsilon)],_k dx\,, \end{aligned} \tag{76}$$

where we have dropped the explicit chart dependence on l and where again, the boundary integral terms have vanished due to (23d). We integrate (76) over the time interval $[0,T]$:

$$\|\zeta\bar\partial^2[v(t) - u_0^\epsilon]\|^2_{0,B^+} + \int_0^T \|\zeta\bar\partial^2[v(t) - u_0^\epsilon]\|^2_{1,B^+} \leqslant |\mathcal{K}_1| + |\mathcal{K}_2| + |\mathcal{K}_3| + |\mathcal{K}_4|\,,$$

where we are writing u_0^ϵ for $u_0^\epsilon \circ \theta_l$, and where

$$\mathcal{K}_1 = \int_0^T \int_{B^+} \bar\partial^2[A_i^k q]\, \bar\partial^2[\zeta^2 (v - u_0^\epsilon)^i],_k \, dx dt \,,$$

$$\mathcal{K}_2 = \int_0^T \int_{B^+} \bar\partial^2[(A_s^k A_s^j - \delta^{kj})(v - u_0^\epsilon),_j] \cdot \bar\partial^2[\zeta^2 (v - u_0^\epsilon)],_k \, dx dt \,,$$

$$\mathcal{K}_3 = \int_0^T \int_{B^+} \bar\partial^2 (v - u_0^\epsilon)^i,_k \, [\, [\bar\partial^2 \zeta^2 (v - u_0^\epsilon)^i + 2\bar\partial\zeta^2 \bar\partial (v - u_0^\epsilon)^i],_k + \zeta^2,_k \bar\partial^2 v^i] dx dt \,,$$

$$\mathcal{K}_4 = \int_0^T \int_{B^+} \bar\partial^2[(A_s^k A_s^j u_0^\epsilon,_j] \cdot \bar\partial^2[\zeta^2 (v - u_0^\epsilon)],_k \, dx dt \,.$$

We write

$$\mathcal{K}_1 \leqslant \underbrace{\int_0^T \int_{B^+} \bar\partial^2[A_i^k q]\, \bar\partial^2[\zeta^2 v^i],_k \, dx dt}_{\mathcal{K}_1^a} + \underbrace{\int_0^T \int_{B^+} \left| \bar\partial^2[A_i^k q]\, \bar\partial^2[\zeta^2 u_0^{\epsilon\, i}],_k \right| dx dt}_{\mathcal{K}_1^b} \,.$$

By (37) and (43), we see that

$$|\mathcal{K}_1^b| \leqslant \sqrt{T}\mathcal{P}(M_0) \,.$$

For the integral $\mathcal{K}_1^a$, we focus on the integrand that arises when $\bar\partial^2$ acts on *both* q and $v^i,_k$, for all other derivative combinations immediately give an integral bound of $\sqrt{T}\mathcal{P}(M_0)$. Using the Lagrangian divergence-free condition (23c),

$$\left| \int_0^T \int_{B^+} \zeta^2 \bar\partial^2 q \, A_i^k \bar\partial^2 v^i,_k \, dx dt \right| \leqslant \left| \int_0^T \int_{B^+} \zeta^2 \bar\partial^2 q \, \bar\partial^2 A_i^k v^i,_k \, dx dt \right| + 2 \left| \int_0^T \int_{B^+} \zeta^2 \bar\partial^2 q \, \bar\partial A_i^k \bar\partial v^i,_k \, dx dt \right| .$$

An application of the Cauchy–Young inequality together with the Sobolev embedding theorem, shows that

$$|\mathcal{K}_1^a| \leqslant \sqrt{T}\mathcal{P}(M_0) \,.$$

For the integral $\mathcal{K}_2$, we consider the case that $\bar\partial^2$ acts on $(A_s^k A_s^j - \delta^{kj})$, all other terms immediately giving the desired bound. Using (44) and (46), $\|AA^T - \mathrm{Id}\|_{L^\infty(B^+)} \leqslant \sqrt{T}\mathcal{P}(M)$, so that with (43),

$$|\mathcal{K}_2| \leqslant \sqrt{T}\mathcal{P}(M_0) \,.$$

The integral $\mathcal{K}_3$ and $\mathcal{K}_4$ are easily estimated using the Cauchy–Young inequality, the Sobolev embedding theorem, and (43). We have thus established that

$$\|\zeta \bar\partial^2 [v_l(t) - u_0^\epsilon]\|_{0,B^+}^2 + \int_0^T \|\zeta \bar\partial^2 [v_l(t) - u_0^\epsilon \circ \theta_l]\|_{1,B^+}^2 \leqslant \sqrt{T}\mathcal{P}(M_0) \,.$$

Summing over $l = 1, ..., K$ then concludes the proof. □

8. Proof of the main theorem

Using the Lagrangian divergence condition (23c), we have that $\operatorname{div} v = -(A^j_i - \delta^j_i)v^i{,}_j$, which we write as $\operatorname{div} v = -(A - \mathrm{Id}) : \nabla v$. Then, since $\operatorname{div} u_0^\epsilon = 0$, for all $t \in [0, T]$,

$$\|\bar\partial \operatorname{div}(v - u_0^\epsilon)\|^2_{0,\Omega^\epsilon} \leqslant \|\bar\partial (A - \mathrm{Id})\, \nabla v\|^2_{0,\Omega^\epsilon} + \|(A - \mathrm{Id})\, \bar\partial \nabla v\|^2_{0,\Omega^\epsilon} \leqslant \sqrt{T}\mathcal{P}(M_0)\,. \tag{77}$$

Using (77) together with (75), the normal trace theorem (see, for example, (A.6) in [16]) shows that $\bar\partial^2(v - u_0^\epsilon)\cdot N_\epsilon \in C([0,T; H^{-\frac12}(\Gamma^\epsilon))$ and

$$\|\bar\partial^2(v - u_0^\epsilon)\cdot N_\epsilon\|^2_{-1/2,\Gamma^\epsilon} \leqslant \sqrt{T}\mathcal{P}(M_0)\,,$$

so that

$$\|(v - u_0^\epsilon)\cdot N_\epsilon\|^2_{1.5,\Gamma^\epsilon} \leqslant \sqrt{T}\mathcal{P}(M_0)\,,$$

and hence by Lemma 4,

$$\max_{x\in\Gamma^\epsilon} |(v(x,t) - u_0^\epsilon)\cdot N_\epsilon| \leqslant T^{\frac14}\mathcal{P}(M_0) \ \ \forall t \in [0,T]\,. \tag{78}$$

Next, we consider the motion of the points X_+^ϵ and X_- given in Section 6.1 (see Fig. 3). Recall that the unit normal N_ϵ at both the points $X_+^\epsilon = (0,0,\epsilon)$ and $X_- = (0,0,0)$ is vertical, so by definition of u_0^ϵ, we have that

$$u_0^\epsilon(X_+^\epsilon)\cdot N_\epsilon = -1 \quad u_0^\epsilon(X_-)\cdot N_\epsilon = 0\,, \quad \text{and } |X_+^\epsilon - X_-| = \epsilon\,.$$

Using Theorem 6, we choose ϵ so small that $10\epsilon < T$, where $[0,T]$ is the time interval of existence which is independent of ϵ, and we consider the vertical displacement of the falling particle X_+^ϵ. Since $X_+^\epsilon \cdot e_d = \epsilon$, and

$$\eta(X_+^\epsilon, t)\cdot e_d = \epsilon + \int_0^t v^d(X_+^\epsilon, s)ds\,,$$

for $t = 10\epsilon$, we have from (78) that

$$\eta^d(X_+^\epsilon, 10\epsilon) < -8\epsilon\,.$$

Next, let Z denote any point on $\partial\omega_- \cap \{x_d = 0\}$. Since $u_0^\epsilon(Z)\cdot N_\epsilon = 0$ and $\eta(Z, 10\epsilon) = \int_0^{10\epsilon} v(Z,s)ds$, according to (78),

$$\eta(Z, 10\epsilon)\cdot e_d \geqslant -c\epsilon^{\frac54}\,, \quad c = 10^{\frac54}\mathcal{P}(M_0)\,.$$

We then choose $\epsilon > 0$ sufficiently small so that $c\epsilon^{\frac54} < 8\epsilon$. It follows that

$$\eta(X_+^\epsilon, 10\epsilon)\cdot e_d < \eta(Z, 10\epsilon)\cdot e_d\,. \tag{79}$$

We next consider the horizontal displacement of the particle X_+^ϵ and any particle Z on $\partial\omega_- \cap \{x_d = 0\} \times [0, 10\epsilon]$. From the estimate (74), for all time $t \in [0, 10\epsilon]$, $\|v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \mathcal{P}(M_0)$.

Therefore, for any $t \in [0, 10\epsilon]$ and for $\alpha = 1, ..., d-1$,

$$|\eta^\alpha(X_+^\epsilon, t)| \leqslant 10\epsilon\mathcal{P}(M_0) \text{ and } |\eta^\alpha(Z,t) - Z^\alpha| \leqslant 10\epsilon\mathcal{P}(M_0)\,,$$

showing that the distance between the projection of the surface $\eta(\partial\omega_- \cap \{x_d = 0\}, t)$ onto the plane $x_d = 0$ and the set $\partial\omega_- \cap \{x_d = 0\}$ is $O(\epsilon)$. Since by Definition 3, the set $\partial\omega_- \cap \{x_d = 0\}$ contains a $d-1$-dimensional ball of radius $\sqrt{\epsilon}$ centered at the origin, we see that by choosing ϵ sufficiently small the vertical line passing through $\eta(X_+^\epsilon, t)$ must intersect the surface $\eta(\partial\omega_- \cap \{x_d = 0\}, t)$ for any $t \in [0, 10\epsilon]$. Now, since at $t = 0$, X_+^ϵ is directly (vertically) above $\partial\omega_- \cap \{x_d = 0\}$, and at $t = 10\epsilon$, from (79), $\eta(X_+^\epsilon, 10\epsilon)$ is (vertically) below $\eta(\partial\omega_- \cap \{x_d = 0\}, 10\epsilon)$, then by continuity there necessarily exists a time $0 < T^* < 10\epsilon$ at which $\eta(X_+^\epsilon, T^*) = \eta(Z, T^*)$ for some $Z \in \partial\omega_- \cap \{x_d = 0\}$. This concludes the proof of the main theorem.

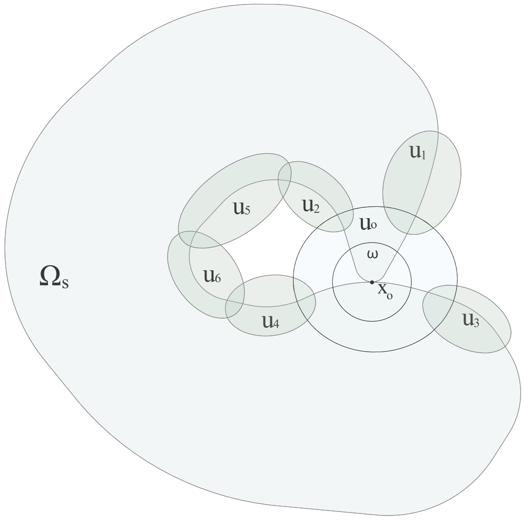

Fig. 4. Splash domain Ω_s, and the collection of open set $\{U_0, U_1, U_2, ..., U_K\}$ covering Γ.

9. The case of a general self-intersection splash geometry

We now show how the analysis presented in the previous sections for the case of the "dinosaur wave" initial domain can be used to establish the existence of a splash singularity in a finite time T^* for any domain whose boundary is arbitrarily close (in the H^3-norm) to any given self-intersecting surface of class H^3. This generalization requires the geometric constructions that we introduced in our previous work [16], coupled with a very minor adaptation of the analysis of the previous sections.

We begin with the definition of the splash domain that we gave in [16].

9.1. The definition of the splash domain

1. We suppose that $x_0 \in \Gamma := \partial\Omega_s$ is the unique boundary self-intersection point, i.e., Ω_s is locally on each side of the tangent plane to $\partial\Omega_s = \Gamma_s$ at x_0. For all other boundary points, the domain is locally on one side of its boundary. Without loss of generality, we suppose that the tangent plane at x_0 is the horizontal plane $x_d - (x_0)_d = 0$.
2. We let U_0 denote an open neighborhood of x_0 in $\mathbb{R}^3$, and then choose an additional L open sets $\{U_l\}_{l=1}^L$ such that the collection $\{U_l\}_{l=0}^K$ is an open cover of Γ_s, and $\{U_l\}_{l=0}^L$ is an open cover of Ω_s and such that there exists a sufficiently small open subset $\omega \subset U_0$ containing x_0 with the property that
$$\overline{\omega} \cap \overline{U_l} = \emptyset \ \text{ for all } \ l = 1, ..., L\,.$$
We set
$$U_0^+ = U_0 \cap \Omega_s \cap \{x_d > (x_0)_d\} \ \text{ and } U_0^- = U_0 \cap \Omega_s \cap \{x_d < (x_0)_d\}\,.$$
Additionally, we assume that $\overline{U_0} \cap \overline{\Omega_s} \cap \{x_d = (x_0)_d\} = \{x_0\}$, which implies in particular that U_0^+ and U_0^- are connected. See Fig. 4.
3. For each $l \in \{1, ..., K\}$, there exists an H^3-class diffeomorphism θ_l satisfying
$$\theta_l : B := B(0,1) \to U_l$$
$$U_l \cap \Omega_s = \theta_l(B^+) \ \text{ and } \ \overline{U_l} \cap \Gamma_s = \theta_l(B^0)\,,$$
where
$$B^+ = \{(x_1, ..., x_d) \in B : x_d > 0\}\,,$$
$$B^0 = \{(x_1, ..., x_d) \in \overline{B} : x_d = 0\}\,.$$

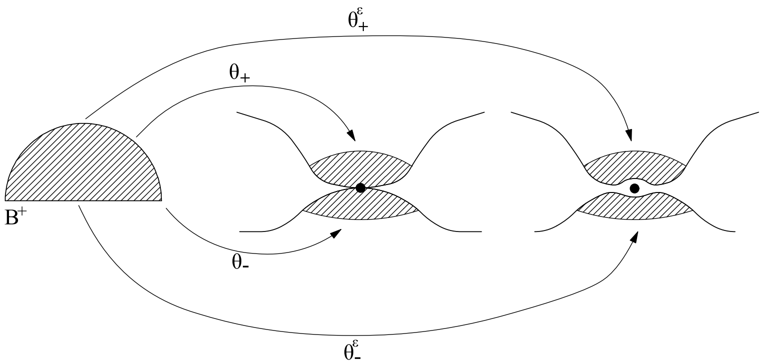

Fig. 5. The black dot denotes the point x_0 where the boundary self-intersects (middle). For $\epsilon > 0$, the approximate domain Ω^ϵ does not intersect itself (right).

4. For $L > K$, let $\{U_l\}_{l=K+1}^L$ denote a family of open sets contained in Ω_s such that $\{U_l\}_{l=0}^L$ is an open cover of Ω_s, and for $l \in \{K+1, ..., L\}$, $\theta_l : B \to U_l$ is an H^3 diffeomorphism.
5. To the open set U_0 we associate two H^3-class diffeomorphisms θ_+ and θ_- of B onto U_0 with the following properties:

$$\theta_+(B^+) = U_0^+ , \qquad \theta_-(B^+) = U_0^- ,$$
$$\theta_+(B^0) = \overline{U_0^+} \cap \Gamma_s , \qquad \theta_-(B^0) = \overline{U_0^-} \cap \Gamma_s ,$$

such that

$$\{x_0\} = \theta_+(B^0) \cap \theta_-(B^0) ,$$

and

$$\theta_+(0) = \theta_-(0) = x_0 .$$

We further assume that

$$\overline{\theta_\pm(B^+ \cap B(0, 1/2))} \cap \overline{\theta_l(B^+)} = \emptyset \text{ for } l = 1, ..., K ,$$

and

$$\overline{\theta_\pm(B^+ \cap B(0, 1/2))} \cap \overline{\theta_l(B)} = \emptyset \text{ for } l = K+1, ..., L .$$

Definition 6 *(Splash domain Ω_s).* We say that Ω_s is a splash domain, if it is defined by a collection of open covers $\{U_l\}_{l=0}^L$ and associated maps $\{\theta_\pm, \theta_1, \theta_2, ..., \theta_L\}$ satisfying the properties (1)–(5) above. Because each of the maps is an H^3 diffeomorphism, we say that the splash domain Ω_s defines a self-intersecting *generalized* $\mathbf{H^3}$-domain.

9.2. An approximating sequence of non self-intersecting domains converging to the splash domain

Following [16], we can then define standard (non self-intersecting) domains Ω^ϵ (for $\epsilon > 0$ small enough) by just modifying $\theta_\pm$, and leaving the other charts unchanged. As shown in Fig. 5, our non self-intersecting domain Ω^ϵ will be defined by associated maps $\{\theta\epsilon_\pm, \theta_1, \theta_2, ..., \theta_L\}$ such that

$$\|\theta^\epsilon_\pm - \theta_\pm\|_{H^3(B^+)} \leqslant C\epsilon , \tag{80}$$

and such that

$$0 < d(\theta^\epsilon_+(B^+), \theta^\epsilon_-(B^+)) \leqslant \epsilon . \tag{81}$$

In summary, we have approximated the self-intersecting splash domain Ω_s with a sequence of H^3-class domains Ω^ϵ converging toward Ω, such that for each $\epsilon > 0$, $\partial\Omega^\epsilon$ does not self-intersect. As such, each one of these domains Ω^ϵ, $\epsilon > 0$, will thus be amenable to our local-in-time well-posedness theory for free-boundary incompressible Navier–Stokes equations.

10. Existence of a splash in finite time in a domain arbitrarily close to a given splash domain

We next define an initial velocity field of the same type as in Section 6.1. Due to (80), the estimates of Section 7 remain unchanged. Similarly, the main proof of Section 8 works in a similar manner due to (81), leading to the necessity of self-intersection at a time $T^\epsilon \in (0, 10\epsilon)$. Note that since the tangent plane at the intended splash singularity x_0 is the horizontal plane $\{x_d = 0\}$, $\partial[\theta_-(B^+)]$ is very close to $\{x_d = 0\}$ in a small ball $B(x_0, \sqrt{\epsilon})$ for ϵ taken sufficiently small; thus, we are using the fact that the almost flat portion of $\theta_-(B^+)$ is very close to $\{x_d = 0\}$ and contains a region of diameter at least $\sqrt{\epsilon}$.

Furthermore,

$$\begin{aligned}\|\eta^\epsilon(\theta^\epsilon_\pm, T^\epsilon) - \theta_\pm\|_3 &\leqslant \|\eta^\epsilon(\theta^\epsilon_\pm, T^\epsilon) - \theta^\epsilon_\pm\|_3 + \|\theta^\epsilon_\pm - \theta_\pm\|_3 \\ &\leqslant \|\int_0^{T^\epsilon} v^\epsilon(\theta^\epsilon_\pm, t)\, dt\|_3 + C\epsilon\,, \end{aligned} \tag{82}$$

where we used the estimate (80) in the above inequality (82); hence, from our estimates in Section 7,

$$\|\eta^\epsilon(\theta^\epsilon_\pm, T^\epsilon) - \theta_\pm\|_3 \leqslant C\mathcal{P}(M_0)\sqrt{T^\epsilon} + C\epsilon \leqslant C\mathcal{P}(M_0)\sqrt{\epsilon}\,. \tag{83}$$

This, therefore, shows that the splash-free surface $\eta^\epsilon(\Omega^\epsilon, T^\epsilon)$ is at a distance less than $C\mathcal{P}(M_0)\sqrt{\epsilon}$ from Ω_s in H^3. We have then established the following:

Theorem 8. *For any given splash domain Ω_s of class H^3, there exists a splash domain $\tilde{\Omega}_s$ arbitrarily close in H^3 to Ω_s, and smooth initial data consisting of a non self-intersecting domain Ω^ϵ of class H^3 and a divergence-free velocity field $u_0^\epsilon \in H^3(\Omega^\epsilon)$ satisfying $[\mathrm{Def}\, u_0^\epsilon \cdot N_\epsilon] \times N_\epsilon = 0$ on $\partial\Omega^\epsilon$, such that the flow map $\eta(x,t)$ solving the Navier–Stokes equations (23) satisfies $\eta(\partial\Omega^\epsilon, T^*) = \tilde{\Omega}_s$. That is, in finite time $T^* > 0$, a splash singularity occurs which is very close to a prescribed self-intersecting geometry.*

Conflict of interest statement

There is no conflict of interest.

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