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A SIMPLE PROOF OF WELL-POSEDNESS FOR THE FREE-SURFACE INCOMPRESSIBLE EULER EQUATIONS

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Abstract. The purpose of this this paper is to present a new simple proof for the construction of unique solutions to the moving free-boundary incompressible 3-D Euler equations in vacuum. Our method relies on the Lagrangian representation of the fluid, and the anisotropic smoothing operation that we call *horizontal convolution-by-layers*. The method is general and can be applied to a number of other moving free-boundary problems.

1. **Introduction.**

Following the framework that we developed in [3], we present a new simple proof for the construction of unique solutions to the moving free-boundary incompressible 3-D Euler equations in vacuum. Our method relies on the Lagrangian representation of the fluid, and the anisotropic smoothing operation that we call *horizontal convolution-by-layers*. While we present the material for the case of three space dimensions, the same presentation covers the case of plane flow also. We will not consider the presence of surface tension in this paper, and we will not make any assumptions of irrotationality.

1.1. **The Eulerian description.**

For $0 \le t \le T$, the evolution of a three-dimensional incompressible fluid with a moving free-surface is modeled by the incompressible Euler equations:

$$u_t + u \cdot Du + Dp = 0 \quad \text{in} \ \ \Omega(t)\,, \tag{1a}$$

$$\operatorname{div} u = 0 \quad \text{in} \ \ \Omega(t)\,, \tag{1b}$$

$$p = 0 \quad \text{on} \ \ \Gamma(t)\,, \tag{1c}$$

$$\mathcal{V}(\Gamma(t)) = u \cdot n \tag{1d}$$

$$u = u_0 \quad \text{on} \ \ \Omega(0)\,, \tag{1e}$$

$$\Omega(0) = \Omega\,. \tag{1f}$$

The open subset $\Omega(t) \subset \mathbb{R}^3$ denotes the changing volume occupied by the fluid, $\Gamma(t) := \partial\Omega(t)$ denotes the moving free-surface, $\mathcal{V}(\Gamma(t))$ denotes normal velocity of $\Gamma(t)$, and $n(t)$ denotes the exterior unit normal vector to the free-surface $\Gamma(t)$. The vector-field $u = (u_1, u_2, u_3)$ denotes the Eulerian velocity field, and p denotes the pressure function. We use the notation $D = (\partial_1, \partial_2, \partial_3)$ to denote the gradient

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operator. We have normalized the equations to have all physical constants equal to 1.

This is a free-boundary partial differential equation to determine the velocity and pressure in the fluid, as well as the location and smoothness of the a priori unknown free-surface. A recent explosion of interest in the analysis of the free-boundary incompressible Euler equations, particularly in irrotational form, has produced a number of different methodologies for obtaining a priori estimates, and the accompanying existence theories have mostly relied on the Nash-Moser iteration to deal with derivative loss in linearized equations when arbitrary domains are considered, or complex analysis tools for the irrotational problem with infinite depth. We refer the reader to [1], [3], [5], [6], [7], [8], [11], [12], [13], [14] for a partial list of papers on this topic.

The purpose of these lectures is to make our presentation in [3] more accessible, and to explain the basic interaction of the geometry of the free-surface with the motion of the fluid. As is often the case, with the passage of time, a more concise treatment can be presented, and this is indeed the case for the free-surface incompressible Euler problem.

To avoid the use of local coordinate charts necessary for arbitrary geometries, for simplicity, we will assume that the initial domain Ω at time $t = 0$ is given by

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \mathbb{T}^2,\ x_3 \in (0,1)\}\,, \tag{2}$$

where $\mathbb{T}^2$ denotes the 2-torus, which can be thought of as the unit square $(0,1)^2$ with periodic boundary conditions. This permits the use of *one* global Cartesian coordinate system. We only allow the *top* boundary

$$\Gamma = \{x_3 = 1\}$$

to move, while the *bottom* boundary is fixed with boundary condition

$$u_3 = 0 \text{ on } \{x_3 = 0\} \times [0, T]\,.$$

We refer the reader to our paper [3] for the case that the initial domain is an arbitrary bounded, open subset of $\mathbb{R}^3$ with H^3-class boundary.

1.2. **Einstein's summation convention.** Repeated Latin indices $i, j, k,$, etc., are summed from 1 to 3; for example, $F_{,ii} := \sum_{i=1,3} \frac{\partial^2}{\partial x_i \partial x_i}$.

1.3. **The Lagrangian description.** We transform the system (1) into Lagrangian variables. We let $\eta(x,t)$ denote the "position" of the fluid particle x at time t. Thus,

$$\partial_t \eta = u \circ \eta \text{ for } t > 0 \text{ and } \eta(x, 0) = x$$

where $\circ$ denotes composition so that $[u \circ \eta](x,t) := u(\eta(x,t),t)$. We set

$$\begin{aligned}
&v = u \circ \eta \text{ (Lagrangian velocity)},\\
&q = p \circ \eta \text{ (Lagrangian pressure)},\\
&A = [D\eta]^{-1} \text{ (inverse of the deformation tensor)},\\
&J = \det[D\eta] \text{ (Jacobian determinant of the deformation tensor)},\\
&a = J\,A \text{ (cofactor of the deformation tensor)}.
\end{aligned}$$

Since $\operatorname{div} u = 0$, we have that $\det D\eta = 1$, and hence the cofactor matrix of $D\eta$ is equal to $[D\eta]^{-1}$. Using Einstein's summation convention, and using the notation

$F,_k$ to denote $\frac{\partial F}{\partial x_k}$, the k$th$-partial derivative of F for $k = 1, 2, 3$, the Lagrangian version of equations (1) is given on the fixed reference domain Ω by

$$v^i_t + A^k_i q,_k = 0 \qquad \text{in } \Omega \times (0,T]\,, \tag{3a}$$
$$\operatorname{div}_\eta v = 0 \qquad \text{in } \Omega \times (0,T]\,, \tag{3b}$$
$$q = 0 \qquad \text{on } \Gamma \times (0,T]\,, \tag{3c}$$
$$v_3 = 0 \qquad \text{on } \{x_3 = 0\} \times (0,T]\,, \tag{3d}$$
$$(\eta, v) = (e, u_0) \qquad \text{in } \Omega \times \{t = 0\}\,, \tag{3e}$$

where $e(x) = x$ denotes the identity map on Ω. Notice that the free suface $\Gamma(t)$ is given by

$$\Gamma(t) = \eta(t)(\Gamma)\,.$$

The Lagrangian divergence $\operatorname{div}_\eta v = A^j_i v^i,_j$. Equation (3a) can be written in vector form as

$$v_t + A^T Dq = 0 \quad \text{in } \Omega \times (0,T]\,, \tag{3a'}$$

where A^T denotes the transpose of A. Solutions to (3) which are sufficiently smooth to ensure that $\eta(t)$ are diffeomorphisms, give solutions to (1) via the change of variables indicated above.

1.4. **The Lagrangian vorticity equation.** We make use of the permutation symbol

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation of } \{1,2,3\}, \\ -1, & \text{odd permutation of } \{1,2,3\}, \\ 0, & \text{otherwise}\,, \end{cases}$$

and the basic identity regarding the ith component of the curl of a vector field u:

$$(\operatorname{curl} u)_i = \varepsilon_{ijk} u^k,_j \ .$$

Defining $\operatorname{curl}_\eta v = \operatorname{curl} u \circ \eta$, the chain rule shows, by taking the curl of the Euler equations (3a), that

$$(\operatorname{curl}_\eta v_t)_i = \varepsilon_{ijk} A^s_j v^k_t,_s = 0. \tag{4}$$

2. Notation.

2.1. **Differentiation and norms.** For integers $k \geq 0$ and a smooth, open domain Ω of $\mathbb{R}^3$, we define the Sobolev space $H^k(\Omega)$ ($H^k(\Omega; \mathbb{R}^3)$) to be the completion of $C^\infty(\Omega)$ ($C^\infty(\Omega; \mathbb{R}^3)$) in the norm

$$\|u\|^2_k := \sum_{|a| \leq k} \int_\Omega |D^a u|^2 dx,$$

where D^a denotes all partial derivatives of order a. For real numbers $s \geq 0$, the Sobolev spaces $H^s(\Omega)$ and the norms $\|\cdot\|_s$ are defined by interpolation. We will write $H^s(\Omega)$ instead of $H^s(\Omega; \mathbb{R}^3)$.

We define the horizontal derivative by $\bar\partial = (\partial_1, \partial_2)$, and define the Sobolev space $H^k(\Gamma)$ to be the completion of $C^\infty(\Gamma)$ in the norm

$$|u|^2_k := \sum_{|a| \leq k} \int_\Gamma |\bar\partial^a u|^2 dS\,,$$

where $\bar\partial^a$ denotes all horizontal partial derivatives of order a, and $dS = dx_1 dx_2$ denotes the 'surface measure.' For real $s \geq 0$, the Hilbert space $H^s(\Gamma)$ and the boundary norm $|\cdot|_s$ is defined by interpolation. The negative-order Sobolev spaces $H^{-s}(\Gamma)$ are defined via duality: for real $s \geq 0$, $H^{-s}(\Gamma) := [H^s(\Gamma)]'$.

2.2. **The space of divergence-free vectors on Ω.** In order to specify our initial velocity field, we introduce the following subspace of H^s vector-fields on Ω for $s \geq 0$:

Definition 2.1 (H^s-class divergence free vectors)**.**

$$H^s_{\rm div}(\Omega) = \{u \in H^s(\Omega;\mathbb{R}^3) \ : \ u^3 = 0 \text{ on } \{x_3 = 0\},\, x_h \mapsto u(x_h, x_3) \text{ periodic} \\ \operatorname{div} u = 0\}\,.$$

3. **Properties of the cofactor matrix a, and a polynomial-type inequality.**

3.1. **Differentiating the inverse matrix A.** Using that $D\eta\, A = \mathrm{Id}$, we have the following identities

$$\bar\partial A^k_i = -A^s_i \bar\partial \eta^r,_s A^k_r\,, \tag{5}$$

$$D A^k_i = -A^s_i D \eta^r,_s A^k_r\,, \tag{6}$$

$$\partial_t A^k_i = -A^s_i v^r,_s A^k_r\,. \tag{7}$$

3.2. **Relating the cofactor matrix and the unit normal $n(t)$.** With $N = (0,0,1)$ the outward unit normal to Γ, we have the identity

$$n(\eta) = a^T N/|a^T N| \quad \text{or} \quad n_i(\eta) = a^3_i/|a^3_\cdot|\,.$$

For $\alpha, \beta = 1, 2$, we define the components of the *induced metric* on $\Gamma(t)$ by

$$g_{\alpha\beta}(x_1, x_2, t) = \eta,_\alpha (x_1, x_2, t) \cdot \eta,_\beta (x_1, x_2, t),$$

and let $\sqrt{g}$ denote $\sqrt{\det(g_{\alpha\beta})}$. The metric g is a 2×2 matrix defined on Γ. It is an elementary computation (see [3]) to verify that

$$\sqrt{g} = |\eta,_1 \times \eta,_2| = |a^3_\cdot| \text{ on } \Gamma\,,$$

and hence that $a^3_i = \sqrt{g} n_i(\eta)$.

$$A^3_i = J^{-1}\sqrt{g} n_i(\eta) \text{ on } \Gamma\,. \tag{8}$$

3.3. **A polynomial-type inequality.** For a constant $M_0 \geq 0$, suppose that $f(t) \geq 0$, $t \mapsto f(t)$ is continuous, and

$$f(t) \leq M_0 + C\, t\, P(f(t))\,, \tag{9}$$

where P denotes a polynomial function, and C is a generic constant. Then for t taken sufficiently small, we have the bound

$$f(t) \leq 2M_0\,.$$

We use this type of inequality (see [3]) in place of nonlinear Gronwall-type of inequalities.

4. **Trace estimates and the Hodge decomposition elliptic estimates.** The normal trace theorem states that the existence of the normal trace $w \cdot N|_\Gamma$ of a velocity field $w \in L^2(\Omega)$ relies on the regularity of $\text{div} w \in L^2(\Omega)$ (see, for example, [10]). If $\text{div} w \in L^2(\Omega)$, then $w \cdot N$ exists in $H^{-0.5}(\Gamma)$. We will use the following variant:

$$\|\bar\partial w \cdot N\|^2_{H^{-0.5}(\Gamma)} \le C\Big[\|\bar\partial w\|^2_{L^2(\Omega)} + \|\text{div} w\|^2_{L^2(\Omega)}\Big] \tag{10}$$

for some constant C independent of w.

The construction of our higher-order energy function is based on the following Hodge-type elliptic estimate:

Proposition 1. *For an H^r domain Ω with $\Gamma = \partial\Omega$, $r \ge 3$, if $F \in L^2(\Omega;\mathbb{R}^3)$ with $\operatorname{curl} F \in H^{s-1}(\Omega;\mathbb{R}^3)$, $\text{div} F \in H^{s-1}(\Omega)$, and $\bar\partial F \cdot N|_\Gamma \in H^{s-\frac{3}{2}}(\Gamma)$ for $1 \le s \le r$, then there exists a constant $\bar C > 0$ depending only on Ω such that*

$$\begin{aligned} \|F\|_s &\le \bar C \left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + |\bar\partial F \cdot N|_{s-\frac{3}{2}} \right), \\ \|F\|_s &\le \bar C \left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + \textstyle\sum_{\alpha=1}^2 |\bar\partial F \cdot T_\alpha|_{s-\frac{3}{2}} \right), \end{aligned} \tag{11}$$

where N denotes the outward unit-normal to Γ, and T_α, $\alpha = 1, 2$, denotes the two tangent vectors to Γ.

The first estimate is well-known and follows from the identity $-\Delta F = \operatorname{curl}\operatorname{curl} F - D\text{div} F$; a convenient reference is Taylor [9]. The second estimate immediately follows.

5. **Horizontal convolution-by-layers and commutation estimates.**

5.1. **Horizontal convolution-by-layers.** With $x_h = (x_1, x_2)$, we define $\rho \in C_0^\infty(\mathbb{R}^2)$ by $\rho(x_h) = C \exp\left(\frac{1}{|x_h|^2-1}\right)$ if $|x_h| < 1$ and $\rho(x_h) = 0$ if $|x_h| \ge 1$; we then select the constant C so that $\int_{\mathbb{R}^2} \rho dx_h = 1$. We define $\rho_\kappa(x_h) = \frac{1}{\kappa^2}\rho(\frac{x}{\kappa})$. It follows that for $\kappa > 0$, $0 \le \rho_\kappa \in C_0^\infty(\mathbb{R}^2)$ with $\text{spt}(\rho_\kappa) \subset \overline{B(0,\kappa)}$. (Here, spt stands for support.) We define the operation of *horizontal convolution-by-layers* as follows:

$$\Lambda_\kappa f(x_h, x_3) = \int_{\mathbb{R}^2} \rho_\kappa(x_h - y_h) f(y_h, x_3) dy_h \quad \text{for } f \in L^1(\mathbb{R}^2).$$

We introduced this type of smoothing operation in Section 2 of [3], but in the setting of a general initial domain Ω, which required a partition of unity and a collection of local charts to define properly.

By standard properties of convolution, there exists a constant C which is independent of κ, such that for $s \ge 0$,

$$\|\Lambda_\kappa F\|_s \le C\|F\|_s \quad \forall\ F \in H^s(\Omega),$$

and

$$|\Lambda_\kappa F|_s \le C|F|_s \quad \forall\ F \in H^s(\Gamma).$$

Furthermore,

$$\kappa\|\bar\partial\Lambda_\kappa F\|_0 \le C\|F\|_0 \quad \forall\ F \in L^2(\Omega). \tag{12}$$

5.2. **Commutation estimates.**

Lemma 5.1. *For $f \in W^{1,\infty}(\Gamma)$ and $g, \bar\partial g \in L^2(\Gamma)$, there is a generic constant C independent of κ such that*

$$|\Lambda_\kappa(f\,\bar\partial g) - f\,\Lambda_\kappa \bar\partial g|_0 \le C |f|_{W^{1,\infty}(\Gamma)}\, |g|_0\,,$$

where $W^{1,\infty}(\Gamma)$ denotes the Sobolev space of functions $u \in L^\infty(\Gamma)$ with weak derivative $\bar\partial u \in L^\infty(\Gamma)$.

Proof. We have that

$$\begin{aligned}
[\Lambda_\kappa(f\,\bar\partial g) - f\,\Lambda_\kappa\bar\partial g](x) &= \int_{B(x_h,\kappa)} \frac{1}{\kappa^2}\rho(\frac{x_h-y_h}{\kappa})[f(y_h)-f(x_h)]\bar\partial g(y_h) dy_h \\
&= \underbrace{\int_{B(x_h,\kappa)} \frac{1}{\kappa^3}\bar\partial\rho(\frac{x_h-y_h}{\kappa})[f(y_h)-f(x_h)] g(y_h) dy_h}_{\mathcal{I}_1} \\
&\qquad - \underbrace{\int_{B(x_h,\kappa)} \rho_\kappa(x_h-y_h)\bar\partial f(y_h) g(y_h) dy_h}_{\mathcal{I}_2}\,,
\end{aligned}$$

where we have used integration-by-parts in order to obtain the second equality. From Morrey's inequality, for all $y_h \in B(x_h,\kappa)$,

$$|f(x_h) - f(y_h)| \le \kappa \|\bar\partial f\|_{L^\infty(\mathbb{T}^2)} \le \kappa |f|_{W^{1,\infty}(\Gamma)}\,,$$

so that with $\mathcal{K}(x_h) = \frac{1}{\kappa^2}|\bar\partial\rho(\frac{x_h}{\kappa})|$,

$$|\mathcal{I}_1| \le |f|_{W^{1,\infty}(\Gamma)}\, \mathcal{K} * |g|\,,$$

so by Young's inequality for convolution integrals,

$$|\mathcal{I}_1|_0 \le |f|_{W^{1,\infty}(\Gamma)} \|\mathcal{K}\|_{L^1(\mathbb{T}^2)}\, |g|_0\,.$$

Similarly,

$$|\mathcal{I}_2|_0 \le |f|_{W^{1,\infty}(\Gamma)} \|\rho_\kappa\|_{L^1(\mathbb{T}^2)} |g|_0\,.$$

The assertion is proved, given that

$$\|\mathcal{K}\|_{L^1(\mathbb{T}^2)} = \int_{\mathbb{T}^2} \frac{1}{\kappa^2}|\bar\partial\rho(\frac{y_h}{\kappa})| dy_h = \int_{\mathbb{T}^2} |\bar\partial\rho(z)| dz < \infty\,.$$

□

Lemma 5.2. *For $\kappa > 0$, there exists $C > 0$ independent of κ, such that for any $g \in H^{\frac12}(\Omega)$ and $f \in H^3(\Omega)$, we have that*

$$\left\|\Lambda_\kappa(fg) - f\Lambda_\kappa g\right\|_{\frac12} \le C\kappa \|g\|_{\frac12} \|f\|_3 + C\kappa^{\frac12}\|g\|_0 \|f\|_3.$$

Proof. Let $\Delta = \Lambda_\kappa(fg) - f\ \Lambda_\kappa g$. Then, we have that

$$\Delta(x) = \int_{B(x_h,\kappa)} \rho_\kappa(x_h-y_h)[f(y_h,x_3) - f(x_h,x_3)]\ g(y_h,x_3)\ dy_h\,.$$

Using the fact that $H^2(\Omega)$ is embedded in $L^\infty(\Omega)$, we have that

$$|\Delta(x)| \le C\kappa \|f\|_3 \int_{B(x_h,\kappa)} \rho_\kappa(x_h-y_h)\ |g(y_h,x_3)|\ dy_h\,,$$

showing that

$$\begin{aligned}\|\Delta\|_0 &\le C\kappa \|f\|_3 \|\Lambda_\kappa |g|\|_0 \\ &\le C\kappa \|f\|_3 \|g\|_0. \end{aligned} \tag{13}$$

Now, for $j = 1, 2, 3$,

$$\Delta,_j = \Lambda_\kappa(fg,_j) - f\,\Lambda_\kappa g,_j + \Lambda_\kappa(f,_j\, g) - f,_j\,\Lambda_\kappa g.$$

The difference between the two first terms of the right-hand side of this identity can be treated in a similar fashion as (13), leading us to:

$$\begin{aligned}\|\Delta,_j\|_0 &\le C\kappa \|f\|_3 \|g\|_1 + \|\Lambda_\kappa(f,_j\, g)\|_0 + \|f,_j\,\Lambda_\kappa g\|_0 \\ &\le C\kappa \|f\|_3 \|g\|_1 + \|f,_j\, g\|_0 + \|f,_j\|_{L^\infty(\Omega)} \|\Lambda_\kappa g\|_0 \\ &\le C\kappa \|f\|_3 \|g\|_1 + 2\|f,_j\|_{L^\infty(\Omega)} \|g\|_0 \\ &\le C\kappa \|f\|_3 \|g\|_1 + C\|f\|_3 \|g\|_0. \end{aligned} \tag{14}$$

Consequently, we obtain by interpolation from (13) and (14):

$$\|\Delta\|_{\frac12} \le C\kappa \|f\|_3 \|g\|_{\frac12} + C\kappa^{\frac12} \|f\|_3 \|g\|_0.$$

□

Following the estimate (14), we can similarly obtain estimates for $\|\Delta,_{jk}\|_0$ and $\|\Delta,_{jkl}\|_0$, $j,k,l = 1,2,3$, and thus estimate the $H^2(\Omega)$-norm as well as the $H^3(\Omega)$-norm of Δ. Interpolation then yields the following:

Lemma 5.3. *For $\kappa > 0$ and $s = \frac32$ or $\frac52$, there exists $C > 0$ independent of κ, such that for any $g \in H^s(\Omega)$ and $f \in H^3(\Omega)$, we have that*

$$\left\|\Lambda_\kappa(fg) - f\Lambda_\kappa g\right\|_s \le C\kappa \|g\|_s \|f\|_3 + C\kappa^{\frac12} \|g\|_{s-\frac12} \|f\|_3.$$

6. An asymptotically consistent κ-approximation of the Euler equations.

6.1. The smoothed flow map η_κ.

Definition 6.1 (The horizontally-smoothed flow map η_κ)**.** Suppose the Lagrangian flow map η is in $C([0,T], H^3(\Omega))$. For $\kappa \in (0, \kappa_0)$, we set on $[0,T]$

$$\eta_\kappa = \Lambda_\kappa^2 \eta \quad \text{and} \quad v_\kappa = \partial_t \eta_\kappa\,.$$

By choosing $T > 0$ and κ_0 sufficiently small, we can ensure that

$$A_\kappa(x,t) = [D\eta_\kappa(x,t)]^{-1}$$

is well-defined on $[0,T]$. We then define $J_\kappa = \det[D\eta_\kappa]$, and $a_\kappa = J_\kappa A_\kappa$.

6.2. **Smoothing the initial data.** In order to construct solutions to (3), we will introduce an approximation scheme below, and it will be convenient to smooth the initial velocity field.

Let $0 \le \varrho_\kappa \in C_0^\infty(\mathbb{R}^3)$ denote the standard family of mollifiers with $\operatorname{spt}(\varrho_\kappa) \subset \overline{B(0,\kappa)}$, and let $\mathcal{E}_\Omega$ denote the Sobolev extension operator mapping $H^s(\Omega)$ to $H^s(\mathbb{R}^3)$ for $s \ge 0$.

We set $w_0^\kappa = \varrho_\kappa \mathcal{E}_\Omega(u_0)$, and define

$$u_0^\kappa = w_0^\kappa - Dr^\kappa\,,$$

where the scalar function r^κ is the solution to the elliptic problem

$$\Delta r^\kappa = \operatorname{div} w_0^\kappa \quad \text{in } \Omega\,, \tag{15a}$$
$$x_h \mapsto r^\kappa(x_h, x_3) \text{ periodic} \quad x_3 \in [0,1]\,, \tag{15b}$$
$$r^\kappa = 0 \quad \text{on } \Gamma\,, \tag{15c}$$
$$\frac{\partial r^\kappa}{\partial x_3} = (w_0^\kappa)^3 \quad \text{on } \{x_3 = 0\}\,. \tag{15d}$$

Thus, given $u_0 \in H^4_{\text{div}}(\Omega)$, it follows that $u_0^\kappa \in H^s_{\text{div}}(\Omega)$ for all $s \geq 4$, and that $u_0^\kappa \to u_0$ in $H^4(\Omega)$ as $\kappa \to 0$.

6.3. **The approximate κ-problem.** For $\kappa \in (0, \kappa_0)$, we consider the following sequence of approximate problems:

$$v_t + A_\kappa^T Dq = 0 \quad \text{in } \Omega \times (0, T_\kappa]\,, \tag{16a}$$
$$\operatorname{div}_{\eta_\kappa} v = 0 \quad \text{in } \Omega \times (0, T_\kappa]\,, \tag{16b}$$
$$q = 0 \quad \text{on } \Gamma \times (0, T_\kappa]\,, \tag{16c}$$
$$v^3 = 0 \quad \text{on } \{x_3 = 0\} \times (0, T_\kappa]\,, \tag{16d}$$
$$(v, \eta) = (u_0^\kappa, e_\kappa) \quad \text{on } \Omega \times \{t = 0\}\,, \tag{16e}$$

where the solution $\eta = \eta(\kappa)$ depends on the parameter κ, but for notational simplification, we do not explicitly write this dependence, and where $\operatorname{div}_{\eta_\kappa} v = (A_\kappa)^j_i v^i,_j$, and $e_\kappa = \Lambda_\kappa^2 e$, where $e(x) = x$ denotes the identity map on Ω. We refer to the approximation (16) as the κ-problem. (Note that our solution is periodic in the x_1 and x_2 directions.)

7. Construction of smooth solutions to κ-approximate Euler equations.

Definition 7.1 (The manifold of volume-preserving embeddings)**.** For $s \geq 3$, we let

$$\mathcal{D}^s_{\text{vol}} = \{\eta \in H^s(\Omega; \mathbb{R}^3) \ : \ \eta(\{x_3 = 0\}) \subset \{x_3 = 0\},\, x_h \mapsto \eta(x_h, x_3) \text{ periodic} \\ \det D\eta = 1\,, \eta^{-1} \in H^s(\eta(\Omega); \Omega)\,\}\,.$$

For $s \geq 3$, $\mathcal{D}^s_{\text{vol}}$ is a infinite-dimensional C^∞ Hilbert manifold (see, for example, [4]). We let $T\mathcal{D}^s_{\text{vol}}$ denote the tangent bundle over $\mathcal{D}^s_{\text{vol}}$, and $TT\mathcal{D}^s_{\text{vol}}$ the second tangent bundle. Locally, elements of $T\mathcal{D}^s_{\text{vol}}$ consist of pairs $(\eta, v) \in \mathcal{D}^s_{\text{vol}} \times H^s_{\text{div}}(\Omega) \circ \eta$, where

$$H^s_{\text{div}}(\Omega) \circ \eta = \{u \circ \eta \ : u \in H^s_{\text{div}}(\Omega)\}\,.$$

Theorem 7.2. *Suppose that for $s \geq 4$, $u_0^\kappa \in H^s(\Omega)$ with $\operatorname{div} u_0 = 0$. Then for each $\kappa \in (0, \kappa_0)$, there exists a unique solution $(\eta(\kappa), v(\kappa)) \in C^\infty([0, T_\kappa]; T\mathcal{D}^s_{\text{vol}})$ to (16) with $T_\kappa > 0$ depending on $\|u_0^\kappa\|_s$ an on $\kappa > 0$.*

Proof. With η_κ and v_κ defined in Definition 6.1, we define

$$u = v \circ \eta_\kappa^{-1}\,, \quad u_\kappa = v_\kappa \circ \eta_\kappa^{-1}\,, \quad \text{and } p = q \circ \eta_\kappa^{-1}\,.$$

It follows from the chain-rule that (16) can be written on the time dependent domain $\eta_\kappa(t,\Omega)$ as

$$
\begin{aligned}
u_t + u_\kappa \cdot Du + Dp &= 0 && \text{in } \eta_\kappa(t,\Omega)\,, && \text{(17a)}\\
\operatorname{div} u &= 0 && \text{in } \eta_\kappa(t,\Omega)\,, && \text{(17b)}\\
p &= 0 && \text{on } \eta_\kappa(t,\Gamma)\,, && \text{(17c)}\\
(u,\eta) &= (u_0^\kappa, e) && \text{on } \Omega\times\{t=0\}\,, && \text{(17d)}
\end{aligned}
$$

Taking the divergence of (17a), we see that $p(t)$ satisfies

$$
\begin{aligned}
-\Delta p &= u^i,_j\, u^j_\kappa,_i && \text{in } \eta_\kappa(t,\Omega)\,, && \text{(18a)}\\
p &= 0 && \text{on } \eta_\kappa(t,\Gamma)\,, && \text{(18b)}\\
\frac{\partial p}{\partial N} &= 0 && \text{on } \{x_3=0\}\,. && \text{(18c)}
\end{aligned}
$$

The estimates for $p(t)$ can be easily obtained by transforming (18), set on the smoothed moving domain $\eta_\kappa(t,\Omega)$, to an elliptic equation on the fixed domain Ω. It is important to note that this transformation should not be made with the map $\eta_\kappa(t)$, but rather with a family of diffeomorphisms which inherits the smoothness of $\eta_\kappa|_\Gamma$. To this end, consider the solution to $\Delta\Phi(t) = 0$ in Ω with $\Phi(t) = \eta_\kappa(t)$ on Γ. It follows that

$$\|\Phi(t)\|_{s+1} \le C|\eta_\kappa|_{s+1/2} \le C(1+\frac{1}{\kappa})|\eta|_{s-1/2} \le C(1+\frac{1}{\kappa})\|\eta\|_s\,, \tag{19}$$

where the first inequality is the standard elliptic estimate for the Dirichlet problem, the second follows from (12), and the third from the trace theorem. For κ and T_κ taken sufficiently small $\|\Phi(t)-e\|_s$ can be made arbitrarily small on $[0,T_\kappa]$, from which it follows that each such $\Phi(t):\Omega\to\eta_\kappa(t,\Omega)$ is a diffeomorphism.

Next, define the matrix $B=[D\Phi]^{-1}$ and the pressure function $Q=p\circ\Phi$. Using the chain-rule, (18) is transformed to

$$
\begin{aligned}
-B^j_i[B^k_i Q,_k],_j &= \big(u^i,_j\, u^j_\kappa,_i\big)\circ\Phi && \text{in } \Omega\,, && \text{(20a)}\\
Q &= 0 && \text{on } \Gamma\,, && \text{(20b)}\\
Q,_k\, B^k_i B^3_i &= 0 && \text{on } \{x_3\}=0\,, && \text{(20c)}
\end{aligned}
$$

for $0\le t\le T_\kappa$. Elliptic estimates (in conjunction with the Sobolev embedding theorem) show that

$$\|Q(t)\|_{s+1} \le CP(\|\Phi(t)\|_{s+1})\|u^i,_j\, u^j_\kappa,_i \circ\Phi\|_{s-1}$$

where P is a polynomial function of its argument; hence, together with (19), we have the estimate

$$\|Q(t)\|_{s+1} \le C_\kappa P(\|\eta(t)\|_s)\cdot\|v\|_s^2\,,$$

where the constant C_κ depends on $\kappa>0$ and, in fact, blows-up as $\kappa\to0$. Since $p=Q\circ\Phi^{-1}$, we see that

$$\|Dp(t)\circ\eta(t)\|_s \le C_\kappa P(\|\eta(t)\|_s, \|v\|_s)\,. \tag{21}$$

We define the function $\mathcal{F}$ on $T\mathcal{D}^s_{\text{vol}}$ by

$$\mathcal{F}(\eta,v) = -Dp\circ\eta_\kappa\,,$$

and write (16a) as the coupled system of first-order ordinary differential equations written on $T\mathcal{D}^s_{\rm vol}$:

$$\partial_t(\eta, v) = (v, \mathcal{F})\,,$$
$$(\eta, v)|_{t=0} = (e, u_0)\,.$$

According to the estimate (21), $(v, \mathcal{F}) : T\mathcal{D}^s_{\rm vol} \to TT\mathcal{D}^s_{\rm vol}$ continuously; moreover, since composition on the right is a smooth operation,

$$(v, \mathcal{F}) : T\mathcal{D}^s_{\rm vol} \to TT\mathcal{D}^s_{\rm vol} \text{ is a } C^\infty \text{ map.}$$

Thus, the fundamental theorem of ordinary differential equations (or Picard iteration) shows that there exists a time $T_\kappa > 0$ depending on the initial data (and, of course, $\kappa > 0$) such that

$$(\eta, v) = (\eta(\kappa), v(\kappa)) \in C^\infty([0, T_\kappa]; T\mathcal{D}^s_{\rm vol})$$

is a unique solution of (16). Since, by definition $Q = 0$ on Γ, and

$$q = Q \circ \Phi^{-1} \circ \eta_\kappa\,, \tag{22}$$

it follows that $q = 0$ on Γ as well. □

8. **Asymptotic estimates which are independent of the smooth parameter κ.** According to Theorem 7.2, we have unique solutions to (16),

$$(\eta(\kappa), v(\kappa)) \in C^\infty([0, T_\kappa]; T\mathcal{D}^s_{\rm vol})\,,$$

and $q(\kappa)$ given by (22). We will take $s \geq 6$, and for notational convenience we will denote $(\eta(\kappa), v(\kappa))$ by $(\tilde\eta, \tilde v)$, and write $\tilde A$ for $[D\tilde\eta]^{-1}$. We use the notation $\tilde\eta_\kappa$ to denote $\Lambda^2_\kappa \tilde\eta$ and set $\tilde A_\kappa = [D\tilde\eta_\kappa]^{-1}$.

8.1. **A continuous-in-time energy function appropriate for the asymptotic process $\kappa \to 0$.**

Definition 8.1. We set on $[0, T_\kappa]$

$$E_\kappa(t) = 1 + \|\Lambda_\kappa \tilde\eta(t)\|^2_{4.5} + \|\tilde v(t)\|^2_4 + \|\kappa \tilde v(t)\|^2_{4.5} + \|\tilde v_t(t)\|^2_{3.5} + \|\sqrt{\kappa}\tilde v_t(t)\|^2_4\,. \tag{23}$$

The function $E_\kappa(t)$ is the *higher-order energy function* which we will prove remains bounded on a time-interval which is independent of κ. Given $(\tilde\eta, \tilde v) \in T\mathcal{D}^6_{\rm vol}$, the $E_\kappa(t)$ is continuous on $[0, T_\kappa]$.

Definition 8.2. We set $E(t) = E_{\kappa=0}(t)\,.$

Definition 8.3. We set the constant M_0 to be a polynomial function of $E(0)$ so that

$$M_0 = P(E(0), \|\operatorname{curl} u^\kappa_0\|^2_{3.5})\,. \tag{24}$$

8.2. **Statement of the main result.** Given an initial velocity field $u_0 \in H^4_{\rm div}$, we obtain the initial pressure function p_0 as the solution to the elliptic equation

$$\begin{aligned} -\Delta p_0 &= u^i_{0,j}\, u^j_{0,i} && \text{in } \ \Omega\,,\\ p_0 &= 0 && \text{on } \ \Gamma\,,\\ p_{0,3} &= 0 && \text{on } \ \{x_3 = 0\}\,. \end{aligned}$$

Theorem 8.4 (Main Result). *Given initial data* $u_0 \in H^4_{\rm div}(\Omega)$ *with* $\operatorname{curl} u_0 \in H^{3.5}(\Omega)$ *such that*

$$-\frac{\partial p_0}{\partial x_3}(x) \geq \lambda > 0 \quad \textit{for} \quad x \in \Gamma\,,$$

there exists a solution to (3) verifying

$$\sup_{t\in[0,T]} E(t) \leq P(E(0))\,.$$

If $u_0 \in H^5_{\rm div}(\Omega)$ *and* $\operatorname{curl} u_0 \in H^{4.5}(\Omega)$*, then the solution is unique.*

Remark 1. The same theorem and proof hold in the case that $\Omega \subset \mathbb{R}^2$. We refer the reader to our paper [3] for the case of a general initial domain Ω.

Remark 2. The regularity for the existence theory is not optimal. In fact, for our domain Ω, all that is necessary to establish existence and uniqueness of solutions to (3) is an initial velocity field $u_0 \in H^3_{\rm div}(\Omega)$; nevertheless, the assumptions of Theorem 8.4 allow for the most transparent proof.

8.3. **Conventions about constants.** As noted above, Theorem 7.2 provides us with solutions $(\tilde\eta, \tilde v) \in T\mathcal{D}^6_{\rm vol}$, and hence $\sup_{t\in[0,T_\kappa]} E_\kappa(t)$ is continuous.

We take $T_\kappa > 0$ sufficiently small so that, using the fundamental theorem of calculus, for constants c_1, c_2 and $t \in [0, T_\kappa]$,

$$\begin{aligned}
-\tilde q,_3(t) &\geq \lambda/2\,,\\
c_1 \det \tilde g_\kappa(0) \leq \det \tilde g_\kappa(t) &\leq c_2 \det \tilde g_\kappa(0) \text{ on } \Gamma\,,\\
c_1 \det \tilde J_\kappa(0) \leq \det \tilde J_\kappa(t) &\leq c_2 \det \tilde J_\kappa(0) \text{ in } \Omega\,,\\
\|\tilde\Lambda_\kappa \eta(t)\|_4 \leq |\Lambda_\kappa e|_4 + 1\,, &\quad \|\tilde q(t)\|_4 \leq \|\tilde q(0)\|_4 + 1\,,\\
\|\tilde v(t)\|_{3.5} \leq \|u_0\|_{3.5} + 1\,, &\quad \|\tilde v_t(t)\|_3 \leq \|\tilde v_t(0)\|_3 + 1\,.
\end{aligned}$$

The right-hand sides appearing in the last three inequalities shall be denoted by a generic constant C in the estimates that we will perform.

8.4. **Curl and divergence estimates for $\tilde\eta$, $\tilde v$, and $\tilde v_t$.**

Proposition 2. *For all* $t \in (0, T)$*, with* $T \leq T_\kappa$*,*

$$\begin{aligned}
\|\operatorname{curl}\Lambda_\kappa \tilde\eta(t)\|^2_{3.5} + \|\operatorname{curl}\tilde v(t)\|^2_3 + \|\kappa \operatorname{curl}\tilde v(t)\|^2_{3.5} + \|\sqrt{\kappa}\operatorname{curl}\tilde v_t(t)\|^2_{2.5}&\\
\leq M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,.& \qquad (25)
\end{aligned}$$

Proof. By taking the curl of (16a), we have that

$$\operatorname{curl}_{\eta_\kappa} \tilde v_t = 0\,.$$

From (4), it follows that $(\operatorname{curl}_{\tilde\eta_\kappa} \tilde v)^k_t = B(\tilde A_\kappa, D\tilde v)$, where

$$B(\tilde A_\kappa, D\tilde v) = \varepsilon_{kji} (\tilde A_\kappa)_{t}{}^{s}_{j} \tilde v^i,_s = \varepsilon_{kij} \tilde v^i,_s (\tilde A_\kappa)^s_p \, \tilde v^p_\kappa,_l \, (\tilde A_\kappa)^l_j\,;$$

hence,

$$\operatorname{curl}_{\tilde\eta_\kappa} \tilde v(t) = \operatorname{curl} u_0^\kappa + \int_0^t B(\tilde A_\kappa(t'), D\tilde v(t'))dt'\,. \qquad (26)$$

Step 1. Estimate for $\operatorname{curl}\Lambda_\kappa\tilde\eta$. Computing the gradient of (26) yields

$$\operatorname{curl}_{\tilde\eta_\kappa} D\tilde v(t) = D \operatorname{curl} u_0^\kappa - \varepsilon_{\cdot ji} D(\tilde A_\kappa)^s_j \tilde v^i,_s + \int_0^t DB(\tilde A_\kappa(t'), D\tilde v(t'))dt'\,.$$

Applying the fundamental theorem of calculus once again, shows that

$$\operatorname{curl}_{\tilde\eta_\kappa} D\tilde\eta(t) = tD\operatorname{curl} u_0^\kappa + \varepsilon_{\cdot ji}\int_0^t [(\tilde A_\kappa)_t{}_j^s D\tilde\eta^i,_s - D(\tilde A_\kappa)_j^s \tilde v^i,_s]dt' \\ + \int_0^t\int_0^{t'} DB(\tilde A_\kappa(t''), D\tilde v(t''))dt''dt' ,$$

and finally that

$$D\operatorname{curl}\tilde\eta(t) = tD\operatorname{curl} u_0^\kappa - \varepsilon_{\cdot ji}\int_0^t (\tilde A_\kappa)_t{}_j^s(t')dt'\, D\tilde\eta^i,_s \\ + \varepsilon_{\cdot ji}\int_0^t [(\tilde A_\kappa)_t{}_j^s D\tilde\eta^i,_s - D(\tilde A_\kappa)_j^s \tilde v^i,_s]dt' + \int_0^t\int_0^{t'} DB(\tilde A_\kappa(t''), D\tilde v(t''))dt''dt' . \tag{27}$$

Using the fact that $\partial_t(\tilde A_\kappa)_j^s = -(\tilde A_\kappa)_l^s \tilde v_\kappa^l,_p (\tilde A_\kappa)_j^p$ and $D(\tilde A_\kappa)_j^s = -(\tilde A_\kappa)_l^s D\tilde\eta_\kappa^l,_p (\tilde A_\kappa)_j^p$, we see that

$$DB(\tilde A, D\tilde v) = -\varepsilon_{kji}[D\tilde v^i,_s (\tilde A_\kappa)_l^s \tilde v_\kappa^l,_p (\tilde A_\kappa)_j^p + \tilde v^i,_s (\tilde A_\kappa)_l^s D\tilde v_\kappa^l,_p (\tilde A_\kappa)_j^p \\ + \tilde v^i,_s \tilde v_\kappa^l,_p D((\tilde A_\kappa)_l^s (\tilde A_\kappa)_j^p)] .$$

The precise structure of the right-hand side is not very important; rather, the derivative count is the focus, and as such we write

$$DB(\tilde A, \tilde D v) \sim D^2\tilde v\, D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa + D^2\tilde v_\kappa\, D\tilde v\, \tilde A_\kappa\, \tilde A_\kappa + D^2\tilde\eta_\kappa\, D\tilde v\, D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa .$$

Integrating by parts in time in the last term of the right-hand side of (27), we see that

$$\int_0^t\int_0^{t'} DB(\tilde A_\kappa, D\tilde v)\, dt''dt' \\ \sim -\int_0^t\int_0^{t'} \Big[D^2\tilde\eta\,(D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa)_t + D^2\tilde\eta_\kappa\,(D\tilde v\, \tilde A_\kappa\, \tilde A_\kappa)_t\Big]\, dt''dt' \\ + \int_0^t\int_0^{t'} D^2\tilde\eta_\kappa\, D\tilde v\, D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa dt''dt' \\ + \int_0^t \Big[D^2\tilde\eta\, D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa + D^2\tilde\eta_\kappa\, D\tilde v\, \tilde A_\kappa\, \tilde A_\kappa\Big]\, dt' .$$

Thus, we can write

$$D\operatorname{curl}\tilde\eta(t) \sim tD\operatorname{curl} u_0^\kappa + \underbrace{D^2\tilde\eta\int_0^t D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa dt'}_{\mathcal I_1} + \underbrace{\int_0^t D^2\tilde\eta D\tilde v_\kappa \tilde A_\kappa \tilde A_\kappa dt'}_{\mathcal I_2} \\ + \underbrace{\int_0^t D^2\tilde\eta_\kappa D\tilde v\, \tilde A_\kappa \tilde A_\kappa dt'}_{\mathcal I_3} + \underbrace{\int_0^t\int_0^{t'} D^2\tilde\eta_\kappa\, D\tilde v\, D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa dt''dt'}_{\mathcal I_4} \\ + \underbrace{\int_0^t\int_0^{t'} D^2\tilde\eta_\kappa\,(D\tilde v\, \tilde A_\kappa\, \tilde A_\kappa)_t dt''dt'}_{\mathcal I_5} + \underbrace{\int_0^t\int_0^{t'} D^2\tilde\eta\,(D\tilde v_\kappa\, \tilde A_\kappa\, \tilde A_\kappa)_t dt''dt'}_{\mathcal I_6} .$$

Our goal is to estimate $\|D\operatorname{curl}\Lambda_\kappa\eta\|^2_{2.5}$, which in turn requires us to estimate $\|\Lambda_\kappa\mathcal{I}_i\|^2_{2.5}$ for $i=1,...,6$. We begin with $i=1$:

$$\|\Lambda_\kappa\mathcal{I}_1\|^2_{2.5} \le \|D^2\Lambda_\kappa\tilde\eta\int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\|^2_{2.5} + \|\Lambda_\kappa\left(D^2\tilde\eta\int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\right) - \int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\Lambda_\kappa D^2\tilde\eta\|^2_{2.5}$$

It is easy to see that

$$\|D^2\Lambda_\kappa\tilde\eta\int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\|^2_{2.5} \le CTP(\sup_{t\in[0,T]} E_\kappa(t))\,,$$

and by Lemma 5.3

$$\begin{aligned}&\|\Lambda_\kappa\left(D^2\tilde\eta\int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\right) - \int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\Lambda_\kappa D^2\tilde\eta\|^2_{2.5}\\ &\qquad\le C\kappa^2\|D^2\tilde\eta\|^2_{2.5}\,\|\int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\|^2_3 + C\kappa\|D^2\tilde\eta\|^2_2\,\|\int_0^t D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa dt'\|^2_3\\ &\qquad\le CTP(\sup_{t\in[0,T]} E_\kappa(t))\end{aligned}$$

The same type of commutation estimate shows that

$$\|\Lambda_\kappa\mathcal{I}_2\|^2_{2.5} \le CTP(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

For $i=3,4,5$, we use that $\|\Lambda_\kappa\mathcal{I}_i\|_{2.5}\le\|\mathcal{I}_i\|_{2.5}$ and as $H^{2.5}(\Omega)$ is a multiplicative algebra, we see that for $i=3,4,5$,

$$\|\Lambda_\kappa\mathcal{I}_i\|^2_{2.5} \le CTP(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

Finally, we consider the case that $i=6$:

$$\begin{aligned}\|\Lambda_\kappa\mathcal{I}_6\|^2_{2.5} &\le \|\int_0^t\int_0^{t'} D^2\Lambda_\kappa\tilde\eta\,(D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa)_t dt''dt'\|^2_{2.5}\\ &\quad + \|\int_0^t\int_0^{t'}\left(\Lambda_\kappa\left[D^2\tilde\eta\,(D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa)_t\right] - \left(D\tilde v_\kappa\,\tilde A_\kappa\,\tilde A_\kappa\right)_t D^2\Lambda_\kappa\tilde\eta\right)dt''dt'\|^2_{2.5}\\ &\le CTP(\sup_{t\in[0,T]} E_\kappa(t))\,,\end{aligned}$$

where we have used Lemma 5.3 for the last inequality together with the fact that $\|\kappa^{\frac12}\tilde v_t\|^2_4$is contained in the energy function $E_\kappa(t)$. Therefore, we have proven that

$$\|D\operatorname{curl}\Lambda_\kappa\eta\|^2_{2.5} \le M_0 + CTP(\sup_{t\in[0,T]} E_\kappa(t))\,,$$

and hence with $\operatorname{curl}_{\tilde\eta_\kappa}\tilde v_t = 0$, that

$$\|\operatorname{curl}\Lambda_\kappa\eta\|^2_{3.5} \le M_0 + CTP(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

Step 2. Estimate for $\operatorname{curl}\tilde v$. From (26),

$$\operatorname{curl} v(t) = \operatorname{curl} u_0^\kappa + \int_0^t B(\tilde A_\kappa(t'), D\tilde v(t'))dt' - \varepsilon_{\cdot jk}\tilde v^k,_r\int_0^t \tilde\partial_t(A_\kappa)^r_j dt'\,.$$

As $H^3(\Omega)$ is a multiplicative algebra, it follows that on $[0,T]$,

$$\|\operatorname{curl}\tilde v(t)\|^2_3 \le M_0 + CTP(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

Similarly,

$$\|\kappa \operatorname{curl} \tilde{v}(t)\|^2_{3.5} \leq M_0 + CTP(\sup_{t\in[0,T]} E_\kappa(t)) .$$

Step 3. Estimate for $\operatorname{curl} \tilde{v}_t$**.** From (26),

$$\operatorname{curl} \tilde{v}_t(t) = \varepsilon_{ikj} \int_0^t \partial_t (\tilde{A}_\kappa)^r_j(t') dt' \, \tilde{v}^i_t,_r \ ,$$

from which it follows that on $[0,T]$,

$$\|\sqrt{\kappa} \operatorname{curl} \tilde{v}_t(t)\|^2_3 \leq CTP(\sup_{t\in[0,T]} E_\kappa(t)) .$$

□

Proposition 3. *For all* $t \in (0,T)$*, with* $T \leq T_\kappa$*,*

$$\|\operatorname{div} \Lambda_\kappa \tilde{\eta}(t)\|^2_{3.5} + \|\operatorname{div} \tilde{v}(t)\|^2_3 + \|\kappa \operatorname{div} \tilde{v}(t)\|^2_{3.5} + \|\sqrt{\kappa} \operatorname{div} \tilde{v}_t(t)\|^2_3 \\ \leq M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t)) . \tag{28}$$

Proof. Since $(\tilde{A}_\kappa)^j_i \tilde{v}^i,_j = 0$, we see that

$$(\tilde{A}_\kappa)^j_i D\tilde{v}^i,_j = -D(\tilde{A}_\kappa)^j_i \, \tilde{v}^i,_j \ . \tag{29}$$

Step 1. Estimate for $\operatorname{div} \Lambda_\kappa \tilde{\eta}$**.** It follows that

$$[(\tilde{A}_\kappa)^j_i D\tilde{\eta}^i,_j]_t = \partial_t (\tilde{A}_\kappa)^j_i D\tilde{\eta}^i,_j - D(\tilde{A}_\kappa)^j_i \, \tilde{v}^i,_j$$

so that

$$[(\tilde{A}_\kappa)^j_i D\tilde{\eta}^i,_j](t) = \int_0^t \Big(\partial_t (\tilde{A}_\kappa)^j_i D\tilde{\eta}^i,_j - D(\tilde{A}_\kappa)^j_i \, \tilde{v}^i,_j \Big) dt' ,$$

and hence

$$D \operatorname{div} \tilde{\eta}(t) = \underbrace{\int_0^t \partial_t (\tilde{A}_\kappa)^j_i D\tilde{\eta}^i,_j \, dt'}_{\mathcal{I}_1} - \underbrace{\int_0^t D(\tilde{A}_\kappa)^j_i \, \tilde{v}^i,_j \, dt'}_{\mathcal{I}_2} - \underbrace{\int_0^t \partial_t (\tilde{A}_\kappa)^j_i dt' \, D\tilde{\eta}^i,_j}_{\mathcal{I}_3} .$$

Thus,

$$\|D \operatorname{div} \Lambda_\kappa \tilde{\eta}(t)\|^2_{2.5} \leq \sum_{i=1}^3 \|\Lambda_\kappa \mathcal{I}_i(t)\|^2_{2.5} .$$

Using Lemma 5.3 in the same fashion as was used for Propostion 2, we see that

$$\sum_{i=1}^3 \|\Lambda_\kappa \mathcal{I}_i(t)\|^2_{2.5} \leq CTP(\sup_{t\in[0,T]} E_\kappa(t)) ,$$

from which it follows that

$$\|\operatorname{div} \Lambda_\kappa \tilde{\eta}(t)\|^2_{3.5} \leq CTP(\sup_{t\in[0,T]} E_\kappa(t)) .$$

Step 2. Estimate for $\operatorname{div} \tilde{v}$**.** From $(\tilde{A}_\kappa)^j_i \tilde{v}^i,_j = 0$, we see that

$$\operatorname{div} \tilde{v}(t) = -\int_0^t \partial_t (\tilde{A}_\kappa)^j_i dt' \, \tilde{v}^i,_j \ . \tag{30}$$

Hence, it is clear that

$$\|\operatorname{div} \tilde{v}(t)\|^2_3 + \|\kappa \operatorname{div} \tilde{v}(t)\|^2_{3.5} \leq CTP(\sup_{t\in[0,T]} E_\kappa(t)) .$$

Step 3. Estimate for $\operatorname{div} \tilde{v}_t$**.** From (30),

$$\operatorname{div} \tilde{v}_t(t) = -\partial_t(\tilde{A}_\kappa)^j_i \tilde{v}^i,_j - \int_0^t \partial_t(\tilde{A}_\kappa)^j_i dt' \, \tilde{v}^i_t,_j \; .$$

It follows that

$$\| \operatorname{div} \tilde{v}_t(t)\|^2_{2.5} + \|\sqrt{\kappa} \operatorname{div} \tilde{v}_t(t)\|^2_3 \leq CTP(\sup_{t\in[0,T]} E_\kappa(t)) \, .$$

□

8.5. **Pressure estimates.** Letting $(\tilde{A}_\kappa)^j_i \frac{\partial}{\partial x_j}$ act on (16a), for $t \in [0, T_\kappa]$, the pressure function $q(x,t)$ satisfies the elliptic equation

$$\begin{aligned} -(\tilde{A}_\kappa)^j_i \left[(\tilde{A}_\kappa)^k_i \tilde{q},_k \right],_j &= \tilde{v}^i,_j (\tilde{A}_\kappa)^j_r \tilde{v}^r_\kappa,_s (\tilde{A}_\kappa)^s_i \text{ in } \Omega \, , \\ \tilde{q} &= 0 \qquad\qquad\qquad\quad \text{on } \Gamma \, , \\ \tilde{q},_k (\tilde{A}_\kappa)^k_i (\tilde{A}_\kappa)^3_i &= \tilde{v} \cdot \partial_t (\tilde{A}_\kappa)^3_\cdot = 0 \qquad \text{on } \{x_3 = 0\} \, . \end{aligned}$$

Using our conventions of Section 8.3 concerning the generic constant C, we have the standard elliptic estimate (see [3]) on $[0, T_\kappa]$

$$\|\tilde{q}(t)\|_{4.5} \leq C \|\Lambda_\kappa \tilde{\eta}(t)\|_{4.5} \, . \tag{31}$$

Similarly by time-differentiating the above elliptic equation for q, elliptic estimates show that

$$\|\tilde{q}_t(t)\|^2_4 \leq C \left(\|\Lambda_\kappa \tilde{\eta}(t)\|^2_{4.5} + \|\tilde{v}(t)\|^2_4 \right) \, .$$

8.6. **Technical lemma.** Our energy estimates require the use of the following

Lemma 8.5. *Let* $H^{\frac{1}{2}}(\Omega)'$ *denote the dual space of* $H^{\frac{1}{2}}(\Omega)$*. There exists a positive constant* C *such that*

$$\|\bar{\partial} F\|_{H^{\frac{1}{2}}(\Omega)'} \leq C \, \|F\|_{\frac{1}{2}} \quad \forall F \in H^{\frac{1}{2}}(\Omega) \, .$$

Proof. Integrating by parts with respect to the horizontal derivative yields for all $G \in H^1(\Omega)$,

$$\int_\Omega \bar{\partial} F \, G \, dx = - \int_\Omega F \; \bar{\partial} G \, dx \leq C \|F\|_0 \, \|G\|_1 \, ,$$

which shows that there exists $C > 0$ such that

$$\forall F \in L^2(\Omega), \quad \|\bar{\partial} F\|_{H^1(\Omega)'} \leq C \|F\|_0 \, . \tag{32}$$

Interpolating with the obvious inequality

$$\forall F \in H^1(\Omega), \quad \|\bar{\partial} F\|_{L^2(\Omega)} \leq C \|F\|_1$$

proves the lemma. □

8.7. **Energy estimates.** In this section, we take $T \in (0, T_\kappa)$.

Proposition 4. *For* $t \in [0, T_\kappa]$,

$$|\Lambda_\kappa \tilde{\eta}^3(t)|^2_4 + |\tilde{v}^3(t)|^2_{3.5} \leq M_0 + C \, T \, P(\sup_{t\in[0,T]} E_\kappa(t)) \, . \tag{33}$$

Proof. Taking the $L^2(\Omega)$ inner-produce $\bar\partial^4$ of (16a) with $\bar\partial^4\tilde v^i$ yields

$$0=\underbrace{\frac12\frac{d}{dt}\|\bar\partial^4\tilde v(t)\|_0^2}_{\mathcal{I}_1}+\underbrace{\int_\Omega\bar\partial^4(\tilde A_\kappa)^k_i\;\tilde q,_k\;\bar\partial^4\tilde v^i dx}_{\mathcal{I}_2}+\underbrace{\int_\Omega(\tilde A_\kappa)^k_i\bar\partial^4\tilde q,_k\;\bar\partial^4\tilde v^i dx}_{\mathcal{I}_3}+\mathcal{R}\,, \tag{34}$$

where $\mathcal{R}$ denotes integrals consisting of lower-order terms which can easily be shown, via the Cauchy-Schwarz inequality, to satisfy

$$\int_0^T|\mathcal{R}(t)|dt\le M_0+C\,T\,P(\sup_{t\in[0,T]}E(t))\,.$$

Using the identity (5), we see that

$$\begin{aligned}\mathcal{I}_2&=-\int_\Omega(\tilde A_\kappa)^k_r\bar\partial^4\tilde\eta^r_\kappa,_s(\tilde A_\kappa)^s_i\tilde q,_k\bar\partial^4\tilde v^i dx+\mathcal{R}\\&=-\underbrace{\int_\Gamma(\tilde A_\kappa)^k_r\bar\partial^4\eta^r_\kappa(\tilde A_\kappa)^3_i\tilde q,_k\bar\partial^4\tilde v^i dx_h}_{\mathcal{I}_{2_a}}+\underbrace{\int_\Omega(\tilde A_\kappa)^k_r\bar\partial^4\eta^r_\kappa(\tilde A_\kappa)^s_i q,_k\bar\partial^4\tilde v^i,_s dx}_{\mathcal{I}_{2_b}}+\mathcal{R}\,.\end{aligned}$$

Since $\tilde q=0$ on Γ, so that $\tilde q,_k=\tilde q,_3$, we see that

$$-\mathcal{I}_{2a}=\int_\Gamma(-\tilde q,_3)\bar\partial^4\tilde\eta^r_\kappa(\tilde A_\kappa)^3_r\bar\partial^4\;\tilde v^i(\tilde A_\kappa)^3_i\,dx_h\,,$$

Recalling that $\tilde\eta_\kappa=\Lambda_\kappa\Lambda_\kappa\tilde\eta$,

$$\begin{aligned}-\mathcal{I}_{2a}=&\underbrace{\int_\Gamma(-\tilde q,_3)\bar\partial^4\Lambda_\kappa\tilde\eta^r(\tilde A_\kappa)^3_r\;\bar\partial^4\Lambda_\kappa\tilde v^i(\tilde A_\kappa)^3_i dx_h}_{\mathcal{K}_1}\\&+\underbrace{\int_\Gamma\bar\partial^4\Lambda_\kappa\tilde\eta^r\left[\Lambda_\kappa\left((-\tilde q,_3)(\tilde A_\kappa)^3_i(\tilde A_\kappa)^3_r\;\bar\partial^4\tilde v^i\right)-(-\tilde q,_3)(\tilde A_\kappa)^3_i(\tilde A_\kappa)^3_r\;\Lambda_\kappa\bar\partial^4\tilde v^i\right]dx_h}_{\mathcal{K}_2},\end{aligned}$$

According to Lemma 5.1,

$$\begin{aligned}\mathcal{K}_2(t)&\le|\bar\partial^4\Lambda_\kappa\tilde\eta^r|_0\;\left|\Lambda_\kappa\left((-\tilde q,_3)(\tilde A_\kappa)^3_i(\tilde A_\kappa)^3_r\;\bar\partial^4\tilde v^i\right)-(-\tilde q,_3)(\tilde A_\kappa)^3_i(\tilde A_\kappa)^3_r\;\Lambda_\kappa\bar\partial^4\tilde v^i\right|_0\\&\le C|\bar\partial^4\Lambda_\kappa\tilde\eta^r|_0\,|\tilde q,_3\;(\tilde A_\kappa)^3_i\,(\tilde A_\kappa)^3_r|_{W^{1,\infty}(\Gamma)}\,|\bar\partial^3\tilde v^i|_0\,.\end{aligned}$$

By the Sobolev embedding theorem,

$$|\tilde q,_3\;(\tilde A_\kappa)^3_i\,(\tilde A_\kappa)^3_r|_{W^{1,\infty}(\Gamma)}\le C|\tilde q,_3|_{1.5}\,|(\tilde A_\kappa)^3_i|_{1.5}\,|(\tilde A_\kappa)^3_r|_{1.5}\,,$$

so that

$$\int_0^T\mathcal{K}_2(t)dt\le C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))\,.$$

To study the integral $\mathcal{K}_1$, we define $\tilde n_\kappa$ to be the outward unit normal to the moving surface $\tilde\eta_\kappa(t,\Gamma)$, given by $\tilde n_\kappa=(\tilde a_\kappa)^3_i/|(\tilde a_\kappa)^3_\cdot|$. From (8),

$$(\tilde A_\kappa)^3_\cdot=\tilde J_\kappa^{-1}\sqrt{\tilde g_\kappa}n_\kappa\text{ on }\Gamma\,,$$

where

$$\sqrt{\tilde g_\kappa}=|\tilde\eta_\kappa,_1\times\tilde\eta_\kappa,_2|=|(\tilde a_\kappa)^3_\cdot|\text{ on }\Gamma\,.$$

It follows that

$$\begin{aligned}\mathcal{K}_1(t) &= \int_\Gamma (-\tilde{q},_3)\bar{\partial}^4\Lambda_\kappa\tilde{\eta}\cdot\tilde{n}_\kappa\ \bar{\partial}^4\Lambda_\kappa\tilde{v}\cdot\tilde{n}_\kappa\,|\det\tilde{g}_\kappa|\tilde{J}_\kappa^{-2}\,dx_h\\ &= \underbrace{\frac{1}{2}\frac{d}{dt}\int_\Gamma(-\tilde{q},_3)|\bar{\partial}^4\Lambda_\kappa\tilde{\eta}\cdot\tilde{n}_\kappa|^2\,|\det\tilde{g}_\kappa|\tilde{J}_\kappa^{-2}\,dx_h}_{\mathcal{K}_{1a}}\\ &\qquad - \underbrace{\int_\Gamma \frac{1}{2}\bar{\partial}^4\Lambda_\kappa\tilde{\eta}^i\bar{\partial}^4\Lambda_\kappa\tilde{\eta}^j\,\partial_t[(\tilde{n}_\kappa)_i(\tilde{n}_\kappa)_j\,|\det\tilde{g}_\kappa|\,\tilde{J}_\kappa^{-2}]dx_h}_{\mathcal{K}_{1b}}\,.\end{aligned}$$

By the assumption of Section 8.3,

$$\Big|\partial_t[(\tilde{n}_\kappa)_i(\tilde{n}_\kappa)_j\,|\det\tilde{g}_\kappa|\,\tilde{J}_\kappa^{-2}]\Big|_{L^\infty(\Gamma)}\le C\,,$$

from which it follows that

$$\int_0^T \mathcal{K}_{1b}(t)dt \le C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))\,.$$

Using our assumed bounds for $-\tilde{q},_3(t), \det\tilde{g}_\kappa(t), \tilde{J}_\kappa$ on $[0,T_\kappa]$, we see that

$$\bar{c}|\bar{\partial}^4\Lambda_\kappa\tilde{\eta}(t)\cdot\tilde{n}_\kappa(t)|_0^2 \le \int_0^T \mathcal{K}_1(t)dt + M_0 + C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))\,,$$

for a constant $\bar{c}$ which depends on $\lambda, \tilde{g}_\kappa(0), \tilde{J}_\kappa(0)$.

Notice that $N=(0,0,1)=\tilde{n}_\kappa(t)-\int_0^t\partial_t\tilde{n}_\kappa(t')dt'$, and by our assumptions in Section 8.3, $|\partial_t\tilde{n}_\kappa(t)|_{L^\infty(\Gamma)}\le C$, so that with $\tilde{\eta}^3=\tilde{\eta}\cdot N$,

$$\bar{c}|\Lambda_\kappa\tilde{\eta}^3(t)|_4^2 \le \int_0^T \mathcal{K}_1(t)dt + M_0 + C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))\,,$$

and hence

$$\bar{c}|\Lambda_\kappa\tilde{\eta}^3(t)|_4^2 \le -\int_0^T \mathcal{I}_{2a}(t)dt + M_0 + C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))\,,$$

It remains to show that the integrals $\int_0^T\mathcal{I}_{2b}(t)dt$ and $\int_0^T\mathcal{I}_3(t)dt$ are both bounded by $C\,T\,P(\sup_{t\in[0,T]}E_\kappa(t))$. Using (16b),

$$\begin{aligned}\mathcal{I}_{2b}(t) &= -\int_\Omega(\tilde{A}_\kappa)_r^k\,\bar{\partial}^4\tilde{\eta}_\kappa^r\,\tilde{q},_k\ \tilde{v}^i,_s\ \bar{\partial}^4(\tilde{A}_\kappa)_i^s\,dx + \mathcal{R}\\ &\le C\|\bar{\partial}^4\tilde{\eta}_\kappa(t)\|_{\frac{1}{2}}\|\bar{\partial}^4\tilde{A}_\kappa(t)\|_{H^{\frac{1}{2}}(\Omega)'} + \mathcal{R}\\ &\le C\|\bar{\partial}^4\tilde{\eta}_\kappa(t)\|_{\frac{1}{2}}\|\bar{\partial}^3\tilde{A}_\kappa(t)\|_{H^{\frac{1}{2}}(\Omega)} + \mathcal{R}\\ &\le C\sup_{t\in[0,T]}E(t) + \mathcal{R}\,,\end{aligned}$$

where we have used Lemma 8.5 for the second inequality.

Finally,

$$\begin{aligned}\mathcal{I}_3(t) &= -\int_\Omega \bar\partial^4 \tilde q\, \bar\partial^4 \tilde v^i,_k\, (\tilde A_\kappa)^k_i\, dx = \int_\Omega \bar\partial^4 \tilde q\, \tilde v^i,_k\, \bar\partial^4 (\tilde A_\kappa)^k_i\, dx + \mathcal{R}\\ &\le C\|\bar\partial^3 \tilde q(t)\|_{\frac12} \|\bar\partial^4 \tilde A_\kappa(t)\|_{H^{\frac12}(\Omega)'} + \mathcal{R}\\ &\le C \sup_{t\in[0,T]} E(t) + \mathcal{R}\,,\end{aligned}$$

where we have used the pressure estimate (31) and Lemma 8.5 for the last inequality.

Summing the estimates for $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and integrating (34) from 0 to T, we obtain the inequality,

$$\sup_{t\in[0,T]} \left(|\tilde\eta^3|^2_4 + \|\bar\partial^4 \tilde v(t)\|^2_4 \right) \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

According to Proposition 3,

$$\sup_{t\in[0,T]} \|\operatorname{div} \tilde v(t)\|^2_3 \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,,$$

from which it follows that

$$\sup_{t\in[0,T]} \|\bar\partial^4 \operatorname{div} \tilde v(t)\|^2_{H^1(\Omega)'} \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

Hence, the normal trace estimate (10) shows that

$$\sup_{t\in[0,T]} \left(|\tilde\eta^3(t)|^2_4 + |\tilde v^3(t)|^2_{3.5} \right) \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{35}$$

□

Combining Proposition 4 with the curl estimates in Proposition 2 and the divergence estimates in Proposition 3 for $\eta(t)$ and $v(t)$ and using (11) provides the following

Proposition 5. *For $t \in [0, T_\kappa]$,*

$$\|\Lambda_\kappa \tilde\eta(t)\|^2_{4.5} + \|\tilde v(t)\|^2_4 \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{36}$$

Since $\tilde v_t = -\tilde A^T_\kappa D\tilde q$, we also obtain

Proposition 6. *For $t \in [0, T_\kappa]$,*

$$\|\tilde v_t(t)\|^2_{3.5} \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{37}$$

Proposition 7. *For $t \in [0, T_\kappa]$ and $\alpha = 1, 2$,*

$$|\sqrt{\kappa}\tilde v^\alpha_t(t)|^2_{3.5} + |\kappa \tilde v^\alpha(t)|^2_4 \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{38}$$

Proof. Multiplying (16a) by $\tilde\eta^j_\kappa$, we find the identity

$$\tilde v_t \cdot \tilde\eta_\kappa,_k = -\tilde q,_k \ \text{ for } k = 1, 2, 3\,.$$

It follows that

$$\tilde v_t \cdot \bar\partial \eta_\kappa = -\bar\partial \tilde q = 0 \text{ on } \Gamma\,,$$

so that the tangential component (with respect to the moving boundary $\tilde\eta_\kappa(t, \Gamma)$) of $\tilde v_t$ vanishes on Γ. Hence,

$$\bar\partial^3 \tilde v_t \cdot \bar\partial \tilde\eta_\kappa = -\tilde v_t \cdot \bar\partial^4 \tilde\eta_\kappa - 3\bar\partial^2 \tilde v_t \cdot \bar\partial^2 \tilde\eta_\kappa - 3\bar\partial \tilde v_t \cdot \bar\partial^3 \tilde\eta_\kappa\,.$$

Thus,

$$|\bar\partial^3 \tilde v_t \cdot \bar\partial \tilde\eta_\kappa|_0 \le C|\tilde v_t|_{2.5}|\tilde\eta_\kappa|_4 \le C|\tilde\eta_\kappa|_4\,,$$

so that using the fundamental theorem of calculus, together with the fact that $\eta_\kappa,_\alpha(0)$ is proportional to T_α for $\alpha = 1, 2$, we find that

$$|\tilde v_t^\alpha|_3 \le C|\tilde\eta_\kappa|_4 + C\,T\,|\tilde v_t|_3\,, \tag{39}$$

Similarly,

$$|\bar\partial^4 \tilde v_t \cdot \bar\partial \tilde\eta_\kappa|_0 \le C|\tilde v_t|_3|\tilde\eta_\kappa|_5 \le C|\tilde\eta_\kappa|_5\,,$$

from which it follows that

$$|\tilde v_t^\alpha|_4 \le C|\tilde\eta|_5 + C\,T|\tilde v_t|_4\,. \tag{40}$$

Interpolation between (39) and (40) yields

$$|\tilde v_t^\alpha|_{3.5} \le C|\tilde\eta_\kappa|_{4.5} + C\,T|\tilde v_t|_{3.5}\,.$$

Interpolation between the inequalities $|\tilde\eta_\kappa|_4 \le C|\Lambda_\kappa\tilde\eta|_4$ and $|\tilde\eta_\kappa|_5 \le \frac{C}{\kappa}|\Lambda_\kappa\tilde\eta|_4$ shows that $\sqrt{\kappa}|\tilde\eta_\kappa|_{4.5} \le C|\Lambda_\kappa\tilde\eta|_4$. It thus follows from (36) and (37) that

$$\sqrt{\kappa}|\tilde v_t^\alpha(t)|_{3.5} \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

A similar argument shows that

$$\kappa|\tilde v^\alpha(t)|_4 \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

□

Combining the estimate (38) together with the curl estimates in Proposition 2 and divergence estimates in Proposition 3 for $\kappa\tilde v(t)$ and $\sqrt{\kappa}\tilde v_t(t)$ proves the following

Proposition 8. *For $t \in [0, T_\kappa]$ and $\alpha = 1, 2$,*

$$\|\sqrt{\kappa}\tilde v_t(t)\|_4^2 + \|\kappa\tilde v(t)\|_{4.5}^2 \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{41}$$

9. Proof of the Main Theorem.

9.1. **Time of existence and bounds independent of κ and existence of solutions to (3).** Summing the inequalities provided by the above Propositions, we find that

$$\sup_{t\in[0,T]} E_\kappa(t) \le M_0 + C\,T\,P(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

Just as in Section 9 of [2], this provides us with a time of existence T_1 independent of κ and an estimate on $(0, T_1)$ independent of κ of the type:

$$\sup_{t\in[0,T_1]} E_\kappa(t) \le 2M_0\,. \tag{42}$$

In particular, our sequence of solutions $(\tilde\eta, \tilde v)$ satisfy the κ-independent bound (42) on the κ-independent time-interval $(0, T_1)$.

9.2. **The limit as $\kappa \to 0$.** By the κ-independent estimate (42), there exists a subsequence of $\{\tilde v_t, \tilde A_\kappa, D\tilde q\}$ which converges uniformly to (v_t, A, Dq) where $A = [D\eta]^{-1}$, and $\eta = e + \int_0^t v dt'$. Standard arguments show that (η, v) solve (3), and that

$$\sup_{t\in[0,T_1]} E(t) \le P(E(0))\,.$$

9.3. **Uniqueness.** Suppose that on $[0, T_1]$, (η^1, v^1, q^1) and (η^2, v^2, q^2) are both solutions of (3) with initial data $u_0 \in H^5_{\mathrm{div}}(\Omega)$ and $\operatorname{curl} u_0 \in H^{4.5}(\Omega)$, and with both q_1 and q_2 satisfying the stability condition $-\partial q_i/\partial N > 0$ on Γ for $i = 1, 2$.

Setting

$$\mathcal{E}_\eta(t) = 1 + \|\eta(t)\|^2_{5.5} + \|v(t)\|^2_5 \,,$$

by the method of the previous section (with $\kappa = 0$), we infer that both $\mathcal{E}_{\eta^1}(t)$ and $\mathcal{E}_{\eta^2}(t)$ are bounded by a constant $\mathcal{M}_0$ depending on the data u_0 and Γ on a time interval $0 \le t \le T_1$ for T_1 small enough.

Let

$$w := v^1 - v^2, \quad r := q^1 - q^2, \quad \text{and } \xi := \eta^1 - \eta^2 \,.$$

Then (ξ, w, r) satisfies

$$\xi = \int_0^t w \qquad \text{in } \Omega \times (0, T] \,, \tag{43a}$$

$$\partial_t w^i + (a^1)^k_i \, r,_k = (a^2 - a^1)^k_i \, q^2,_k \qquad \text{in } \Omega \times (0, T] \,, \tag{43b}$$

$$(a^1)^j_i w^i,_j = (a^2 - a^1)^j_i {v^2}^i,_j \qquad \text{in } \Omega \times (0, T] \,, \tag{43c}$$

$$r = 0 \qquad \text{on } \Gamma \times (0, T] \,, \tag{43d}$$

$$(\xi, w) = (0, 0) \qquad \text{on } \Omega \times \{t = 0\} \,. \tag{43e}$$

We set

$$E(t) = 1 + \|\xi(t)\|^2_{4.5} + \|w(t)\|^2_4 \,.$$

We will show that $E(t) = 0$, which shows that $w = 0$. We follow the identical analysis as in the previous section, and estimate the new error terms, arising from the difference of two solutions, using the additional space regularity in our assumptions. With (43e), we find that $\sup_{t \in [0,T]} E(t) \le C \, T \, P(\sup_{t \in [0,T]} E(t))$.

9.4. **Optimal regularity for initial data.** We smoothed our initial data u_0 in order to construct solutions to the κ-approximation (16). Having obtained solutions which depend only on $E(0)$, a standard density argument shows that the initial data needs only to satisfy $E(0) < \infty$. All assumptions on from Section 8.3 can now be verified by the fundamental theorem of calculus, and taking T_1 even small if necessary.

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