SIAM J. MATH. ANAL.
Vol. 49, No. 6, pp. 4942–5006

LOCAL WELL-POSEDNESS AND GLOBAL STABILITY OF THE TWO-PHASE STEFAN PROBLEM*

MAHIR HADŽIĆ[†], GUSTAVO NAVARRO[‡], AND STEVE SHKOLLER[‡]

Abstract. The two-phase Stefan problem describes the temperature distribution in a homogeneous medium undergoing a phase transition such as ice melting to water. This is accomplished by solving the heat equation on a time-dependent domain, composed of two regions separated by an a priori unknown moving boundary which is transported by the difference (or jump) of the normal derivatives of the temperature in each phase. We establish local-in-time well-posedness and a global-in-time stability result for arbitrary sufficiently smooth domains and small initial temperatures. To this end, we develop a higher-order energy with natural weights adapted to the problem and combine it with Hopf-type inequalities. This extends the previous work by Hadžić and Shkoller [*Comm. Pure Appl. Math.*, 68 (2015), pp. 689–757; *Philos. Trans. A*, 373 (2015), 20140284] on the one-phase Stefan problem to the setting of two-phase problems, and simplifies the proof significantly.

Key words. Stefan problem, two-phase problem, interface motion, free-boundary problem

AMS subject classifications. 35B35, 85A05, 35Q75

DOI. 10.1137/16M1083207

1. Introduction to the problem.

1.1. Problem formulation and the reference domain. We consider the local and global well-posedness and interface regularity of solutions to the classical *two-phase Stefan problem*, describing the evolving interface, separating a freezing liquid and a melting solid. The temperature of the liquid-solid phase $p^{\pm}(t,x)$ and the *a priori unknown moving interface* $\Gamma(t)$ must satisfy the following system of equations:

$$p_t^{\pm} - \Delta p^{\pm} = 0 \quad \text{in } \Omega^{\pm}(t)\,, \tag{1.1a}$$

$$[\partial_n p]^{\pm} = -V_{\Gamma(t)} \quad \text{on } \Gamma(t)\,, \tag{1.1b}$$

$$p^{+} = p^{-} = 0 \quad \text{on } \Gamma(t)\,, \tag{1.1c}$$

$$p^{\pm}(0,\cdot) = p_0^{\pm}\,,\ \Gamma(0) = \Gamma_0\,, \tag{1.1d}$$

where for each time $t \in [0,T]$, $\Omega^{+}(t)$ and $\Omega^{-}(t)$ denote two evolving open and bounded domains as shown in Figure 1, and $\Gamma(t)$ denotes the moving interface separating $\Omega^{+}(t)$ and $\Omega^{-}(t)$, so that $\Gamma(t) = \overline{\Omega^{-}(t)} \cap \overline{\Omega^{+}(t)}$.

DEFINITION 1.1 (the domains Ω and $\Omega^{\pm}(t)$). *For $d \geq 2$, we denote by $\Omega \subset \mathbb{R}^d$, a fixed, open, and bounded set such that*

$$\Omega = \overline{\Omega^{-}(t)} \cup \Omega^{+}(t)\,,$$

as shown in Figure 1. *We assume that the fixed boundary $\partial\Omega$ is C^{∞}.*

*Received by the editors July 5, 2016; accepted for publication (in revised form) September 6, 2017; published electronically December 12, 2017.
http://www.siam.org/journals/sima/49-6/M108320.html
Funding: The work of the first author was supported by the EPSRC grant EP/N016777/1. The work of the second author was was supported by the National Science Foundation under grant MS-1301380 and by a Royal Society Wolfson Merit Award.
[†]Department of Mathematics, King's College London, London WC2R 2LS, UK (mahir.hadzic@kcl.ac.uk).
[‡]Department of Mathematics, UC Davis, Davis, CA 95616 (aramis.entreri@gmail.com, shkoller@math.ucdavis.edu).

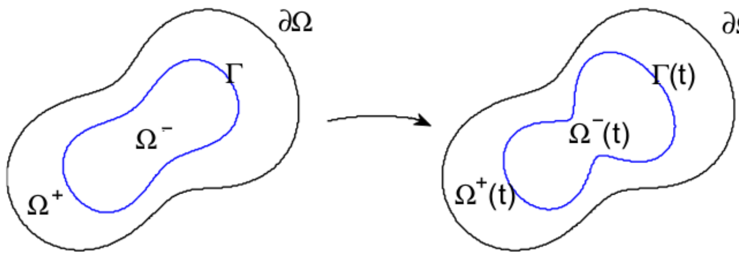

FIG. 1. *The two-phase Stefan problem. Displayed on the left side of the figure are the reference domains $\Omega^{\pm}$ and reference interface Γ. The time-dependent domains $\Omega^{\pm}(t)$ and the moving interface $\Gamma(t)$ are shown on the right side of the figure. The domain $\Omega^{-}(t)$ denotes the solid phase, while the domain $\Omega^{+}(t)$ denotes the liquid phase.*

Equation (1.1a) models temperature diffusion in the bulk $\Omega^{\pm}(t)$, while the interface jump condition (1.1b) states that the jump in temperature gradients evolves the interface; that is, $[\partial_n p]^{\pm} := \partial_n p^{+} - \partial_n p^{-} = \nabla(p^{+} - p^{-}) \cdot n$ on $\Gamma(t)$, where $n(\cdot, t)$ denotes the outward unit normal on $\Gamma(t)$ (pointing into $\Omega^{+}(t)$), and $V_{\Gamma(t)}$ denotes the speed or normal velocity of the interface $\Gamma(t)$. Note that the freezing of the liquid and the melting of the solid occur at a constant temperature $p = 0$ as seen from the Dirichlet boundary condition (1.1c). Initial conditions are prescribed in (1.1d): the initial interface Γ_0 and the initial temperature functions p_0^- and p_0^+ are specified.

Herein, we shall, for simplicity, consider the two-dimensional Stefan problem $d = 2$, although all of our methods easily extend in a straightforward manner to the case that $d \geq 3$.[1] No convexity assumptions are made on the initial interface Γ_0, but we shall assume that Γ_0 is diffeomorphic to the unit circle $\mathbb{S}^1$.

Remark 1.2. Surface tension effects can be included as well by replacing (1.1c) with

$$p^{\pm} = \gamma H_{\Gamma(t)} \quad \text{on} \quad \Gamma(t), \tag{1.2}$$

where $\gamma \geq 0$ is the surface tension parameter and $H_{\Gamma(t)}$ is the mean curvature of $\Gamma(t)$. Herein, we shall study the case that $\gamma = 0$. We shall also consider the two-dimensional problem $d = 2$, although all of our results extend in a straightforward manner to the case that $d \geq 3$.

1.2. Specifying a smooth reference interface Γ and reference domains $\Omega^{\pm}$. In order to describe our initial interface Γ_0, we employ an H^6-class parametrization $z_0 : \mathbb{S}^1 \to \Gamma_0$, where $\mathbb{S}^1$ is identified with the period $[0, 2\pi]$. To construct a smooth reference interface, we consider a C^∞ nearby interface Γ_σ which is constructed by smoothing $z_0(\theta)$, $\theta \in \mathbb{S}^1$, using a standard mollification approach. For $\sigma > 0$ taken

[1] As we shall explain, our method relies on having bounds for $D^2 q$ in $L^\infty_{t,x}$ which follow from the Sobolev embedding theorem and the fact that we shall require $q \in L^\infty_t H^s_x$ for $s > d/2 + 2$. In the case that $d = 2$, we require $s > 3$. Solutions to the heat equation are naturally studied in the L^2 framework in integer Sobolev spaces, so for dimension d, we shall require $q \in L^\infty_t H^k_x$, where k is the smallest integer greater than $d/2 + 2$. As such, our framework will work in any space dimension, merely by making such a modification in the definition of the Sobolev norms used for the analysis. Furthermore, our methodology for smoothing the initial data is independent of the dimension d, and our energy method is based on the use of tangential derivatives to the fixed reference boundary, and hence it also does not require any modification for higher dimension, other than the use of $d - 1$ partial derivatives in the tangential derivative operator.

sufficiently small, we define the convolution operator Λ_σ as follows:

$$\Lambda_\sigma z_0(\theta) = \int_{\mathbb{R}} \rho_\sigma(\theta - \vartheta) z_0(\vartheta) d\vartheta \,, \tag{1.3}$$

where $\rho_\sigma(\theta) = \sigma^{-1}\rho(\theta/\sigma)$, and ρ is the standard mollifier on $\mathbb{R}$, given by

$$\rho(x) = \begin{cases} Ce^{\frac{-1}{1-|x|^2}} \,, & |x| \leq 1 \,, \\ 0 \,, & |x| > 1 \,. \end{cases}$$

Definition 1.3 (C^∞ reference interface and domains). *For $\sigma > 0$ taken sufficiently small and fixed, we set*

$$z_0^\sigma(\theta) = \Lambda_\sigma z_0(\theta),$$

and we define the nearby C^∞ curve

$$\Gamma := z_\sigma\left(\mathbb{S}^1\right) \,.$$

The curve Γ is the reference interface, and define the reference domain Ω^- to be the open set enclosed by Γ, and we set $\Omega^- = \Omega - \overline{\Omega^+}$.

The initial interface Γ_0 is in the normal bundle of Γ; hence there exists a signed height function $h_0 \in H^6(\Gamma_\sigma)$ such that

$$z_0(\theta) = z_0^\sigma(\theta) + h_0(z_\sigma(\theta)) N_\sigma(z_0^\sigma(\theta)), \;\; z_0^\sigma(\theta) \in \Gamma_\sigma,$$

where $N_\sigma(z_0^\sigma(\theta))$ is the unit normal vector to Γ (pointing into Ω^+) at the point $z_0^\sigma(\theta)$. It follows that the initial height function h_0 has amplitude of order σ, and that $h_0 \to 0$ as $\sigma \to 0$.

As time evolves, if the interface $\Gamma(t)$ stays in the normal bundle of Γ_σ, then for each time t, we can define the corresponding signed height function $h(t, z_\sigma)$ as follows:

$$\Gamma(t) := \{y \mid y = z_0^\sigma + h(t, z_0^\sigma) N_\sigma(z_0^\sigma), \;\; z_0^\sigma \in \Gamma_\sigma\} \tag{1.4}$$

with the initial condition

$$h(0, z_\sigma) = h_0(z_\sigma).$$

We note, that while it is not essential, it is convenient to use the C^∞ curve Γ_σ as the reference interface. This allows us to use the normal bundle of Γ_σ with a C^∞ unit normal vector field. If we had instead worked with the initial interface Γ_0 as the reference interface, we would have been forced to use a different (from the normal) transverse vector field to define the height function due to the limited regularity of Γ_0 and the fact that the regularity of the normal would have a one derivative loss.

For notational clarity, we shall henceforth drop the explicit dependence on σ in our parametrization, and write $z_0(\theta)$ for $z_0^\sigma(\theta)$.

1.3. Notation. We denote the identity map $x \mapsto x$ by e and the identity (2×2)-matrix $(\delta_{ij})_{i,j=1,2}$ by Id. A constant C is a generic constant and may change from line to line, and we write $X \lesssim Y$ to denote $X \leq CY$. Similarly, we use the notation $P(\cdot)$ to denote a generic polynomial of the form $P(x) = Cx^p$ with $p \geq 1$, and the constants C and p may also change from line to line.

We use $\nabla = (\partial_{x_1}, \partial_{x_2})$ to denote the gradient operator. For $i = 1, 2$, we shall abbreviate partial differentiation of a function f as $f_{,k} = \frac{\partial f}{\partial x^k}$, and for time-differentiation we let $F_t := \partial_t F$. We shall use the Einstein summation convention, where repeated indices are summed from 1 to 2. Furthermore, given a function $F(t,x)$, we shall often write $F(t)$ instead of $F(t,\cdot)$ and $F(0)$ instead of $F(0,x)$.

To deal with lower-order terms in energy estimates, we shall use the abbreviation l.o.t. for spacetime integrals in which the integrand has sufficiently few derivatives so as to be bounded by a simple application of Hölder's inequality; this is made precise in (3.51).

It will be useful for some estimates to set the following notation for functions evaluated at time $t=0$:

$$\Psi_0^{\pm} := \Psi^{\pm}(0),\ \ {}^{\kappa}\Psi_0{}^{\pm} := {}^{\kappa}\Psi^{\pm}(0),$$
$$A_0{}^{\pm} := A^{\pm}(0) = \left[\nabla\Psi_0^{\pm}\right]^{-1}, {}^{\kappa}\!A_0{}^{\pm} := {}^{\kappa}\!A^{\pm}(0) = \left[\nabla {}^{\kappa}\Psi_0^{\pm}\right]^{-1},$$

and we can define as well the respective differential operators

$$\Delta_{\Psi_0^{\pm}} f := A_0{}^{\pm}{}^i_j \left(A_0{}^k_j f,_k\right),_i\,,$$
$$\Delta_{{}^{\kappa}\Psi_0{}^{\pm}} f := {}^{\kappa}\!A_0{}^{\pm}{}^i_j \left({}^{\kappa}\!A_0{}^{\pm}{}^k_j f,_k\right),_i\,,$$
$$(\nabla_{\Psi_0^{\pm}} f)^j := A_0{}^{\pm}{}^i_j f,_i\,,$$
$$(\nabla_{{}^{\kappa}\Psi_0{}^{\pm}} f)^j := {}^{\kappa}\!A_0{}^{\pm}{}^i_j f,_i\,,$$

which are the generalizations of the Laplacian and the gradient, respectively, for a scalar function f over the regions $\Omega^{\pm}$. For a vector field F, we define the matrices

$$[\nabla_{\Psi_0^{\pm}} F]^i_j := A_0{}^{\pm}{}^k_j F^i,_k\,,$$
$$[\nabla_{{}^{\kappa}\Psi_0{}^{\pm}} F]^i_j := {}^{\kappa}\!A_0{}^{\pm}{}^k_j F^i,_k\,.$$

1.4. Sobolev norms. For any $s \ge 0$ and given functions $F^{\pm} : \Omega^{\pm} \to \mathbb{R}$, $\varphi : \Gamma \to \mathbb{R}$, we denote the norms in the standard Sobolev spaces $H^s(\Omega^{\pm})$, $H^s(\Gamma)$ as

$$\left\|F^{\pm}\right\|_s := \left\|F^{\pm}\right\|_{H^s(\Omega^{\pm})},\quad |\varphi|_s := \|\varphi\|_{H^s(\Gamma)},$$

where $\|F^{\pm}\|_s$ is either $\|F^+\|_{H^s(\Omega^+)}$ or $\|F^-\|_{H^s(\Omega^-)}$ depending on which domain we are considering.

If $F : [0,T]\times\Omega \to \mathbb{R}$, $\varphi : [0,T]\times\Gamma \to \mathbb{R}$ are given time-dependent functions, then

$$\|F\|_{L^2_t H^s} := \left(\int_0^t \|f(s)\|^2_{H^s(\Omega)} ds\right)^{1/2},$$
$$\|F\|_{L^\infty_t H^s} := \sup_{0\le s\le t} \|F(s)\|_{H^s(\Omega)},$$
$$|\varphi|_{L^2_t H^s} := \left(\int_0^t |\varphi(s)|^2_{H^s(\Gamma)} ds\right)^{1/2},$$
$$|\varphi|_{L^\infty_t H^s} := \sup_{0\le s\le t} |\varphi(s)|_{H^s(\Gamma)}.$$

For given weight functions $W^{\pm} : \Omega^{\pm} \to \mathbb{R}$ such that $W^{\pm} > 0$, we define the *weighted L^2 norm* as

$$\left\|F^{\pm}\right\|^2_{L^2,W^{\pm}} := \int_{\Omega^{\pm}} \left|F^{\pm}(s,x)\right|^2 W^{\pm}\, dx. \tag{1.5}$$

1.5. Tangential derivatives. For a given $0 < \nu \ll 1$, we define a smooth cutoff function $\mu : \bar{\Omega} \to \mathbb{R}_+$ satisfying

$$\mu(x) \equiv 1 \text{ if } \operatorname{dist}(x, \Gamma \cup \partial\Omega) \le \nu \text{ and } \mu(x) \equiv 0 \text{ if } \operatorname{dist}(x, \Gamma \cup \partial\Omega) \ge 2\nu. \tag{1.6}$$

This will allow us to localize the analysis to a neighborhood of the interface, wherein we define the *tangential derivative* as $\bar{\partial} f = \nabla f \cdot \tau$, where τ is the smooth extension of the tangent vector to Γ into that neighborhood. In the case of functions defined solely on Γ, we define the tangential derivative naturally as $\bar{\partial} g = \frac{1}{\|z_0'\|} \frac{d}{d\theta} g(z_0(\theta))$, where $z_0(\theta)$ is the parametrization of Γ described in section 1.2.

1.6. Steady states. Let $\bar{\Gamma}$ be any given closed C^1-curve separating Ω into two connected components Ω^+ and Ω^-. Then the triple $(u^+, u^-, \Gamma) \equiv (0, 0, \bar{\Gamma})$ constitutes a steady state solution to (1.1). The space of steady states is therefore infinite dimensional and NOT parametrized by finitely many parameters. The main goal of this article is to understand the nonlinear stability of these steady states.

1.7. Pulling-back to the reference domains $\Omega^\pm$. To develop a well-posedness theory for (1.1), we pull-back the equations to the reference domains $\Omega^\pm$. It is convenient to construct harmonic diffeomorphisms. Hence, for each $t \in [0, T]$, we define the diffeomorphisms $\Psi^\pm(\cdot, t) : \Omega^\pm \to \Omega^\pm(t)$ as the solution of

$$\Delta \Psi^\pm = 0 \quad \text{in } \Omega^\pm, \tag{1.7a}$$

$$\Psi^\pm(t, x) = x + h(t, x) N(x) \quad \text{on } \Gamma, \tag{1.7b}$$

$$\Psi^+ = \mathrm{e} \quad \text{on } \partial\Omega, \tag{1.7c}$$

where e is the identity map on $\partial\Omega$. Elliptic estimates show that for $k \ge 1$,

$$\left\| \Psi^\pm - e \right\|_{6.5} \le C(|h|_6)\,. \tag{1.8}$$

When $|h|_6 \le \epsilon \ll 1$, the inverse function theorem, together with the Sobolev embedding theorem, show that $\Psi^\pm(\cdot, t) : \Omega^\pm \to \Omega^\pm(t)$ are $H^{6.5}$-class diffeomorphisms. Of course, we could have used any Sobolev space H^s, in place of $H^{6.5}$, but our analysis will make use of the latter.

We next introduce our physical variables set on the fixed reference domains $\Omega^\pm$. We set

$$q^\pm := p^\pm(t, \cdot) \circ \Psi^\pm, \tag{1.9a}$$

$$v^\pm := \nabla p^\pm(t, \cdot) \circ \Psi^\pm, \tag{1.9b}$$

$$A^\pm := \left(\nabla \Psi^\pm\right)^{-1}, \tag{1.9c}$$

$$w^\pm := \Psi^\pm_t. \tag{1.9d}$$

In the parlance of fluid dynamics, the mappings $\Psi^\pm$ are often called arbitrary Lagrangian Eulerian (ALE) coordinates. The Laplace operator in ALE coordinates is given by

$$\Delta_{\Psi^\pm} := A^{\pm}{}^i_j \partial_i \left(A^{\pm}{}^k_j \partial_k \right) .$$

Therefore, on the fixed reference domains $\Omega^\pm$, the Stefan problem (1.1) has the following form:

$$
\begin{aligned}
&\text{(1.10a)} && q^\pm_t - \Delta_{\Psi^\pm} q^\pm = -v^\pm \cdot w^\pm && \text{in } \Omega^\pm,\\
&\text{(1.10b)} && v^\pm + A^{\pm T}\nabla q^\pm = 0 && \text{in } \Omega^\pm,\\
&\text{(1.10c)} && q^\pm(t,x) = 0 && \text{on } \Gamma,\\
&\text{(1.10d)} && h_t = [v\cdot\tilde n]^+_- && \text{on } \Gamma,\\
&\text{(1.10e)} && v^+\cdot \mathbf{N}^+ = 0 && \text{on } \partial\Omega,\\
&\text{(1.10f)} && q^\pm(0,x) = q_0^\pm(x) && \text{on } \{t=0\}\times\Omega^\pm,\\
&\text{(1.10g)} && h = h_0 && \text{on } \{t=0\}\times\Gamma.
\end{aligned}
$$

The motion of the interace $\Gamma(t)$ is given by (1.10d), which is an equivalent form of (1.1b), since the speed $V_\Gamma(t)$ of $\Gamma(t)$ is equal to $\Psi_t\cdot n = h_t N\cdot n$, where n is the outward normal vector to $\Omega^-(t)$ to be defined below, and $\tilde n := \frac{n}{n\cdot N}$. Observe that the matrices $A^\pm$ depend on $\Psi^\pm$, and the $\Psi^\pm$ are extensions of $e + hN$ obtained from (1.7).

Notice that using the description of the reference interface Γ as a curve $z(\theta)$, the moving interface $\Gamma(t)$ is described as $y(\theta) = z(\theta) + h(t, z(\theta))N(z(\theta))$, and so the normal vector n is given by

$$
n(t, y(\theta)) = \frac{-\bar\partial h\tau + (1 + H(\theta)h(t, z(\theta)))N}{\sqrt{(\bar\partial h)^2 + (1 + H(\theta)h)^2}} \tag{1.11}
$$

where $H(\theta) := \frac{z_2' z_1'' - z_1' z_2''}{\|z'\|^3}$ is the signed curvature of Γ at the point $z(\theta)$ and τ is defined in section 1.3.

1.8. Higher-order norm used for our analysis.

1.8.1. Local well-posedness theory. We will develop the local-in-time well-posedness theory with respect to the following norm:

$$
\mathcal{S}(t) := \underbrace{\mathcal{E}^+(t) + \mathcal{E}^-(t) + \mathcal{E}^\Gamma_{\text{loc}}(t)}_{L^\infty\text{-in time control}} + \underbrace{\int_0^t \left(\mathcal{D}^+(s) + \mathcal{D}^-(s) + \mathcal{D}^\Gamma_{\text{loc}}(s)\right) ds}_{L^2\text{-in time control}}, \tag{1.12}
$$

where

$$
\begin{aligned}
\mathcal{E}^\pm(t) &:= \sum_{l=0}^{3} \left\|\partial_t^l q^\pm\right\|^2_{L^\infty_t H^{6-2l}(\Omega^\pm)} + \left\|\bar\partial^{5-2l}\partial_t^l v^\pm\right\|^2_{L^\infty_t L^2(\Omega^\pm)},\\
\mathcal{D}^\pm(t) &:= \sum_{l=0}^{3} \left\|\partial_t^l q^\pm(t)\right\|^2_{H^{6.5-2l}(\Omega^\pm)} + \left\|\bar\partial^{6-2l}\partial_t^l v^\pm(t)\right\|^2_{L^2(\Omega^\pm)},\\
\mathcal{E}^\Gamma_{\text{loc}}(t) &:= \sum_{l=0}^{3} \sup_{0\le s\le t} \left|\partial_t^l h(s)\right|^2_{H^{6-2l}(\Gamma)},\\
\mathcal{D}^\Gamma_{\text{loc}}(t) &:= \sum_{l=0}^{2} \left|\partial_t^{l+1} h\right|^2_{H^{5-2l}(\Gamma)}.
\end{aligned}
$$

1.8.2. Global well-posedness and nonlinear stability theory. We will develop the global-in-time well-posedness and our nonlinear stability theory with respect to the following norm:

(1.13)
$$S(t) := \underbrace{\mathcal{E}^+(t) + \mathcal{E}^-(t) + \mathcal{E}^\Gamma(t)}_{L^\infty\text{-in-time control}} + \underbrace{\int_0^t \left(\mathcal{D}^+(s) + \mathcal{D}^-(s) + \mathcal{D}^\Gamma(s)\right) ds}_{L^2\text{-in-time control}} + \underbrace{E^+_\beta(t) + E^-_\beta(t)}_{\text{exponential-decay-in-time}} ,$$

where

$$\mathcal{E}^\pm(t) := \sum_{l=0}^{3} \left\|\partial_t^l q^\pm\right\|^2_{L^\infty_t H^{6-2l}(\Omega^\pm)} + \left\|\bar\partial^{5-2l}\partial_t^l v^\pm\right\|^2_{L^\infty_t L^2(\Omega^\pm)} ,$$
$$\mathcal{D}^\pm(t) := \sum_{l=0}^{3} \left\|\partial_t^l q^\pm(t)\right\|^2_{H^{6.5-2l}(\Omega^\pm)} + \left\|\bar\partial^{6-2l}\partial_t^l v^\pm(t)\right\|^2_{L^2(\Omega^\pm)} ,$$
$$\mathcal{E}^\Gamma(t) := \sum_{l=0}^{3} \sup_{0\le s\le t} e^{(-\lambda_1+\eta)s} \left|\partial_t^l h(s)\right|^2_{H^{6-2l}(\Gamma)} ,$$
$$\mathcal{D}^\Gamma(t) := e^{(-\lambda_1+\eta)t} \sum_{l=0}^{2} \left|\partial_t^{l+1} h\right|^2_{H^{5-2l}(\Gamma)} ,$$
$$E^\pm_\beta(t) := e^{\beta^\pm t} \sum_{l=0}^{2} \left\|\partial_t^l q^\pm(t)\right\|^2_{H^{4-2l}(\Omega^\pm)} ,$$

where $\lambda_1 = \min\{\lambda_1^+, \lambda_1^-\}$ is the smaller of the two first eigenvalues $\lambda_1^\pm$ of the *Dirichlet–Laplacian* on the reference domains $\Omega^\pm$, $\eta > 0$ is a small constant relative to λ_1 to be fixed later, and

$$\beta^\pm = 2\lambda_1^\pm - \eta\,. \tag{1.14}$$

Remark 1.4. The definition of our higher-order energy function requires the definition of the terms $\partial_t^l q$ and $\partial_t^l h$ at time $t = 0$. These are computed using the time-differentiated version of (1.10a) at time $t = 0$. For example,

$$q_1^\pm := q_t^\pm(0) = \Delta_{\Psi^\pm(0)} q_0^\pm + A^\pm(0)^\top \nabla q_0^\pm \cdot \Psi_t^\pm(0). \tag{1.15}$$

The other time derivatives follow the same procedure. The terms $\partial_t^l h|_{t=0}$ follow similarly by means of taking time derivatives of the evolution equation (1.10d) and restricting it at time $t = 0$. We define the functions

$$g_1 := h_t(0) = [v(0)\cdot \tilde n(0)]^+_-, \tag{1.16a}$$
$$g_2 := h_{tt}(0) = [v_t(0)\cdot \tilde n(0)]^+_- + [v(0)\cdot \tilde n_t(0)]^+_-, \tag{1.16b}$$
$$g_3 := h_{ttt}(0) = [v_{tt}(0)\cdot \tilde n(0)]^+_- + 2[v_t(0)\cdot \tilde n_t(0)]^+_- + [v(0)\cdot \tilde n_{tt}(0)]^+_-, \tag{1.16c}$$

where $v(0), v_t(0), v_{tt}(0)$ are computed from (1.10b) by taking time derivatives and restricting to $t = 0$. Notice that they depend only on h_0 and $q_0^\pm$. The derivatives of

$\tilde{n}(0)$ are given by

$$\tilde{n}(0) := \left(N - \tau \frac{\bar{\partial} h_0}{(1+Hh_0)} \right),$$
$$\tilde{n}_t(0) := \tau \left(\frac{\bar{\partial} g_1}{(1+Hh_0)} - \frac{\bar{\partial} h_0 H g_1}{(1+Hh_0)^2} \right),$$
$$\tilde{n}_{tt}(0) := \tau \left(\frac{\bar{\partial} g_2}{(1+Hh_0)} + \frac{\bar{\partial} g_1 H g_1}{(1+Hh_0)^2} - \frac{(\bar{\partial} g_1 H g_1 + \bar{\partial} h_0 H g_2)}{(1+Hh_0)^2} + \frac{2\bar{\partial} h_0 H^2 (g_1)^2}{(1+Hh_0)^3} \right).$$

1.9. Compatibility conditions. Since we study the twice time-differentiated problem, in order to ensure the continuity of the solution in time, we have to impose certain compatibility conditions. Since $q_t^\pm|_\Gamma = q_{tt}^\pm|_\Gamma \equiv 0$, by restricting time derivatives of (1.15) to Γ at time $t=0$, we obtain that $q_0^\pm$ must satisfy

$$\Delta_{\Psi_0^\pm} q_0^\pm = \nabla_{\Psi_0} q_0^\pm \cdot N g_1 \text{ on } \Gamma, \tag{1.17a}$$

$$\begin{aligned} -\Delta^2_{\Psi_0^\pm} q_0^\pm = &- \left(A_0{}^\pm\right)^i_l \Psi_t^\pm(0)^l,_s \left(A_0{}^\pm\right)^s_j \left(\left(A_0{}^\pm\right)^k_j q_0^\pm,_k \right),_i \\ &- \left(A_0{}^\pm\right)^i_j \left(\left(A_0{}^\pm\right)^k_l \Psi_t^\pm(0)^l,_s \left(A_0{}^\pm\right)^s_j q_0^\pm,_k \right),_i \\ &- \left(A_0{}^\pm\right)^i_k \Psi_t^\pm(0)^k,_l \left(A_0{}^\pm\right)^l_j q_0^\pm,_i N^j g_1 \\ &+ \nabla_{\Psi_0^\pm} \left(\Delta_{\Psi_0^\pm} q_0 + \nabla_{\Psi_0^\pm} q_0^\pm \cdot \Psi_t^\pm(0) \right) \cdot N g_1 \\ &+ \nabla_{\Psi_0^\pm} q_0^\pm \cdot N g_2 + \Delta_{\Psi_0^\pm} \left(\nabla_{\Psi_0^\pm} q_0 \cdot \Psi_t^\pm(0) \right), \end{aligned} \tag{1.17b}$$

and over $\partial\Omega$,

$$\nabla_{\Psi_0^+} q_0^+ \cdot \mathbf{N}^+ = 0, \tag{1.17c}$$

$$\begin{aligned} \nabla_{\Psi_0^+} \left(\Delta_{\Psi_0^+} q_0^+ \right) \cdot \mathbf{N}^+ = &-\nabla_{\Psi_0^+} \left(\nabla_{\Psi_0^+} q_0^+ \cdot \Psi_t^+(0) \right) \cdot \mathbf{N}^+ \\ &+ \left(A_0{}^+\right)^i_l (\Psi^+)^l,_t (0),_s \left(A_0{}^+\right)^s_j q_0^+,_i \mathbf{N}^{+j}. \end{aligned} \tag{1.17d}$$

Remark 1.5. Unlike the analysis of [31], herein, the matrix $A^\pm(0) \neq \mathrm{Id}$, but it is nevertheless a very small perturbation of the identity matrix. We have the following estimate:

$$\left\| A^\pm(0) - \mathrm{Id} \right\|_s \lesssim |h_0|_{s-0.5} \lesssim \sigma,$$

which follows from the boundary condition (1.7b) restricted at time $t=0$. Recall that h_0 is defined on the smooth reference curve Γ, and the graph of h_0 defines the initial interface Γ_0 in the normal bundle over Γ.

1.10. Initial data satisfying the compatibility conditions. We shall now provide examples of initial data which satisfy the compatibility conditions. If the initial data $h_0 \equiv 0$ and $\Gamma = \mathbb{S}^1$, then (1.17a) takes the form

$$\Delta q_0^\pm = \partial_N q_0^\pm [\partial_N q_0]_+^- \text{ on } \mathbb{S}^1, \tag{1.18}$$

and (1.17b) reduces to

$$\begin{aligned} -\Delta^2 q_0^\pm = &- 2\Psi_t^\pm(0)^i,_j q_{0,ij}^\pm - \Psi_t^\pm(0)^i,_j q_{0,i}^\pm N^j \\ &+ g_1 \nabla \left(\Delta q_0^\pm + g_1 \nabla q_0^\pm \cdot N \right) \cdot N + g_2 \nabla q_0^\pm \cdot N + \Delta \left(g_1 \nabla q_0^\pm \cdot N \right) \text{ on } \mathbb{S}^1, \end{aligned} \tag{1.19}$$

where g_1, g_2 are defined in (1.16a) and (1.16b). To find functions q_0^+ and q_0^- that satisfy conditions (1.18) and (1.19), it is sufficient to prescribe the behavior of $q_0^\pm$ in a small neighborhood of the interface $\mathbb{S}^1$. Locally, near the set $r = 1$, we choose

$$q_0^\pm = \alpha^\pm (r-1) + \beta^\pm (r-1)^3 + O\left(|r-1|^5\right), \quad \alpha^\pm, \beta^\pm > 0, \tag{1.20}$$

so that q_0^+ and $-q_0^-$ are both positive in Ω^+ and Ω^-, respectively. Next we choose α^+ and α^- such that

$$\alpha^- = \alpha^+ + 1\,, \tag{1.21}$$

and so (1.18) is satisfied. With these assumptions, it is straightforward to check that $\Psi_t^\pm(0,x) = x$ and therefore $D\Psi_t^\pm(0)_j^k = \delta_j^k$, $k, j = 1, 2$; moreover $g_1 = 1$ and $g_2 = -1$. Then, the condition (1.19) reduces to a simple identity for the polynomial $q_0^\pm$ locally around $r = 1$, which is satisfied if $\beta^\pm = \frac{1}{6}\alpha^\pm$.

We have just provided one of many nontrivial examples of initial data which satisfy the compatibility conditions. For compatibility conditions of higher order, the polynomial $q_0^\pm$ must employ a higher-order expansion in $(r-1)$ for $|r-1| \ll 1$; see (1.20).

Yet another family of initial data is provided by functions $q_0^\pm$ that, locally about $r = 1$, take the form

$$q_0^\pm = e^{p^\pm (r-1)} - 1, \tag{1.22}$$

where $p^\pm$ have the Taylor expansion

$$p^\pm = \alpha_1^\pm (r-1) + \alpha_2^\pm (r-1)^2 + \alpha_3^\pm (r-1)^3 + \alpha_4^\pm (r-1)^4 + O\left(|r-1|^5\right). \tag{1.23}$$

A direct check shows that the first compatibility condition reduces to any choice of coefficients $\alpha_i^\pm > 0$, $i = 1, 2$, satisfying

$$\left(\alpha_1^\pm\right)^2 + \alpha_1^\pm + 2\alpha_2^\pm = \left[\alpha_1^- - \alpha_1^+\right] \alpha_1^\pm.$$

The second compatibility condition (1.19) then simplifies to a polynomial equation of the form

$$Q\left(\alpha_1^\pm, \alpha_2^\pm, \alpha_2^\pm, \alpha_4^\pm\right) = 0,$$

wherein the polynomial Q depends linearly on $\alpha_4^\pm$. This allows us to consistently solve for $\alpha_4^\pm$ and therefore find a family of initial data satisfying all compatibility conditions. The choice (1.22) is somewhat advantageous to (1.20), as the restrictive assumption (1.21) can be avoided. In particular, for any given $\epsilon > 0$ we can choose, for example,

$$\alpha_1^- = 2\epsilon,\ \alpha_1^+ = \epsilon,\ \alpha_2^- = 2\epsilon^2,\ \alpha_2^+ = \frac{1}{2}\epsilon^2. \tag{1.24}$$

With this choice, the first compatibility condition is satisfied. To satisfy the second compatibility condition, both $\alpha_3^\pm$ and $\alpha_4^\pm$ are then chosen to be at least $O(\epsilon)$. This example shows that the smallness of the initial data can be enforced, together with the compatibility conditions, and it is clear from the construction that such a choice is stable under small perturbations.

1.11. Nondegeneracy or Rayleigh–Taylor stability condition. To ensure the local-in-time well-posedness of the Stefan problem, we impose the well-known nondegeneracy condition that has been used by Meirmanov [43], Prüss, Saal, and Simonett [44], Hadžić and Shkoller [31, 32], and other authors; specifically, we make the following requirement on the initial temperature function:

$$\partial_N q_0^\pm \geq \delta > 0, \text{ uniformly on } \Gamma \tag{1.25}$$

for some constant $\delta > 0$. Condition (1.25) is the classical Rayleigh–Taylor sign condition that naturally appears in the context of many free boundary and moving interface problems in fluid dynamics as a stability condition for well-posedness [16], wherein the function q_0 is the initial pressure function rather than temperature function. For the one-phase water waves equations with an interface that does not self-intersect, (1.25) was shown to always hold in [51] using the Hopf lemma. For the incompressible Euler equations with vorticity, it is essential to chose an initial velocity profile that provides a pressure function satisfying (1.25), for if (1.25) holds, then the free boundary incompressible Euler equations are well-posed (see, for example, [16] and the references therein), while if (1.25) does not hold, then the problem is ill-posed, as shown in [23]. In the setting of compressible flows with the so-called physical vacuum boundary, condition (1.25) is equivalent to the sound speed of the gas vanishing as the square root of the distance function to the vacuum boundary, and is also required for well-posedness as shown in [18, 19]. This condition also appears in both the Hele-Shaw and Muskat problems (see, for example, [10] and [11]). In all of these problems, the natural control of the second-fundamental form of the moving interface $\Gamma(t)$ can be obtained in a somewhat similar fashion (at least for short time), and we will discuss this further in section 2.2.

To ensure the global-in-time nonlinear stability of the steady state solutions of the Stefan problem (1.1) described in section 1.6, we shall demand natural sign assumptions on the initial temperatures in the liquid and the solid phases, respectively,

$$q_0^+ > 0 \;\text{ in }\; \Omega^+ \quad \text{and} \quad q_0^- < 0 \;\text{ in }\; \Omega^-. \tag{1.26}$$

In addition to this, given some universal constant $C^* > 0$, we consider initial temperature distributions $q_0^\pm$ satisfying

$$k\left(q_0^\pm\right) := \frac{\inf_{x\in\Gamma} \partial_N q_0^\pm(x)}{\int_{\Omega^\pm} q_0^\pm \varphi_1^\pm dx} \geq C^*, \tag{1.27}$$

where $\varphi_1^\pm \geq 0$ denotes the first eigenfunction of the Dirichlet–Laplacian on $\Omega^\pm$. The quantity $k(q_0)$ is dimensionless and it is invariant under scaling, i.e., $k(\varepsilon q_0) = k(q_0)$, $\varepsilon \neq 0$. We also denote

$$c_1^\pm := \int_{\Omega^\pm} q_0^\pm \varphi_1^\pm dx, \tag{1.28}$$

i.e., $c_1^\pm$ is the projection of $q_0^\pm$ onto the first eigenfunction of the Dirichlet–Laplacian. Observe that (1.27) implies (1.25). Notice that under the sign assumptions (1.26) the parabolic Hopf lemma implies that $\partial_N q^\pm(t) > 0$ for some period of time $0 < t < T$; however, as $t \to 0^+$, there is no uniformity on this lower bound. Condition (1.27) is designed to ensure a uniform lower bound on $\partial_N q^\pm(t)$ as $t \to 0$ in a certain quantified manner, involving the quantity $c_1^\pm$. This will be crucial in obtaining a sharp lower bound for $\partial_N q^\pm(t)$, $t > 0$, which is used in the proof of the global-in-time stability.

2. Main results. Our first result is a local well-posedness theorem in Sobolev spaces.

THEOREM 2.1 (local well-posedness). *With Ω and $\Omega^\pm$ as given in Definition* 1.1, *and with $(q_0^\pm, h_0)$ satisfying the initial data compatibility conditions* (1.17a)–(1.17d), *the Rayleigh–Taylor sign condition* (1.25), *and*

$$\mathcal{S}(0) < \infty\,,$$

where $\mathcal{S}(t)$ is defined in (1.12), *there exists a time $T > 0$ and a universal constant $C > 0$, such that there exists a unique solution to the two-phase Stefan problem* (1.10), *the map $t \to S(t)$ is continuous on $[0, T]$, and the solution verifies the following estimate:*

$$\mathcal{S}(t) \le C\mathcal{S}(0) \quad \textit{for all} \quad t \in [0, T]\,.$$

Having established short-time existence for arbitrarily large data, we next consider the nonlinear stability of equilibria. To do so, we introduce the following dimensionless quantity:

$$K(q_0^\pm) := \frac{\|q_0^\pm\|_4}{\|q_0^\pm\|_0}, \tag{2.1}$$

which is invariant under the rescaling $q_0^\pm \mapsto \varepsilon q_0^\pm$. Note that $K > 1$, since, in the standard definition of the norm in $H^4(\Omega^\pm)$, the $L^2(\Omega^\pm)$ norm is contained in the sum.

THEOREM 2.2 (global existence, nonlinear stability, and decay). *For $K > 1$, suppose that the initial data $(q_0^\pm, h_0)$ satisfy the conditions* (1.17a)–(1.17d), (1.26), *and* (1.27), *as well as the condition*

$$\max\left\{K\left(q_0^+\right), K(q_0^-)\right\} \le K\,,$$

where $K(q_0^\pm)$ is defined in (2.1). *Then, there exists an $\epsilon_0 > 0$ and a monotonically increasing function $F : (1, \infty) \to \mathbb{R}_+$ which is independent of ϵ_0 and K, such that if*

$$S(0) < \frac{\epsilon_0^2}{F(K)}\,, \tag{2.2}$$

then there exist a unique global-in-time solution $(q^\pm, h)$ to problem (1.10) *satisfying*

$$S(t) < C\epsilon_0^2, \quad t \in [0, \infty), \tag{2.3}$$

for some universal constant $C > 0$. Moreover, the temperature $q^\pm(t) \to 0$ as $t \to \infty$ with a decay rate

$$\left\|q^\pm(t)\right\|^2_{H^4(\Omega^\pm)} \le Ce^{-\beta^\pm t}, \tag{2.4}$$

where $\beta^\pm = 2\lambda_1^\pm - \eta$ is defined in (1.14). *The moving boundary $\Gamma(t)$ converges asymptotically to some nearby time-independent hypersurface $\bar\Gamma$, and*

$$\sup_{0\le t<\infty} |h(t,\cdot) - h_0(\cdot)|_{4.5} \le C\sqrt{\epsilon_0}. \tag{2.5}$$

Remark 2.3. Other existing global stability results for the Stefan problem contain an effective heat source most commonly introduced through the presence of nontrivial Dirichlet boundary conditions [43, 46]. Such stability questions are simpler than our problem, as the presence of a heat source makes the family of possible steady states finite dimensional. This allows one to a priori guess a possible asymptotic attractor for the nonlinear dynamics. In our case, due to an abundance of possible steady states, small perturbations converge to some nearby element of the set of steady states. Characterization of such a nearby asymptotic state in terms of initial data is a difficult problem.

Remark 2.4 (on the existence of initial data satisfying the assumptions of Theorem 2.2). As explained in section 1.10 initial data satisfying compatibility conditions (1.17a)–(1.17d) can be constructed in a robust way with $q_0^\pm$ satisfying the required smallness assumptions. Condition (1.26) is also satisfied by any choice of data in section 1.10; see, e.g., the choice (1.22)–(1.23). Given a constant C^* it remains to show that such a choice of initial data will satisfy (1.27) for a suitable choice of coefficients $\alpha_i^\pm$, $i = 1, 2, 3, 4$. Choose $\alpha_i^\pm$ as in (1.24). Note that by (1.22)–(1.23), $\partial_N q_0^\pm = \alpha_1^\pm = D^\pm \epsilon$, for $D^+ = 1$ and $D^- = 2$. Taylor expanding $q_0^\pm$ around $r = 1$ we see that

$$q_0^\pm = \alpha_1^\pm (r-1) + O\left((r-1)^2\right).$$

Choosing the above expansion to be valid in a suitably small region $|r-1| \ll 1$ and then extending $q_0^\pm$ in the remainder of the domain $\Omega^\pm$ to be of order ϵ^2 we can ensure that

$$\int_{\Omega^\pm} q_0^\pm\, \varphi_1^\pm\, dx = \frac{1}{C}\alpha_1^\pm \quad \text{for any} \quad C > 1.$$

In particular, for C large enough, we can enforce the compatibility condition (1.27).

2.1. A brief history of prior results. The Stefan problem was introduced by Stefan in 1889 as a model for the melting of ice caps [48, 49], and is now considered a prototype free boundary problem in the area of nonlinear partial differential equations; a historical account of the analysis of related free boundary is given in [25, 39] for results prior to the 1980s. An account of more recent results is provided in [43, 47, 25, 50]; see also the introduction to [31].

Weak solutions to the classical Stefan problem were shown to exist in [34, 24, 41], for both the one-phase and the two-phase problems. In the one-phase case, the problem lends itself to a variational approach that was successfully used in [26] to study the existence and regularity of solutions. Important regularity results were established in [6, 37, 38, 7, 9]. The continuity of the temperature function for the weak solutions of the two-phase classical Stefan problem in any dimension was proved in [8]. Another notion of a generalized solution for the classical Stefan problem, called the *viscosity* solution, was introduced and studied in the seminal works [1, 2, 3, 4], while the existence proof and further regularity results can be found in [35, 36, 12, 13]. An overview of various regularity results for viscosity solutions prior to 2005 can be found in the monograph [5].

Short-time existence of classical solutions of the one-phase problem was established in [33] under sufficient regularity assumptions and higher-order compatibility conditions. In [27] the authors prove local existence for the one-phase classical Stefan problem in higher dimension. In the two-phase case, local existence and uniqueness of classical solutions was proven in [43]. Neither of these papers, however, established the full well-posedness in the sense of Hadamard, as the constructed solutions experience a potential derivative loss. Under mild regularity assumptions on the initial data and a more general domain, the local-in-time solutions were shown to exist in [44], proving additionally the space-time analyticity of the solutions. Smoothness of the free boundary and the temperature were also shown in [40].

Using initial domains $\mathbb{T}^{d-1} \times (0,1)$ and $\mathbb{T}^{d-1} \times (-1,0)$ and temperature profiles that allow for only one steady state solution to the two-phase Stefan problem, global-in-time stability was established in [43] by imposing Dirichlet boundary conditions on the two fixed boundaries: $\mathbb{T}^{d-1} \times \{x_d = 1\}$ and $\mathbb{T}^{d-1} \times \{x_d = -1\}$. In such a setting, the solution to the nonlinear problem, can be treated as a small perturbation of the known linear solution by contrast to our problem.

A similar strategy is taken in [46], where a global-in-time description of the dynamics for the one-phase classical Stefan problem is given. Therein, the authors study the *exterior* problem in the presence of a nontrivial heat source, modeled again through the imposition of an appropriate Dirichlet boundary condition. The corresponding free boundary expands to infinity and the asymptotic rate is given. That method relies on the availability of a nontrivial background Hele-Shaw solution.

Global existence of classical solutions of the one-phase problem was proved in [22] for log-concave initial temperatures, and hence for convex initial domains.

In the presence of surface tension, the families of steady states to the Stefan problem are parametrized by finitely many parameters and therefore the problem does not exhibit the same type of difficulty as the Stefan problem in the absence of surface tension. The global-in-time nonlinear stability of flat steady states was established in [29]. In the more complicated case of steady spheres, the nonlinear stability was first proved in [28], and by a different method in [45].

In the absence of surface tension, the nonlinear stability of nearly spherical steady state solutions to the one-phase Stefan problem was proved in [31] and the authors generalized that result to allow for arbitrary (bounded) initial domains in [32]. Due to the infinite-dimensional space of steady states, the nonlinear stability theory cannot be viewed as a perturbation of a given linear profile; thus, a novel hybrid methodology was developed in [31, 32], which combined energy methods with pointwise maximum principle techniques to establish exponential-in-time lower bounds on the Rayleigh–Taylor stability condition. Maximum principles together with energy estimates were also used in [14] for the analysis of the related Muskat problem.

2.2. Methodology and outline of the paper. Our first main result is Theorem 2.1 proving the local well-posedness for the two-phase classical Stefan problem. Our methodology extends the hybrid method developed in [30, 31] for the one-phase problem in a fundamental way.

Following the energy method of [16] for the incompressible Euler equations, tangential and temporal energy estimates on the problem (1.10) lead to control of the interface regularity via an integral of the type $\int_\Gamma \mathcal{F}(x,t)|\bar{\partial}^k h|^2\,dx$ for some function $\mathcal{F}(x,t) > 0$. Unlike the one-phase problem wherein $\Gamma(t)$ moves with speed $v \cdot n$ and so $\mathcal{F}(x,t) = -\partial_N q$, in the two-phase setting, $\Gamma(t)$ moves with the jump of $v \cdot n$, and hence weight functions must be introduced into the energy method to obtain the function $\mathcal{F}(x,t)$. Specifically, since on Γ,

$$\partial_N q^+ \neq \partial_N q^- \,, \tag{2.6}$$

we introduce the weight functions $W^\pm$ (3.36) in the interior of the two phases $\Omega^\pm$ designed to resolve the mismatch in (2.6), and to allow us to form a common factor in the difference of the two boundary integrals arising from integration-by-parts in both phases Ω^+ and Ω^-. With our weighted energy method, we obtain control on the boundary integral

$$\varepsilon^\Gamma = e^{-(\lambda_1+\eta)t} \int_\Gamma |\bar\partial^6 h|^2 \, dx, \tag{2.7}$$

where λ_1 is the smaller of the two first eigenvalues of the Dirichlet–Laplacian on the domains $\Omega^\pm$. In order to use this energy control to prove the existence of solutions, we will regularize in the tangential directions with the convolution-by-layers smoothing operator introduced in [16], and study the regularized problem. We obtain that for a short time T, depending on the smoothing parameter κ, there exists a solution. The aforementioned energy control will give us a uniform bound, which will guarantee that the time of existence does not vanish as the smoothing parameter goes to zero. Taking a limit as $\kappa \to 0$ leads to a local-in-time solution to (1.10).

In order to prove global-in-time stability of a given steady state, we need to contend with the exponentially decaying weight present in (2.7). Its presence suggests that a bound on (2.7) implies that $|\bar\partial^6 h|_0$ can grow exponentially fast. A related issue is also present in the one-phase case [31] and our general strategy is similar; whenever we have to bound the top-order norms of h we do that at the cost of an exponentially growing factor, since

$$\left|\bar\partial^6 h\right|_0 \le e^{(\lambda_1+\eta)t/2} (\varepsilon^\Gamma)^{1/2}. \tag{2.8}$$

On the other hand, we *do expect* that the temperature q will decay exponentially fast to the equilibrium, as it solves a heat equation. Therefore, each time we use (2.8) we have to make sure that it comes coupled with a lower-order derivative of q which decays sufficiently fast to counterbalance a possible growth coming from (2.8). While this strategy works for most of the error terms, there are certain *energy-critical* error terms with no room left to obtain the desired exponentially decaying factor in the error terms.

In [31] this critical term took the form

$$\int_\Gamma \partial_N q_t \left|\bar\partial^6 h\right|^2, \tag{2.9}$$

which could not be treated as an error term since the expected decay rate of $\partial_N q_t$ is exactly the same as the decay rate of $\partial_N q$. To resolve this issue, the authors proved that after a sufficiently long time interval, the term (2.9) is sign definite, with a favorable sign. This required a complicated usage of comparison principles and a decomposition of the temperature into the eigenfunctions of the Dirichlet–Laplacian.

In our current treatment, we circumvent this difficulty through the introduction of the weights $W^\pm$ in the definition of the natural energy $\mathcal{E}$. As a consequence, the corresponding "critical" term takes the form

$$\int_\Gamma \partial_t \left(e^{-(\lambda_1+\eta)t}\right) |\bar\partial^6 h|^2 = -(\lambda_1+\eta) \int_\Gamma e^{-(\lambda_1+\eta)t} \left|\bar\partial^6 h\right|^2 < 0. \tag{2.10}$$

The simplification in our analysis caused by the estimate (2.10) is very substantial, but it does come with a small price. The terms $\partial_N q_t^\pm$ are implicitly hidden in the terms $\partial_t W^\pm$ which appear inside some of the interior error terms involving integration over $\Omega^\pm$. However, the dissipative effects are stronger inside $\Omega^\pm$ and we combine norm interpolation and energy estimates to overcome a potentially exponential growth in our estimates caused by $\partial_t W^\pm$. This simplifies the proof significantly, as we no longer need to *wait* until the dynamics settle into a regime dominated by the first eigenfunction of the Dirichlet–Laplacian as in [31, section 4.3].

To get quantitative lower bounds on the weights $W^{\pm}$, we must obtain sharp quantitative lower bounds for the quantities $\partial_N q^{\pm}$. We implement a bootstrap scheme, where we first assume such bounds and use them to prove important energy-norm equivalence lemmas in section 4.1. Just like in [32], to show that the lower bound is dynamically preserved, we make a very sharp use of the Pucci operators and comparison principles as explained in section 4.4. Finally, using standard continuity arguments and the improvement of the bootstrap bounds, we present the proof of Theorem 2.2 in section 4.7.

To summarize, a novel aspect of our methodology is the introduction of the weight functions $W^{\pm}(t,x)$ with very specific decay properties. One of its key features is that it measures the boundary energy contribution in terms of a higher-order Sobolev norm weighted by an explicit exponential $e^{(-\lambda_1+\eta)t}$. This simplifies the global-in-time analysis with respect to [31], and provides a tool for studying similar multiphase problems in the absence of surface tension. Equally importantly, using the weighted higher-order energy, we are able to show that the top-order norms $\varepsilon^{\pm}$ also decay in time. The top-order terms decay at a slower rate than predicted by the linear theory, a consequence of the degeneracy caused by the nonlinear and mixed parabolic-hyperbolic character of the equations.

Finally, the perturbation, given by h, from our initial geometry does not decay, but rather it converges, as $t \to \infty$, to some nontrivial h_∞ which is very small in a suitable Sobolev norm.

2.3. Future work. The well-posedness framework introduced in this work is well-suited for the investigation of various singular limits that commonly arise in the study of free boundary problems. We intend to establish that solutions to the one-phase Stefan problem are, in fact, limits of solutions of the two-phase Stefan problem in the limit as the ratio of the diffusion coefficients converges to zero. A second important singular limit amenable to our approach is the problem of the vanishing surface tension limit. Our energy method naturally extends to the surface tension problem, by simply adding new top-order energy terms, weighted by the surface tension coefficient. We intend to examine the possibility of a splash singularity for the one-phase Stefan problem as in [20] and to investigate if a splash singularity can occur for the two-phase Stefan problem following the methodology of [21].

2.4. Outline of the paper. Section 3 is devoted to the proof of the local well-posedness Theorem 2.1. In section 3.1 we regularize the Stefan problem. In section 3.3 we define the energy functionals with the new weights $W^{\pm}$. In sections 3.4 and 3.5 we establish the short-time relationship between the natural energy and the norms and derive the energy identities, respectively. In section 3.6 we prove the energy estimates and in section 3.7 we finally finish the proof of the local existence theorem. In section 4.1 we reintroduce the hypotheses for the global stability theorem and the bootstrap assumptions. In sections 4.2 and 4.3 we obtain global estimates for the weights $W^{\pm}$, energy-norm equivalence, and some a priori estimates for the height function h. Sections 4.4, 4.5, and 4.6 are dedicated to the proof of the dynamic improvement of our the bootstrap assumptions, and in section 4.7 we present the proof of the global stability theorem. Appendix A briefly presents some useful bounds for the change of variables $\Psi^{\pm}$.

3. Local well-posedness: Proof of Theorem 2.1. We begin by constructing a sequence of approximate, so-called κ-problems which retain the nonlinear structure of the original two-phase Stefan problem. The small number $\kappa > 0$ is the radius of convolution, and our κ-problems (3.4) are founded upon the smoothing of the evolving interface $\Gamma(t)$; in particular, in section 3.1, we regularize the height function h using a symmetric horizontal convolution operator Λ_κ and, otherwise, keep the structure of the equations the same. In section 3.2 we establish an existence theorem for our sequence of κ-approximations (3.4) by the contraction mapping principle. The time of existence T_κ, a priori, may shrink to zero as $\kappa \to 0$, but in section 3.6, we establish κ-independent estimates, which allow us to prove that T_κ is, in fact, independent of κ. Passing to the limit as $\kappa \to 0$, we shall obtain solutions to the Stefan problem (1.10).

3.1. Sequence of approximate $\boldsymbol{\kappa}$-problems. For a given parameter $\kappa > 0$ and a height function h, we define its regularization by

$$h^\kappa := \Lambda_\kappa \Lambda_\kappa h, \tag{3.1}$$

where Λ_κ is the smoothing operator defined in (1.3). We introduce the *regularized coordinate transformations* ${}^\kappa\Psi^\pm$ as the solutions to

$$\Delta {}^\kappa\Psi^\pm = 0 \quad \text{in } \Omega^\pm, \tag{3.2a}$$

$${}^\kappa\Psi^\pm(t,x) = x + h^\kappa(t,x)N \quad \text{on } \Gamma, \tag{3.2b}$$

$${}^\kappa\Psi^+ = e \quad \text{on } \partial\Omega, \tag{3.2c}$$

where we recall that e is the identity map on $\partial\Omega$. Similarly as for (1.7), notice that (1.8) for this regularized problem is

$$\left\| {}^\kappa\Psi^\pm - e \right\|_{6.5} \le C|h^\kappa|_6. \tag{3.3}$$

Therefore, the smallness of $|h^\kappa - h_0|_6$ for short time together with the choice of $|h_0|_6 \le C\sigma \ll 1$ gives us that ${}^\kappa\Psi^\pm$ are in fact $H^{6.5}$-class diffeomorphisms. As in (1.9c), we define now, ${}^\kappa\!A^\pm := (\nabla {}^\kappa\Psi^\pm)^{-1}$, and let $w_\kappa^\pm := \partial_t {}^\kappa\Psi^\pm$. We introduce our sequence of approximations to the Stefan problem as the following *κ-problem*:

$$q_t^\pm - \Delta_{{}^\kappa\Psi^\pm} q^\pm = -v^\pm \cdot w_\kappa^\pm + \alpha^\pm \text{ in } \Omega^\pm, \tag{3.4a}$$

$$v^\pm + {}^\kappa\!A^{\pm\top}\nabla q^\pm = 0 \text{ in } \Omega^\pm, \tag{3.4b}$$

$$q^\pm = \mp\kappa^2\left(\left(v^\pm\right)^i \ {}^\kappa\!A_i^j N^j\right) \pm \kappa^2\beta^\pm(t,x) \text{ on } \Gamma, \tag{3.4c}$$

$$h_t = [v \cdot \tilde{n}^\kappa]_-^+ \ \text{ on } \Gamma, \tag{3.4d}$$

$$v^+ \cdot \mathbf{N}^+ = \gamma \text{ on } \partial\Omega, \tag{3.4e}$$

$$q^\pm|_{t=0} = {}^\kappa Q_0{}^\pm \text{ on } \{t=0\}\times\Omega^\pm, \tag{3.4f}$$

$$h|_{t=0} = h_0^\kappa \ \text{ on } \{t=0\}\times\Gamma, \tag{3.4g}$$

where $\mathbf{N}^+$ is the exterior normal vector to the fixed boundary $\partial\Omega$, $\tilde{n}^\kappa := \frac{n^\kappa}{n^\kappa\cdot N}$, n^κ is the normal vector to the regularized interface $\Gamma_\kappa(t)$, given by

$$n^\kappa(t,y(\theta)) = \frac{-\bar{\partial}h^\kappa\tau + (1+H(\theta)h^\kappa(t,z(\theta)))N}{\sqrt{(\bar{\partial}h^\kappa)^2 + (1+H(\theta)h^\kappa)^2}},$$

and ${}^{\kappa}Q_0{}^{\pm}$ is the initial data defined carefully in section 3.1.1. The introduction of this special initial data, and the functions $\alpha^{\pm}$, $\beta^{\pm}$, and γ into (3.4a), (3.4c), (3.4e), respectively, has the purpose of *canceling* the new compatibility conditions that arise in the κ-problem due to the smoothing. The central idea is that (3.4) and its time derivatives, when restricted to time $t = 0$, will produce new terms from the κ-dependent coefficients ${}^{\kappa}\!A_0{}^{\pm}$, that the functions $\alpha^{\pm}(0)$, $\beta^{\pm}(0)$, and $\gamma(0)$ will cancel and replace with the analogous terms of the nonregularized problem (1.10), which in turn corresponds to the original compatibility conditions satisfied by $q_0^{\pm}$ (1.17).

For $s \geq 0$, let $E^{\pm} : H^s(\Omega^{\pm}) \to H^s(\mathbb{R}^2)$ be the Sobolev extension operator of $\Omega^{\pm}$. Then we define the function $\alpha^{\pm}(t,x)$ over $\Omega^{\pm}$ as,

$$\alpha^{\pm}(t,x) = \alpha_0^{\pm}(x) + \int_0^t r^{\pm}(s,x)ds, \tag{3.5}$$

where $\alpha_0^{\pm}$ is given by

$$\alpha_0^{\pm} := -\nabla_{{}^{\kappa}\Psi_0} {}^{\kappa}Q_0{}^{\pm} \cdot {}^{\kappa}\bar{\Psi}_t^{\pm}(0) + \nabla_{\Psi_0} q_0^{\pm} \cdot \Psi_t^{\pm}(0),$$

and $r^{\pm} := \bar{r}^{\pm}|_{\Omega^{\pm}}$ is the restriction to $\Omega^{\pm}$ of the solution to the parabolic problem,

$$\begin{aligned} \bar{r}_t^{\pm} + \Delta^2 \bar{r}^{\pm} &= 0 \text{ in } \mathbb{R}^2, \\ \bar{r}^{\pm}(t=0) &= E^{\pm}\left(\alpha_1^{\pm}\right) \text{ on } \mathbb{R}^2 \times \{t=0\}, \end{aligned}$$

where $\alpha_1^{\pm}$ is defined in $\Omega^{\pm}$ as,

$$\alpha_1^{\pm} := -B_1^{\pm}\left({}^{\kappa}Q_0, h_0^{\kappa}\right) + B_2^{\pm}(q_0, h_0)$$

with

$$\begin{aligned} B_1({}^{\kappa}Q_0, h_0^{\kappa}) :=\ & \Delta_{{}^{\kappa}\Psi_0}\left(\nabla_{{}^{\kappa}\Psi_0} {}^{\kappa}Q_0 \cdot {}^{\kappa}\Psi_t(0) + \alpha_0\right) - {}^{\kappa}\!A_0{}_l^i {}^{\kappa}\Psi_t{}^l{},_s(0) {}^{\kappa}\!A_0{}_j^s \left({}^{\kappa}\!A_0{}_j^k {}^{\kappa}Q_0,{}_k\right),_i \\ & - {}^{\kappa}\!A_0{}_j^i \left({}^{\kappa}\!A_0{}_l^k {}^{\kappa}\Psi_t{}^l,_s {}^{\kappa}\!A_0{}_j^s {}^{\kappa}Q_0,_k\right),_i - {}^{\kappa}\!A_0{}_l^i {}^{\kappa}\Psi_t^l,_s {}^{\kappa}\!A_0{}_j^s {}^{\kappa}Q_0,_i {}^{\kappa}\Psi_t(0)^j \\ & + \nabla_{{}^{\kappa}\Psi_0}\left(\Delta_{{}^{\kappa}\Psi_0} {}^{\kappa}Q_0 + \nabla_{{}^{\kappa}\Psi_0} {}^{\kappa}Q_0 \cdot {}^{\kappa}\Psi_t(0) + v\alpha_0\right) \cdot {}^{\kappa}\Psi_t(0) + \nabla_{{}^{\kappa}\Psi_0} {}^{\kappa}Q_0 \cdot {}^{\kappa}\Psi_{tt}(0), \end{aligned} \tag{3.6a}$$

$$\begin{aligned} B_2(q_0, h_0) :=\ & -A_0{}_l^i \Psi_t(0)^l,_s A_0{}_j^s \left(A_0{}_j^k q_0,_k\right),_i - A_0{}_j^i \left(A_0{}_l^k \Psi_t(0)^l,_s A_0{}_j^s q_0,_k\right),_i \\ & - A_0{}_k^i \Psi_t(0)^k,_l A_0{}_j^l q_0,_i \Psi_t(0)^j + \nabla_{\Psi_0}(\Delta_{\Psi_0} q_0 + \nabla_{\Psi_0} q_0 \cdot \Psi_t(0)) \cdot \Psi_t(0) \\ & + \nabla_{\Psi_0} q_0 \cdot \Psi_{tt}(0) + \Delta_{\Psi_0}(\nabla_{\Psi_0} q_0 \cdot \Psi_t(0)). \end{aligned} \tag{3.6b}$$

Notice that, since q_0 satisfies the compatibility conditions (1.17b), when restricted to Γ, B_1 and B_2 can be writen simply as,

$$\begin{aligned} B_1({}^{\kappa}Q_0, h_0^{\kappa}) &= \partial_t(\Delta_{{}^{\kappa}\Psi} q + \nabla_{{}^{\kappa}\Psi} q \cdot {}^{\kappa}\Psi_t)|_{t=0} - \Delta^2_{{}^{\kappa}\Psi_0} {}^{\kappa}Q_0, \\ B_2(q_0, h_0) &= \Delta^2_{\Psi_0} q_0, \end{aligned}$$

where the value for $q_t(0)$ can be obtained from restricting (3.4a) to $t = 0$ and using that $q(0) = {}^{\kappa}Q_0$.

Remark 3.1. Since, in the right-hand side of (3.4a) we have the term $\alpha^{\pm}(t,x)$, the regularity of $\partial_t^l q^{\pm}$ for $l = 0, \ldots, 3$, depends, among other things, on the regularity of this term and its time derivative. Specifically, we will need to bound $\alpha^{\pm}$ in $H^5(\Omega^{\pm})$ (see section 3.6.1), in order to obtain the desired regularity of $q^{\pm}$, but at the same time we require that $\partial_t \alpha^{\pm}(0) = \alpha_1^{\pm}$ on Γ, which does not have enough derivatives. The introduction of $r^{\pm}$ solves this problem since, from the standard parabolic regularity theory, the solution $r^{\pm}(t,x)$ is in $L^2((0,T);H^5(\Omega^{\pm}))$ since the initial datum $\bar{r}^{\pm}(t=0)$ belongs to $H^3(\Omega)$ by our regularity assumptions on (q_0,h_0).

Now let us define the function γ on $\partial\Omega$ as

$$\gamma := \mathcal{G}^0 \cdot \mathbf{N}^+ + \int_0^t \mathcal{G}^1(s) \cdot \mathbf{N}^+ ds + \int_0^t \int_0^s \mathcal{G}^2(\tau) \cdot \mathbf{N}^+ d\tau ds \text{ in } \Omega^+, \tag{3.7}$$

where $\mathcal{G}^i := \tilde{\mathcal{G}}^i|_{\partial\Omega}$ for $i = 0,1,2$, is the restriction to $\partial\Omega$ of the solution to the following parabolic problem:

$$\tilde{\mathcal{G}}^i_t - \Delta^{2i+1}\tilde{\mathcal{G}}^i = 0 \quad \text{in } \mathbb{R}^2, \tag{3.8}$$

$$\tilde{\mathcal{G}}^i(t=0) = E^+(\gamma_i) \quad \text{on } \mathbb{R}^2 \times \{t=0\} \tag{3.9}$$

with γ_i defined in Ω^+ as

$$\begin{aligned}
&\gamma_0 := -\nabla_{{}^\kappa\Psi_0^+} {}^\kappa Q_0{}^+ + \nabla_{\Psi_0^+} q_0^+,\\
&\gamma_1 = -\left[{}^\kappa A_0{}^+ \nabla {}^\kappa\Psi_t^+(0) {}^\kappa A_0{}^+\right]^\top \nabla {}^\kappa Q_0{}^+ + \nabla_{{}^\kappa\Psi_0^+}\left(\nabla_{\Psi_0^+} q_0^+ \cdot \Psi_t^+(0)\right) + \nabla_{\Psi_0^+}\left(\Delta_{\Psi_0^+} q_0^+\right),\\
&\gamma_2 = \partial_t^2\left(\nabla_{{}^\kappa\Psi^+} q^+\right)|_{t=0} - \nabla_{{}^\kappa\Psi_0^+}\left(\Delta^2_{{}^\kappa\Psi_0^+} {}^\kappa Q_0{}^+\right) + \nabla_{\Psi_0^+}\left(\Delta^2_{\Psi_0^+} q_0^+\right).
\end{aligned}$$

As a consequence of (3.8)–(3.9) and the Sobolev regularity of $\tilde{\mathcal{G}}^i(t=0)$, $i = 0,1,2$, standard parabolic regularity theory gives the bound

$$\left\|\tilde{\mathcal{G}}^0\right\|_{L^\infty_t H^{5.5}} + \sum_{i=0}^{2}\left\|\tilde{\mathcal{G}}^i\right\|_{L^2_t H^6} \lesssim 1. \tag{3.10}$$

Estimate (3.10) plays a crucial role in the nonlinear estimates in section 3.6.1.

The function $\beta^{\pm}(t,x)$ is defined on Γ as

$$\beta^{\pm}(t,x) := \sum_{k=0}^{3} \frac{t^k}{k!}\partial_t^k\left(\left(v^{\pm}\right)^i \;{}^\kappa A_i^j N^j\right)|_{t=0}. \tag{3.11}$$

Note that β is a cubic polynomial in t with space dependent coefficients.

Remark 3.2. The functions $\beta^{\pm}$ serve a similar purpose on the boundary Γ as α does in the interior, and they are used to avoid new compatibility conditions that may appear from the boundary regularization (3.4c). This regularization is needed to overcome a technical difficulty in the higest-in-time energy estimates when we have a term of the form

$$\int_\Gamma \left({}^\kappa\Psi^{\pm}{}_{ttt} \cdot n\right)\left(v_{ttt}^{\pm} \cdot n\right) d\sigma,$$

because the trace of $v_{ttt}^{\pm} \cdot n$ is not necessarily well-defined.

3.1.1. Definition of the smooth initial data ${}^\kappa Q_0{}^{\pm}$. We now construct a smooth version of the initial data $q_0^{\pm}$ that will satisfy the compatibility conditions for the κ-problem (3.4). We solve the tri-Laplacian,

$$\Delta^3_{{}^\kappa\Psi_0^{\pm}} {}^\kappa Q_0{}^{\pm} = \Delta^3_{\Psi_0^{\pm}}\left(\eta_\kappa * E\left(q_0^{\pm}\right)\right) \text{ in } \Omega^{\pm}$$

with specific boundary data designed to satisfy the compatibility conditions. We proceed by solving the equivalent system of elliptic equations:

$$\Delta_{{}^\kappa\Psi_0^\pm} {}^\kappa Q_0{}^\pm = {}^\kappa R_0^\pm \text{ in } \Omega^\pm, \tag{3.12a}$$

$$ {}^\kappa Q_0{}^\pm = 0 \text{ on } \Gamma, \tag{3.12b}$$

$$\nabla_{{}^\kappa\Psi_0^+} {}^\kappa Q_0{}^+ \cdot \mathbf{N}^+ = 0 \text{ on } \partial\Omega, \tag{3.12c}$$

$$\Delta_{{}^\kappa\Psi_0^\pm} {}^\kappa R_0^\pm = {}^\kappa U_0^\pm \text{ in } \Omega^\pm, \tag{3.12d}$$

$$ {}^\kappa R_0^\pm = -\left(\nabla_{\Psi_0^\pm} q_0^\pm \cdot N\right) g_1 \text{ on } \Gamma, \tag{3.12e}$$

$$\nabla_{{}^\kappa\Psi_0^+} {}^\kappa R_0^+ \cdot \mathbf{N}^+ = \nabla_{\Psi_0^+}\left(\Delta_{\Psi_0^+} q_0^+\right) \cdot \mathbf{N}^+ \text{ on } \partial\Omega, \tag{3.12f}$$

$$\Delta_{{}^\kappa\Psi_0^\pm} {}^\kappa U_0^\pm = \Delta^3_{{}^\kappa\Psi_0^\pm}\left(\eta_\kappa * E\left(q_0^\pm\right)\right) \text{ in } \Omega^\pm, \tag{3.12g}$$

$$ {}^\kappa U_0^\pm = B_2\left(q_0^\pm, h_0\right) \text{ on } \Gamma, \tag{3.12h}$$

$$\nabla_{{}^\kappa\Psi_0^+} {}^\kappa U_0^+ \cdot \mathbf{N}^+ = \nabla_{\Psi_0^+}\left(\Delta^2_{\Psi_0^+} q_0^+\right) \cdot \mathbf{N}^+ \text{ on } \partial\Omega. \tag{3.12i}$$

Recall that g_1 is defined in (1.16a). Notice that the system is decoupled, and therefore existence of solutions follows directly. This choice of initial data, and the fact that $q_0^\pm$ satisfy (1.17), shows that the compatibility conditions for (3.4) are automatically satisfied. Moreover, as $\kappa \to 0$,

$$ {}^\kappa Q_0{}^\pm \rightharpoonup q_0^\pm \text{ in } H^6\left(\Omega^\pm\right).$$

Remark 3.3. We actually have strong convergence of ${}^\kappa Q_0{}^\pm$ to $q_0^\pm$ in $H^6(\Omega^\pm)$. The argument is simple, but cumbersome, as it involves elliptic estimates from all three equations (3.12). Consider for example (3.12g) with boundary condition (3.12h), on the region Ω^- (we will omit the index "$-$"). In order to estimate the difference between ${}^\kappa U_0 - \Delta^2_{{}^\kappa\Psi_0} q_0$ we analyze the elliptic problem

$$\begin{aligned} \Delta_{{}^\kappa\Psi_0}\left({}^\kappa U_0 - \Delta^2_{\Psi_0} q_0\right) &= \Delta_{{}^\kappa\Psi_0}\left(\Delta^2_{{}^\kappa\Psi_0}\left(\eta_\kappa * E(q_0)\right) - \Delta^2_{\Psi_0} q_0\right) \quad \text{in } \Omega,\\ {}^\kappa U_0 - \Delta^2_{\Psi_0} q_0 &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Let us define $G := {}^\kappa U_0 - \Delta^2_{\Psi_0} q_0$; then, the system can be rewritten as

$$\begin{aligned} \Delta_{\Psi_0} G &= \Delta_{{}^\kappa\Psi_0}\left(\Delta^2_{{}^\kappa\Psi_0}(\eta_\kappa * E(q_0)) - \Delta^2_{\Psi_0} q_0\right) + (\Delta_{\Psi_0} - \Delta_{{}^\kappa\Psi_0})G \text{ in } \Omega,\\ G &= 0 \text{ on } \Gamma, \end{aligned}$$

and so, by elliptic estimates we have the bound

$$\|G\|_2 \leq \|\Delta_{{}^\kappa\Psi_0}\left(\Delta^2_{{}^\kappa\Psi_0}(\eta_\kappa * E(q_0)) - \Delta^2_{\Psi_0} q_0\right)\|_0 + \|A_0 A_0 - {}^\kappa\!A_0 {}^\kappa\!A_0\|_{L^\infty}\|G\|_2 + \mathcal{O}_\kappa, \tag{3.13}$$

where we have gathered all the lower-order terms coming from the product rule in $\mathcal{O}_\kappa$. Since we have strong convergence of $h_0^\kappa \to h_0$ in $H^6(\Gamma)$, we conclude that ${}^\kappa\Psi_0 \to \Psi_0$ in $H^{6.5}(\Omega)$, and therefore ${}^\kappa\!A_0 \to A_0$ in $H^{5.5}(\Omega)$. Combining this fact together with the strong convergence of $\eta_\kappa * E(q_0)$ to q_0 in $H^6(\Omega)$, the right-hand side of (3.13) goes to zero as $\kappa \to 0$, and therefore, $G \to 0$ in $H^2(\Omega)$ or, equivalently, ${}^\kappa U_0 \to \Delta^2_{\Psi_0} q_0$ in $H^2(\Omega)$. This same procedure applied to the other equations of the system (3.12) gives us the necessary estimates to prove the strong convergence ${}^\kappa Q_0{}^\pm \to q_0^\pm$ in $H^6(\Omega^\pm)$.

3.2. Existence theorem for the κ-problem. In this section, we use the contraction mapping theorem to find solutions to the κ-problem (3.4). We introduce the following normed space of functions:

$$X_M^\kappa = \Big\{ \partial_t^l h^\kappa \in C\left([0,T_\kappa]; H^{6-2l}(\Gamma)\right), \partial_t^{s+1} h^\kappa \in L^2\left([0,T_\kappa]; H^{5-2s}(\Gamma)\right) :$$
$$0 \le l \le 3, 0 \le s \le 2,$$
$$\sum_{l=0}^{3} \left|\partial_t^l h^\kappa(s)\right|^2_{L_t^\infty H^{6-2l}(\Gamma)} + \sum_{l=0}^{2} \left|\partial_t^{l+1} h^\kappa\right|^2_{L_t^2 H^{5-2l}(\Gamma)} \le M,$$
$$h^\kappa(0,x) = h_0^\kappa(x),\ h_t^\kappa(0,x) = g_1^\kappa(x),\ h_{tt}^\kappa(0,x) = g_2^\kappa(x),\ h_{ttt}^\kappa(0) = g_3^\kappa(x) \Big\} \tag{3.14}$$

with $M > 0$ a function of the initial data to be determined, and $g_i^\kappa := \Lambda_\kappa \Lambda_\kappa g_i,\ i = 1,2,3$, the smooth versions of (1.16).

THEOREM 3.4 (solutions to the κ-problem). *For any fixed $\kappa > 0$ there exist a time $T_\kappa > 0$, such that there exists a unique olution (q, h^κ) to the* nonlinear κ-*problem* (3.4) *on the time-interval $[0,T_\kappa]$, and $\mathcal{S}(t) = \mathcal{S}(q^\pm, h^\kappa) \le M$.*

Proof. We will separate the proof into three steps.

3.2.1. Step 1: The linear problem. Assuming that a function $\bar{h} \in X_M$ is given, consider the regularized version of $\bar{h}$:

$$\bar{h}^\kappa := \Lambda_\kappa \Lambda_\kappa \bar{h}. \tag{3.15}$$

Again, we define the regularized coordinate transformations ${}^\kappa\bar{\Psi}^\pm$ as the solutions to

$$\Delta {}^\kappa\bar{\Psi}^\pm = 0 \quad \text{in } \Omega^\pm, \tag{3.16a}$$
$${}^\kappa\bar{\Psi}^\pm(t,\cdot) = x + \bar{h}^\kappa(t,\cdot) N(\cdot) \quad \text{on } \Gamma, \tag{3.16b}$$
$${}^\kappa\bar{\Psi}^+ = e \quad \text{on } \partial\Omega. \tag{3.16c}$$

We define ${}^\kappa\bar{A}^\pm := (\nabla {}^\kappa\bar{\Psi}^\pm)^{-1}$, $\bar{J}_\kappa := \det \nabla {}^\kappa\bar{\Psi}$, the cofactor matrix ${}^\kappa\bar{a} := \bar{J}_\kappa\ {}^\kappa\bar{A}$, and ${}^\kappa\bar{w}^\pm := {}^\kappa\bar{\Psi}_t^\pm$. Recall that (1.8) holds for ${}^\kappa\bar{\Psi}^\pm$ with $\bar{h}^\kappa$ in the right-hand side. This gives us that the transformations ${}^\kappa\bar{\Psi}^\pm$ are in $C^\infty(\Omega^\pm)$. We then define the following *linearization* of the κ-problem:

$$q_t^\pm - \Delta_{{}^\kappa\bar{\Psi}^\pm} q^\pm = -v^\pm \cdot {}^\kappa\bar{w}^\pm + \alpha^\pm \text{ in } \Omega^\pm, \tag{3.17a}$$
$$v^\pm + {}^\kappa\bar{A}^{\pm\top} \nabla q^\pm = 0 \text{ in } \Omega^\pm, \tag{3.17b}$$
$$q^\pm = \mp\kappa^2 \left(\left(v^\pm\right)^i\ {}^\kappa\bar{A}_i^j N^j \right) \pm \kappa^2 \beta^\pm(t,x) \text{ on } \Gamma, \tag{3.17c}$$
$$v^+ \cdot \mathbf{N}^+ = \gamma \text{ on } \partial\Omega, \tag{3.17d}$$
$$q^\pm|_{t=0} = {}^\kappa Q_0{}^\pm \text{ on } \Omega^\pm \times \{t=0\}, \tag{3.17e}$$

where $\alpha^\pm$, $\beta^\pm$, and γ are the functions of the initial data defined in (3.5), (3.11), and (3.7), respectively. Since $\bar{h}^\kappa(t)$ is prescribed, the linear system of (3.17) decouples into two linear heat equations on $\Omega^\pm$, respectively, with $C^\infty(\Omega^\pm)$ coefficients. The initial data for the linear smooth problem are ${}^\kappa Q_0{}^\pm \in H^6(\Omega^\pm)$, which was designed in (3.12), along with the terms $\alpha^\pm$, $\beta^\pm$, γ, to recover the original two-phase compatibility conditions from the decoupled two phases as $\kappa \to 0$.

3.2.2. Step 2: Higher regularity for the linear problem. We want to prove that there exists solutions to (3.17) such that $\partial_t^l q^\pm \in C([0,T_\kappa]; H^{6-2l}(\Omega^\pm)) \cap L^2([0,T_\kappa]; H^{7-2l}(\Omega^\pm))$ for all $l = 0,\ldots,3$. We proceed as in [15]. Since the two phases are decoupled we have the weak formulation of the two different problems on the regions $\Omega^\pm$ separately. For a given function $f \in L^2([0,T_\kappa],\mathbb{R})$ consider the discrete time difference $\delta_t^s f(t) = (\mathfrak{E}f(t+s) - \mathfrak{E}f(t))s^{-1}$, where $\mathfrak{E}$ is a Sobolev extension operator to the positive real line $[0,\infty)$. We define then $q^\pm$ to be a weak solution to (3.17), if for all $\phi^\pm \in H^1(\Omega^\pm)$, the following equations hold pointwise in time for all $t \in [0,T_\kappa]$:

(3.18a)

$$\begin{aligned}\left\langle \partial_t^l(\bar J_\kappa\, q_t), \phi^- \right\rangle + \left(\partial_t^l\left({}^\kappa\bar a_j^i\left({}^\kappa\bar A_j^k q,_k\right)\right), \phi^-,_i\right)_{L^2(\Omega^-)} + \int_\Gamma \partial_t^l\left(\bar J_\kappa\left(\kappa^{-2}q^- + \beta^-\right)\right)\phi^- d\sigma \\ = \left(\partial_t^l\left({}^\kappa\bar a_j^i q,_i\, \delta_t^s\left({}^\kappa\bar\Psi^j\right)\right), \phi^-\right)_{L^2(\Omega^-)} + \left(\partial_t^l(\bar J_\kappa\, \alpha^-), \phi^-\right)_{L^2(\Omega^-)},\end{aligned}$$

(3.18b)

$$\begin{aligned}\left\langle \partial_t^l(\bar J_\kappa\, q_t), \phi^+ \right\rangle + \left(\partial_t^l\left({}^\kappa\bar a_j^i\left({}^\kappa\bar A_j^k q,_k\right)\right), \phi^+,_i\right)_{L^2(\Omega^+)} + \int_\Gamma \partial_t^l\left(\bar J_\kappa\left(\kappa^{-2}q^+ + \beta^+\right)\right)\phi^+ d\sigma \\ = \left(\partial_t^l({}^\kappa\bar a_j^i q,_i\, \delta_t^s({}^\kappa\bar\Psi^j)), \phi^+\right)_{L^2(\Omega^+)} + \int_{\partial\Omega}\left(\partial_t^l(\bar J_\kappa\, \gamma)\right)\phi^+ d\sigma \\ + \left(\partial_t^l(\bar J_\kappa\, \alpha^+), \phi^+\right)_{L^2(\Omega^+)} \quad \text{for } l = 0,\ldots,3\end{aligned}$$

with initial conditions given by $q^\pm(0) = {}^\kappa Q_0{}^\pm$, and for $l = 1,2,3$, $\partial_t^l q^\pm(0) := \partial_t^{l-1}(\Delta_{{}^\kappa\Psi} q + \nabla_{{}^\kappa\Psi} q \cdot {}^\kappa\Psi_t + \alpha)|_{t=0}$. Notice that the solution $q^\pm$ depends on the parameter s, but we will omit its dependence for simplicity of notation, and only at the end of the proof will we take the limit as $s \to 0$. The use of the difference quotient δ_t^s in (3.18a) and (3.18b), is necessary in order to study the third time-differentiated problem; This is due to the fact ${}^\kappa\bar\Psi_{tttt}$ is not well-defined when $\bar h$ is given in X_M^κ.

In what follows, we will omit the upper index $\pm$ for simplicity of notation, but we will keep in mind that we must perform the analogous techniques in the now decoupled regions $\Omega^\pm$. Existence of solutions $q \in C([0,T_\kappa]; L^2(\Omega)) \cap L^2([0,T_\kappa]; H^1(\Omega))$, follow from a Galerkin approximation scheme for parabolic equations, i.e., we consider a solution of the form,

$$q^m(t,x) = \sum_{n=1}^m c_n(t)\varphi_n, \tag{3.19}$$

where $\{\varphi_n\}_{n\in\mathbb{N}}$ is a basis of $H^1(\Omega)$ that, for simplicity, we will choose so that

$$\left(\bar J_\kappa(0)\varphi_n, \varphi_n\right)_{L^2(\Omega)} = 1 \text{ and } \left(\bar J_\kappa(0)\varphi_n, \varphi_s\right)_{L^2(\Omega)} = 0\ \forall s \neq n,$$

and the coefficients $c_n(t)$ satisfy the system of fourth-order differential equations

$$\begin{aligned}\sum_{n=1}^m \left\{ e_{sn}^4 c_n^{(4)}(t) + e_{sn}^3 c_n^{(3)}(t) + e_{sn}^2 c_n^{(2)}(t) + e_{sn}^1 c_n^{(1)}(t) + e_{sn}^0 c_n(t)\right\} \\ = \mathcal{A}_s + \mathcal{B}_s \quad \forall s = 1,\ldots,m\end{aligned} \tag{3.20}$$

with initial data given by

$$q^m(0) = {}^{\kappa}Q_0{}^m = \sum_{n=1}^{m} \left({}^{\kappa}Q_0, \varphi_n\right)_{L^2(\Omega)} \varphi_n, \tag{3.21a}$$

$$\partial_t^k q^m(0) = {}^{\kappa}Q_k^m = \sum_{n=1}^{m} {}^{\kappa}Q_k^m(n)\varphi_n \text{ for } k = 1, 2, 3, \tag{3.21b}$$

where the coefficients ${}^{\kappa}Q_k^m(n)$ are given by

$$\begin{aligned} {}^{\kappa}Q_k^m(n) := \Bigg[& \left([\partial_t^{k-1}, \bar{J}_\kappa] q_t^m, \varphi_n\right)_{L^2} - \left(\partial_t^{k-1}\left({}^{\kappa}\bar{a}_j^i\left({}^{\kappa}\bar{A}_j^k q^m{}_{,k}\right)\right), \varphi_{n,i}\right)_{L^2(\Omega)} \\ & - \int_\Gamma \partial_t^{k-1}(\bar{J}_\kappa(\kappa^{-2} q^m + \beta))\varphi_n d\sigma \\ & + \left(\partial_t^{k-1}\left({}^{\kappa}\bar{a}_j^i q^m{}_{,i}\, \delta_t^s\left({}^{\kappa}\bar{\Psi}^j\right)\right), \varphi_n\right)_{L^2(\Omega)} + \left(\partial_t^{k-1}(\bar{J}_\kappa\, \alpha^-), \varphi_n\right)_{L^2(\Omega)} \Bigg]_{t=0}. \end{aligned}$$

The coefficients e_{sn}^i for $i = 1, \dots, 4$ of the system (3.20) are given by

$$\begin{aligned} e_{sn}^4 &:= \left(\bar{J}_\kappa \varphi_n, \varphi_s\right)_{L^2(\Omega)}, \\ e_{sn}^3 &:= 3\left(\partial_t(\bar{J}_\kappa)\varphi_n, \varphi_s\right)_{L^2(\Omega)} + \left({}^{\kappa}\bar{a}_j^i\, {}^{\kappa}\bar{A}_j^k \varphi_{n,k}, \varphi_{s,i}\right)_{L^2(\Omega)} \\ &\quad + \int_\Gamma \bar{J}_\kappa \kappa^{-2} \varphi_n \varphi_s d\sigma - \left({}^{\kappa}\bar{a}_j^i \varphi_{n,i}\, \delta_t^s\left({}^{\kappa}\bar{\Psi}^j\right), \varphi_l\right)_{L^2(\Omega)}, \\ e_{sn}^2 &:= 3\left(\partial_t^2(\bar{J}_\kappa)\varphi_n, \varphi_s\right)_{L^2(\Omega)} + 3\left(\partial_t\left({}^{\kappa}\bar{a}_j^i\, {}^{\kappa}\bar{A}_j^k\right)\varphi_{n,k}, \varphi_{s,i}\right)_{L^2(\Omega)} \\ &\quad + 3\int_\Gamma \partial_t(\bar{J}_\kappa)\kappa^{-2}\varphi_n\varphi_s d\sigma - 3\left(\partial_t\left({}^{\kappa}\bar{a}_j^i \delta_t^s\left({}^{\kappa}\bar{\Psi}^j\right)\right)\varphi_{n,i}, \varphi_l\right)_{L^2(\Omega)}, \\ e_{sn}^1 &:= 3\left(\partial_t^3(\bar{J}_\kappa)\varphi_n, \varphi_s\right)_{L^2(\Omega)} + 3\left(\partial_t^2\left({}^{\kappa}\bar{a}_j^i\, {}^{\kappa}\bar{A}_j^k\right)\varphi_{n,k}, \varphi_{s,i}\right)_{L^2(\Omega)} \\ &\quad + 3\int_\Gamma \partial_t^2(\bar{J}_\kappa)\kappa^{-2}\varphi_n\varphi_s d\sigma - 3\left(\partial_t^2\left({}^{\kappa}\bar{a}_j^i \delta_t^s\left({}^{\kappa}\bar{\Psi}^j\right)\right)\varphi_{n,i}, \varphi_l\right)_{L^2(\Omega)}, \\ e_{sn}^0 &:= \left(\partial_t^3\left({}^{\kappa}\bar{a}_j^i\, {}^{\kappa}\bar{A}_j^k\right)\varphi_{n,k}, \varphi_{s,i}\right)_{L^2(\Omega)} + \int_\Gamma \partial_t^3(\bar{J}_\kappa)\kappa^{-2}\varphi_n\varphi_s d\sigma \\ &\quad - \left(\partial_t^3\left({}^{\kappa}\bar{a}_j^i \delta_t^s\left({}^{\kappa}\bar{\Psi}^j\right)\right)\varphi_{n,i}, \varphi_l\right)_{L^2(\Omega)} \end{aligned}$$

and

$$\mathcal{A}_s := \left(\partial_t^3(\bar{J}_\kappa\alpha), \varphi_s\right)_{L^2(\Omega)}, \ \mathcal{B}_s := -\int_\Gamma \partial_t^3(\bar{J}_\kappa\beta)\varphi_s d\sigma.$$

Remark 3.5. Notice that we are considering as a generic model the weak formulation in the domain Ω^-, but an analogous process works for Ω^+, with the inclusion of the integral term on the boundary $\partial\Omega$: $\mathcal{C}_s = \int_{\partial\Omega} \partial_t^3(\bar{J}_\kappa\gamma)\varphi_s d\sigma$.

The fundamental theorem of ODEs provides us then with a solution q^m of the form (3.19) that satisfies the system (3.20) in the time interval $[0, T_\kappa^m]$, which a priori depends on the parameter m. Moreover, from linearity, q^m satisfies

(3.22a)

$$\left(\partial_t^3(\bar{J}_\kappa\, q_t^m), \phi^-\right)_{L^2} + \left(\partial_t^3\left({}^\kappa\bar{a}_j^i\left({}^\kappa\bar{A}_j^k q^m,_k\right)\right), \phi^-,_i\right)_{L^2(\Omega^-)} + \int_\Gamma \partial_t^3(\bar{J}_\kappa(\kappa^{-2}q^m+\beta^-))\phi^- d\sigma$$
$$= \left(\partial_t^3\left({}^\kappa\bar{a}_j^i q^m,_i\, \delta_t^s\left({}^\kappa\bar{\Psi}^j\right)\right), \phi^-\right)_{L^2(\Omega^-)} + \left(\partial_t^3(\bar{J}_\kappa\, \alpha^-), \phi^-\right)_{L^2(\Omega^-)},$$

(3.22b)

$$\left(\partial_t^3(\bar{J}_\kappa\, q_t^m), \phi^+\right)_{L^2} + \left(\partial_t^3\left({}^\kappa\bar{a}_j^i\left({}^\kappa\bar{A}_j^k q^m,_k\right)\right), \phi^+,_i\right)_{L^2(\Omega^+)} + \int_\Gamma \partial_t^3\left(\bar{J}_\kappa\left(\kappa^{-2}q^m+\beta^+\right)\right)\phi^+ d\sigma$$
$$= \left(\partial_t^3\left({}^\kappa\bar{a}_j^i q^m,_i\, \delta_t^s\left({}^\kappa\bar{\Psi}^j\right)\right), \phi^+\right)_{L^2(\Omega^+)} + \int_{\partial\Omega}\left(\partial_t^3(\bar{J}_\kappa\, \gamma)\right)\phi^+ d\sigma$$
$$+ \left(\partial_t^3(\bar{J}_\kappa\, \alpha^+), \phi^+\right)_{L^2(\Omega^+)}$$

for all $\phi^\pm$ in the finite-dimensional space generated by $\{\varphi_n^\pm\}_{n\le m}$. In addition, given the definition of the initial data (3.21), we can integrate in time (3.23) as many as three times, to obtain that q^m solves an analogous formulation as (3.18) for $l = 0, 1, 2, 3$:

(3.23a)

$$\left(\partial_t^l(\bar{J}_\kappa\, q_t^m), \phi^-\right)_{L^2} + \left(\partial_t^l\left({}^\kappa\bar{a}_j^i\left({}^\kappa\bar{A}_j^k q^m,_k\right)\right), \phi^-,_i\right)_{L^2(\Omega^-)} + \int_\Gamma \partial_t^l\left(\bar{J}_\kappa\left(\kappa^{-2}q^m+\beta^-\right)\right)\phi^- d\sigma$$
$$= \left(\partial_t^l\left({}^\kappa\bar{a}_j^i q^m,_i\, \delta_t^s\left({}^\kappa\bar{\Psi}^j\right)\right), \phi^-\right)_{L^2(\Omega^-)} + \left(\partial_t^l(\bar{J}_\kappa\, \alpha^-), \phi^-\right)_{L^2(\Omega^-)},$$

(3.23b)

$$\left(\partial_t^l(\bar{J}_\kappa\, q_t^m), \phi^+\right)_{L^2} + \left(\partial_t^l\left({}^\kappa\bar{a}_j^i\left({}^\kappa\bar{A}_j^k q^m,_k\right)\right), \phi^+,_i\right)_{L^2(\Omega^+)} + \int_\Gamma \partial_t^l\left(\bar{J}_\kappa\left(\kappa^{-2}q^m+\beta^+\right)\right)\phi^+ d\sigma$$
$$= \left(\partial_t^l\left({}^\kappa\bar{a}_j^i q^m,_i\, \delta_t^s\left({}^\kappa\bar{\Psi}^j\right)\right), \phi^+\right)_{L^2(\Omega^+)} + \int_{\partial\Omega}\left(\partial_t^l(\bar{J}_\kappa\, \gamma)\right)\phi^+ d\sigma$$
$$+ \left(\partial_t^l\left(\bar{J}_\kappa\, \alpha^+\right), \phi^+\right)_{L^2(\Omega^+)} \quad \forall\phi^\pm \in \langle\{\varphi_n^\pm\}_{n=1}^m\rangle.$$

The next step is to obtain estimates intependent of m. Standard parabolic estimates give us that $q^m \in L^\infty([0,T_\kappa^m]; L^2(\Omega)) \cap L^2([0,T_\kappa^m]; H^1(\Omega))$ with m-independent bounds, which allows us to extend $q^m(t)$ beyond T_κ^m, and up to an m-independent time T_κ. Indeed, substituting $\phi = q^m$ on (3.23) for $l = 0$, we have the bound,

$$\|q^m(t)\|_{L^2}^2 + \|\nabla_{{}^\kappa\bar{\Psi}} q^m\|_{L_t^2L^2}^2 + \kappa^{-2}|q^m|_{L_t^2L^2}^2 \le C(M, q_0)\ \forall t \in [0, T_\kappa], \tag{3.24}$$

where $C(M, q_0)$ is a constant that depends only on M and q_0 (see Lemma A.4 in the apendix for more details). Moreover, given the regularity of ${}^\kappa\bar{\Psi}$, we can improve the bounds so that $q^m \in L^\infty([0,T_\kappa]; H^1(\Omega))$ and $q_t^m \in L^2([0,T_\kappa]; L^2(\Omega))$ by using as a test function $\phi = q_t^m$ in (3.23) with $l = 0$, and following similar estimates as in the proof of (3.24). Consequently, we found that $q^m \in C([0,T_\kappa], L^2(\Omega))$ and, furthermore, using elliptic estimates, we obtain that $q^m \in L^2([0,T_\kappa]; H^2(\Omega))$ and, therefore, $q^m \in C([0,T_\kappa]; H^1(\Omega)) \cap L^2([0,T_\kappa]; H^2(\Omega))$.

Consider now the first time differentiated problem, (3.23) for $l = 1$. Using the previously found regularity of q^m and q_t^m, and repeating the parabolic regularity arguments for $\tilde{q}^m := q_t^m$, we have that $q_t^m \in C([0,T_\kappa]; H^1(\Omega)) \cap L^2([0,T_\kappa]; H^2(\Omega))$ and $q_{tt}^m \in L^2([0,T_\kappa]; L^2(\Omega))$. These estimates for q_t^m, combined again with elliptic estimates for the non-time-differentiated problem (3.23) for $l = 0$, gives us that

$q^m \in C([0,T_\kappa];H^3(\Omega)) \cap L^2([0,T_\kappa];H^4(\Omega))$. Iterating this process one more time for the twice-in-time differentiated problem, we obtain that $q^m_{tt} \in C([0,T_\kappa];H^1(\Omega)) \cap L^2([0,T_\kappa];H^2(\Omega))$, and $q^m_{ttt} \in L^2([0,T_\kappa];L^2(\Omega))$. Elliptic estimates on the one-time-differentiated problem (3.23) for $l=1$, gives us estimates for $q^m_t \in C([0,T_\kappa];H^3(\Omega)) \cap L^2([0,T_\kappa];H^4(\Omega))$, and therefore, using elliptic regularity once again on the non-time-differentiated problem, we obtain estimates for

$$q^m \in C([0,T_\kappa];H^5(\Omega)) \cap L^2([0,T_\kappa];H^6(\Omega)).$$

The final step follows from the triple time-differentiated problem in the same way, but some terms must be treated carefully, and we address them below. First we will show that the triple time-differentiated approximation $q^{m\,\pm}_{ttt}$ to (3.18) satisfies the following inequality:

$$\begin{aligned}(3.25)\quad \frac{1}{2}\left\|q^{m\,\pm}_{ttt}\right\|^2_{L^\infty_t L^2} + \left\|q^{m\,\pm}_{ttt}\right\|^2_{L^2_t H^1} + \kappa^{-2}\left|q^{m\,\pm}_{ttt}\right|^2_{L^2_t L^2} \\ \lesssim C_\kappa M_0 + C_\kappa\sqrt{t}\left(\left\|q^{m\,\pm}_{ttt}\right\|^2_{L^\infty L^2} + \left\|q^{m\,\pm}_{ttt}\right\|^2_{L^2 H^1}\right),\end{aligned}$$

where $M_0 = M_0(q_0^\pm, h_0)$ is a function of the initial data, and C_κ is a constant that depends badly on the smoothing parameter κ. Indeed, the weak form of the triple time differentiated problem can be written as

$$\begin{aligned}\langle \bar{J}_\kappa\, q^m_{tttt}, \phi\rangle + \left(\bar{J}_\kappa\, {}^\kappa\!\bar{A}^i_j q^m_{ttt},_i\,, {}^\kappa\!\bar{A}^k_j \phi,_k\right)_{L^2(\Omega^\pm)} + \int_\Gamma \bar{J}_\kappa\, \left(\kappa^{-2} q^m_{ttt} + \partial_t^3\left(v^m\cdot {}^\kappa\!\bar{A}^\top N\right)\Big|_{t=0}\right)\phi d\sigma \\ = -\left(\bar{J}_\kappa\, v^m\cdot\delta^s_t\left({}^\kappa\bar{\Psi}_{ttt}\right), \phi\right)_{L^2} + \mathcal{Q}^m_3(\phi),\end{aligned}$$

where in the right-hand side we write the highest-order remainder $I := -(\bar{J}_\kappa\, v^m\cdot\delta^s_t({}^\kappa\bar{\Psi}_{ttt}),\phi)_{L^2}$ by itself, and the lower-order terms collected in

$$\begin{aligned}\mathcal{Q}^m_3(\phi) = \left(\left[\partial^3_t, {}^\kappa\bar{a}^i_j {}^\kappa\!\bar{A}^k_j\right] q^m,_k\,,\, \phi,_i\right)_{L^2} - \left(\left[\partial^3_t,\, \bar{J}_\kappa\, v^m\right]\cdot {}^\kappa\bar{\Psi}_t, \phi\right)_{L^2} \\ - \int_\Gamma \left[\partial^3_t,\, \bar{J}_\kappa\right]\left(\kappa^{-2} q^m + \beta^m\right)\phi + \left(\partial^3_t\left(\bar{J}_\kappa\,\alpha^m\right), \phi\right)_{L^2},\end{aligned}$$

where $[a,b]c = a(bc) - b(ac)$ denotes the commutator. We will prove that, choosing $\phi = q^m_{ttt}$, we have the bound

$$(3.26)\qquad \int_0^t \left(I + \mathcal{Q}_3\left(q^m_{ttt}\right)\right) ds \lesssim_M M + C_\kappa\sqrt{t}\left(\|q^m_{ttt}\|^2_{L^2H^1} + \|q^m_{ttt}\|^2_{L^\infty L^2}\right),$$

where C_κ is a constant that depends badly on κ. The integral I contains as a factor the term $\delta^{s\,\kappa}_t\bar{\Psi}_{ttt}$, which, if we were to take the limit as $s\to 0$ right away, it would depend on $\bar{h}^\kappa_{tttt}$, which has too many time derivatives on $\bar{h}^\kappa$ (here lies the necessity to include the discrete operator δ^s_t into the weak formulation). Nonetheless, this problem is easy to overcome. Recall that, since we know that q^m_{ttt} is in $L^2_tL^2$ and q^m_{tt} is in $L^2_tH^2$, we can use the strong form of the twice-in-time differentiated heat equation to obtain

$$\begin{aligned}I_2 &:= -\int_0^t \left(\bar{J}_\kappa\, v^m\cdot\delta^{s\,\kappa}_t\bar{\Psi}_{ttt}, q^m_{ttt}\right)_{L^2} ds \\ &= -\int_0^t \left(\bar{J}_\kappa\, v^m\cdot\delta^{s\,\kappa}_t\bar{\Psi}_{ttt}, -v^m\cdot{}^\kappa\bar{\Psi}_{ttt} + \Delta_{{}^\kappa\bar{\Psi}} q^m_{tt}\right)_{L^2} + \mathcal{Q}_4,\end{aligned}$$

where $\mathcal{Q}_4$ corresponds to the error terms that follow from the L^2-inner product of $v^m \cdot \delta_t^s {}^\kappa\bar{\Psi}_{ttt}$ with the remainder terms from the twice-in-time differentiated heat equation. Estimates for these terms follow from integrating by parts in time to remove a derivative from $\delta_t^s {}^\kappa\bar{\Psi}_{ttt}$, and then using the standard Cauchy–Schwarz inequality. Therefore, we focus on the higher-order terms, which can be rewritten as

$$
\begin{aligned}
I_3 &:= \int_0^t \left(\bar{J}_\kappa\, v^m \cdot \delta_t^s {}^\kappa\bar{\Psi}_{ttt}, -v^m \cdot {}^\kappa\bar{\Psi}_{ttt} + \Delta_{{}^\kappa\bar{\Psi}} q_{tt}^m\right)_{L^2} \\
&= \frac{1}{2}\int_0^t \delta_t^s \left[\left\|\sqrt{\bar{J}_\kappa}\left(v^m \cdot {}^\kappa\bar{\Psi}_{ttt}\right)\right\|_0^2\right] ds - \int_0^t \left(\delta_t^s(\bar{J}_\kappa\, v^m) \cdot {}^\kappa\bar{\Psi}_{ttt}, v^m \cdot {}^\kappa\bar{\Psi}_{ttt}\right)_{L^2} ds \\
&\quad - \int_0^t \left(\bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt}, \delta_t^s(\Delta_{{}^\kappa\bar{\Psi}} q_{tt}^m)\right)_{L^2} ds - \int_0^t \left(\delta_t^s(\bar{J}_\kappa v^m) \cdot {}^\kappa\bar{\Psi}_{ttt}, \Delta_{{}^\kappa\bar{\Psi}} q_{tt}^m\right)_{L^2} ds \\
&\quad + \int_0^t \delta_t^s \left[\left(\bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt}, \Delta_{{}^\kappa\bar{\Psi}} q_{tt}^m\right)_{L^2}\right] ds.
\end{aligned}
$$

Most of these terms are as easily bounded as (3.26) by using the Cauchy–Schwarz inequality, but the fourth and sixth terms require an intermediate step. It is necessary to first integrate by parts in space to remove a derivative from q_{ttt}^m or q_{tt}^m, respectively.

$$
\begin{aligned}
I_4 &:= -\int_0^t \left(\bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt}, \delta_t^s(\Delta_{{}^\kappa\bar{\Psi}} q_{tt}^m)\right)_{L^2} ds \\
&= -\int_0^t \int_\Gamma \bar{J}_\kappa\, \nabla q^m \cdot {}^\kappa\bar{A}^\top N \bar{h}_{ttt}^\kappa \delta_t^s \left({}^\kappa\bar{A}_j^k {}^\kappa\bar{A}_j^i q_{tt}^m{}_{,i}\right) N^k d\sigma ds \\
&\quad + \int_0^t \int_{\Omega^\pm} \left(\bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt} {}^\kappa\bar{A}_{j,i}^i {}^\kappa\bar{A}_j^k + (\bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt})_{,i} {}^\kappa\bar{A}_j^i {}^\kappa\bar{A}_j^k\right) \delta_t^s q_{tt}^m{}_{,k} + \text{l.o.t.} \\
&\le \kappa^{-2} \int_0^t \int_\Gamma \bar{J}_\kappa\, (v^m \cdot N) \bar{h}_{ttt}^\kappa \delta_t^s q_{tt}^m d\sigma ds + C_\kappa \sqrt{t} |\bar{h}_{ttt}^\kappa|_{L_t^\infty L^2} \|\delta_t^s q_{tt}^m\|_{L_t^2 H^1} + \text{l.o.t.} \\
&\le C_M \kappa^{-2} \sqrt{t} |\bar{h}_{ttt}^\kappa|_{L_t^\infty L^2} \left(|q_{ttt}^m|_{L_t^2 L^2(\Gamma)} + \|q_{ttt}^m\|_{L^2 H^1}\right) \sum_{l\le 2} \left\|\partial_t^l q^m\right\|_{L_t^2 H^{6-2l}} \\
&\le C_\kappa \sqrt{t} \left(\|q_{ttt}^m\|_{L_t^\infty L^2}^2 + \|q_{ttt}^m\|_{L_t^2 H^1}^2\right),
\end{aligned}
$$

where in the last inequality we used the previously found bounds for the terms $\|\partial_t^l q^m\|_{L_t^2 H^{6-2l}}$. Now, for the sixth term, we integrate by parts one of the derivatives on q_{tt}^m,

$$
\begin{aligned}
I_5 &:= \int_0^t \delta_t^s \left[\left(\bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt}, \Delta_{{}^\kappa\bar{\Psi}} q_{tt}^m\right)_{L^2}\right] ds \\
&= \int_0^t \delta_t^s \left[-\int_\Omega (\bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt} {}^\kappa\bar{A}_j^i)_{,i} {}^\kappa\bar{A}_j^k q_{tt}^m{}_{,k} + \int_\Gamma \bar{J}_\kappa\, v^m \cdot {}^\kappa\bar{\Psi}_{ttt} {}^\kappa\bar{A}_j^i N^i {}^\kappa\bar{A}_j^k q_{tt}^m{}_{,k}\right] ds \\
&\le C\kappa^{-1} \|q_{tt}^m\|_{L^\infty H^1} |\bar{h}_{ttt}^\kappa|_{L^\infty L^2(\Gamma)} + \int_0^t \delta_t^s \left[\int_\Gamma \bar{J}_\kappa v^m \cdot N \bar{h}_{ttt}^\kappa \left(v_{tt}^{mj} A_j^i N^i\right) d\sigma\right] ds + \text{l.o.t.} \\
&\le C_\kappa M_0 + \int_0^t \delta_t^s \left[\int_\Gamma \bar{J}_\kappa v^m \cdot N \bar{h}_{ttt}^\kappa \kappa^{-2} q_{tt}^m d\sigma\right] ds + \text{l.o.t.} \\
&\le C_\kappa M_0 + C\kappa^{-2} |\bar{h}_{ttt}^\kappa|_{L^\infty L^2} \|q_{tt}^m\|_{L^\infty H^1} + \text{l.o.t.} \\
&\le C_\kappa M_0,
\end{aligned}
$$

where in line three we pulled a coefficient κ^{-1} to absorb a derivative from the norm of $\bar{h}^\kappa_{ttt}$, and in line four we used the strong form of the boundary condition (3.4c) differentiated twice in time. Combining all the estimates together, and using the bounds for $\bar{J}_\kappa$ analogous to the ones in Lemma A.3, we obtain, therefore,

$$\frac{1}{2}\|q^m_{ttt}\|^2_{L^\infty L^2} + \|\nabla_{{}^\kappa\bar{\Psi}} q^m_{ttt}\|^2_{L^2L^2} + \kappa^{-2}|q^m_{ttt}|^2_{L^2L^2} \\ \lesssim C_\kappa M_0 + C_\kappa\sqrt{t}\left(\|q^m_{ttt}\|^2_{L^\infty L^2} + \|q^m_{ttt}\|^2_{L^2H^1}\right).$$

With the bounds for the matrices ${}^\kappa\bar{A}$ being analogous to those of Lemma A.1, combined with a modified Poincaré inequality detailed in [32, (4.6)], we obtain the desired inequality.

Now, taking T_κ small enough on (3.25) gives us m-independent bounds for $q^m_{ttt} \in L^\infty([0,T_\kappa], L^2(\Omega^\pm)) \cap L^2([0,T_\kappa], H^1(\Omega^\pm))$ and, moreover, from the weak formulation (3.23), we can obtain as well that $q^m_{tttt} \in L^2([0,T], H^1(\Omega^\pm)^*)$ with bounds independent of m. As a consequence, $q^m_{ttt} \in C([0,T_\kappa]; L^2(\Omega^\pm)) \cap L^2([0,T_\kappa]; H^1(\Omega^\pm))$, and therefore we can use elliptic regularity in succession on the time differentiated problems to obtain the desired m-independent estimates for $\partial^l_t q^m \in C([0,T_\kappa]; H^{6-2l}(\Omega)) \cap L^2([0,T_\kappa]; H^{7-2l}(\Omega))$ for $l = 0,1,2,3$.

Passing to the limit as $m \to \infty$, we obtain a weak solution ${}^sq$ to (3.18) for $l = 3$ that, from lower semicontinuity, satisfies that $\partial^l_t {}^sq \in C([0,T_\kappa]; H^{6-2l}(\Omega)) \cap L^2([0,T_\kappa]; H^{7-2l}(\Omega))$, and ${}^sq_{tttt} \in L^2([0,T_\kappa]; H^1(\Omega)^*)$. Consider now the the time integral of (3.18) for $l = 3$. Given the compatibility conditions, we will recover that ${}^sq$ satisfies (3.18) for $l = 2$, and for this case, the term containing $\delta^s_t {}^\kappa\bar{\Psi}_{tt}$ converges strongly to ${}^\kappa\bar{\Psi}_{ttt}$ in $L^2(\Omega)$ as $s \to 0$. The estimates that we obtained were independent of the parameter s, therefore, we can pass to the limit as $s \to 0$, to obtain a weak solution $q^\pm$, such that

$$\begin{aligned}
&\left\langle \partial^2_t(\bar{J}_\kappa\, q_t), \phi^-\right\rangle + \left(\partial^2_t({}^\kappa\bar{a}^i_j)\left({}^\kappa\bar{A}^k_j q_{,k}\right), \phi^-{}_{,i}\right)_{L^2(\Omega^-)} + \int_\Gamma \partial^2_t\left(\bar{J}_\kappa\left(\kappa^{-2}q^- + \beta^-\right)\right)\phi^- d\sigma \\
&\quad= \left(\partial^2_t\left({}^\kappa\bar{a}^i_j q_{,i}\, {}^\kappa\bar{\Psi}^j_t\right), \phi^-\right)_{L^2(\Omega^-)} + \left(\partial^2_t\left(\bar{J}_\kappa\,\alpha^-\right), \phi^-\right)_{L^2(\Omega^-)}, \\
&\left\langle \partial^2_t(\bar{J}_\kappa\, q_t), \phi^+\right\rangle + \left(\partial^2_t({}^\kappa\bar{a}^i_j)\left({}^\kappa\bar{A}^k_j q_{,k}\right), \phi^+{}_{,i}\right)_{L^2(\Omega^+)} + \int_\Gamma \partial^2_t\left(\bar{J}_\kappa\left(\kappa^{-2}q^+ + \beta^+\right)\right)\phi^+ d\sigma \\
&\quad= \left(\partial^2_t\left({}^\kappa\bar{a}^i_j q_{,i}\, {}^\kappa\bar{\Psi}^j_t\right), \phi^+\right)_{L^2(\Omega^+)} + \int_{\partial\Omega}\left(\partial^2_t(\bar{J}_\kappa\,\gamma)\right)\phi^+ d\sigma \\
&\qquad+ \left(\partial^2_t(\bar{J}_\kappa\,\alpha^+), \phi^+\right)_{L^2(\Omega^+)},
\end{aligned} \tag{3.27}$$

holds for all $\phi^\pm \in H^1(\Omega^\pm)$, and that it satisfies that $\partial^l_t q^\pm \in C([0,T_\kappa]; H^{6-2l}(\Omega^\pm)) \cap L^2([0,T_\kappa]; H^{7-2l}(\Omega^\pm))$. This finishes the proof of existence of weak solutions to the linear problem (3.17).

3.2.3. Step 3: Contraction mapping theorem. We now define an operator Φ_κ on X_M. Given $\bar{h} \in X_M$ we set

$$\Phi_\kappa(\bar{h}) = h := h_0 + \int_0^t [v\cdot\bar{n}^\kappa]^+_- ds, \tag{3.28}$$

where $v^\pm$ is the solution to the linearized problem (3.17), and $\bar{n}^\kappa := (N - \tau\frac{\bar{\partial}\bar{h}^\kappa}{(1+H\bar{h}^\kappa)})$. Notice that Φ_κ maps X_M to itself as proven in Lemma A.5 in the appendix. To prove that Φ_κ is a contraction, we assume $\bar h^1$, $\bar h^2$ are given, and consider $\Phi_\kappa(\bar h^1) = h^1$, $\Phi_\kappa(\bar h^2) = h^2$ with the associated temperature gradients v_1, v_2. The difference $\delta h_t = h^1_t - h^2_t$ is given by

$$\delta h_t = [\delta v \cdot \bar n^\kappa_1]^+_- + [v_2 \cdot \delta \bar n^\kappa]^+_-, \tag{3.29}$$

where $\bar n^\kappa_\alpha = N - \frac{\bar\partial \bar h^\kappa_\alpha}{(1+H^\kappa h^\kappa_\alpha)}\tau$ for $\alpha = 1, 2$, $\delta v = v_1 - v_2$, and $\delta \bar n^\kappa = \bar n^\kappa_1 - \bar n^\kappa_2$. Taking two time derivatives we obtain that

$$\delta h_{ttt} = [\delta v_{tt} \cdot \bar n^\kappa_1]^+_- + \left[v^2 \cdot \bar n^\kappa_{tt}\right]^+_- + \mathcal{Y},$$

where we gather the lower-order terms in $\mathcal{Y}$,

$$\mathcal{Y} := [v_{2tt} \cdot \delta \bar n^\kappa + \delta v \cdot \bar n^\kappa_{1\,tt} + 2\delta v_t \cdot \bar n^\kappa_{1\,t} + 2 v_{2t} \cdot \delta \bar n^\kappa_t]^+_- .$$

A straightforward bound using Sobolev and trace inequalities gives

$$\begin{aligned} &|\delta h_{ttt}|_{H^k} \\ &\leq C\left(\|\delta v^+_{tt}\|_{k+0.5} + \|\delta v^-_{tt}\|_{k+0.5} + (\|v^+_2\|_{k+0.5} + \|v^-_2\|_{k+0.5})|\delta \bar h^\kappa_{tt}|_{H^{k+1}}\right) \text{ for } k = 0, 1, \end{aligned} \tag{3.30}$$

where $\delta \bar h^\kappa = \bar h^\kappa_1 - \bar h^\kappa_2$. We now obtain the necessary estimates for $\|\delta v^\pm_{tt}\|_{L^2H^{1.5}}$. We will omit the superscript $\pm$ for simplicity of notation. Taking the difference of equations (3.17b) for v_1 and v_2, and taking two time derivatives, we obtain

$$\delta v_{tt} + \partial_t^2\left({}^\kappa\!\bar A_1{}^\top \nabla \delta q\right) = \partial_t^2\left(-\delta\left({}^\kappa\!\bar A^\top\right)\nabla q_2\right)$$

with $\delta({}^\kappa\!A^\top) = {}^\kappa\!A_1{}^\top - {}^\kappa\!A_2{}^\top$. Therefore, using the bounds for ${}^\kappa\!\bar A$ from Lemma A.2 we arrive at

$$\begin{aligned} \|\delta v_{tt}\|_{H^{1.5}} &\leq \|{}^\kappa\!\bar A_1\|_{1.5}\|\delta q_{tt}\|_{2.5} + \|\delta {}^\kappa\!\bar A\|_{1.5}\|q_{2tt}\|_{2.5} + \|\delta {}^\kappa\!\bar A_{tt}\|_{1.5}\|q_2\|_{2.5} \\ &\quad + \|{}^\kappa\!\bar A_{1tt}\|_{1.5}\|\delta q\|_{2.5} + l.o.t. \\ &\leq C_M\left(\|\delta q_{tt}\|_{2.5} + \sqrt{t}|\delta \bar h_{tt}|_{L^2H^2}\right). \end{aligned} \tag{3.31}$$

Furthermore, the difference δq_{tt} satisfies the following parabolic problem:

$$\begin{aligned} \delta q^\pm_{ttt} - {}^\kappa\!\bar A^i_{1j}{}^\pm\left({}^\kappa\!\bar A^k_{1j}{}^\pm \delta q^\pm_{tt},_k\right),_i &= f^\pm \text{ in } \Omega^\pm, \\ \delta q^\pm_{tt} &= 0 \text{ on } \Gamma, \\ \delta v^+_{tt} \cdot \mathbf{N}^+ &= 0 \text{ on } \partial\Omega, \\ \delta q^\pm_{tt}(0, \mathbf{x}) &= 0 \text{ on } \Omega^\pm \times \{t = 0\}, \end{aligned}$$

where

$$\begin{aligned} f = \partial_t^2 &\left(-\delta v \cdot \bar w_{1\kappa} - v_2 \cdot \delta {}^\kappa\!\bar\Psi_t + \delta\left({}^\kappa\!\bar A^i_j\right)\left({}^\kappa\!\bar A^k_{1j} q_2,_k\right),_i + {}^\kappa\!\bar A^i_{2j}\left(\delta\left({}^\kappa\!\bar A^k_j\right) q_2,_k\right),_i\right) \\ &+ \partial_t^2 {}^\kappa\!\bar A_1{}^i_j\left({}^\kappa\!\bar A_1{}^k_j \delta q,_k\right),_i + {}^\kappa\!\bar A_1{}^i_j\left(\partial_t^2 {}^\kappa\!\bar A_1{}^k_j \delta q,_k\right),_i + 2\partial_t {}^\kappa\!\bar A_1{}^i_j\left(\partial_t {}^\kappa\!\bar A_1{}^k_j \delta q, k\right),_i \\ &+ 2\partial_t {}^\kappa\!\bar A_1{}^i_j\left({}^\kappa\!\bar A_1{}^k_j \delta q_t,_k\right),_i + 2 {}^\kappa\!\bar A_1{}^i_j\left(\partial_t {}^\kappa\!\bar A_1{}^k_j \delta q_t,_k\right),_i . \end{aligned} \tag{3.32}$$

Standard parabolic regularity provides

$$\left\|\delta q^\pm_{tt}\right\|_{L^\infty H^2} + \left\|\delta q^\pm_{tt}\right\|_{L^2H^{2.5}} \leq C_\kappa \left\|f^\pm\right\|_{L^2H^{0.5}}, \tag{3.33}$$

where estimates for the source term $f^{\pm}$ given in Lemma A.7 in the appendix gives us

$$\left\|\delta q_{tt}^{\pm}\right\|_{L^\infty H^2} + \left\|\delta q_{tt}^{\pm}\right\|_{L^2_t H^{2.5}} \le C_M \kappa^{-1}\sqrt{T_\kappa}\ \mathcal{S}(\delta q, \delta\bar{h}^\kappa)^{1/2},$$

where $\mathcal{S}(\delta q, \delta\bar{h}^\kappa)$ is the high-order norm defined in (1.12) evaluated in δq and $\delta\bar{h}^\kappa$. Repeating this argument for the parabolic problems associated with δq_t and δq, and adding all the inequalities together we obtain that

(3.34)
$$\mathcal{E}^+(\delta q, \delta\bar{h}^\kappa) + \mathcal{E}^-(\delta q, \delta\bar{h}^\kappa) + \int_0^t (\mathcal{D}^+(\delta q, \delta\bar{h}^\kappa) + \mathcal{D}^-(\delta q, \delta\bar{h}^\kappa))ds \lesssim_M \kappa^{-1}\sqrt{T_\kappa}\mathcal{S}(\delta q, \delta\bar{h}^\kappa),$$

where we recall the definitions of $\mathcal{E}^{\pm}$ and $\mathcal{D}^{\pm}$ from section 1.8 as the higher-order norms of $\partial_t^l \delta q$. A small enough time T_κ allows us to absorb the terms in the right side of (3.34) with the same norms of $\partial_t^l \delta q$ as in the left side, leaving only the boundary norms,

$$\begin{aligned} &\mathcal{E}^+(\delta q, \delta\bar{h}^\kappa) + \mathcal{E}^-(\delta q, \delta\bar{h}^\kappa) + \int_0^t \left(\mathcal{D}^+(\delta q, \delta\bar{h}^\kappa) + \mathcal{D}^-(\delta q, \delta\bar{h}^\kappa)\right) ds \\ &\qquad\qquad \lesssim_M \kappa^{-1}\sqrt{T_\kappa}\left(\mathcal{E}^\Gamma_{\mathrm{loc}}(\delta\bar{h}^\kappa) + \int_0^t \mathcal{D}^\Gamma_{\mathrm{loc}}(\delta\bar{h}^\kappa)ds\right). \end{aligned}$$

Therefore, using this together with (3.31) and (3.30), we obtain

$$|\delta h_{ttt}|^2_{L^\infty L^2(\Gamma)} + |\delta h_{ttt}|^2_{L^2 H^1(\Gamma)} \lesssim_M \kappa^{-1}\left(\mathcal{E}^\Gamma_{\mathrm{loc}}(\delta\bar{h}^\kappa) + \int_0^t \mathcal{D}^\Gamma_{\mathrm{loc}}(\delta\bar{h}^\kappa)ds\right).$$

An analogous estimate for $\partial_t^l \delta h$ in $L^\infty H^{6-2l} \cap L^2 H^{6.5-2l}$ for $l = 0, 1, 2$, respectively, allows us to conclude that

$$\mathcal{E}^\Gamma_{\mathrm{loc}}(\delta h) + \int_0^t \mathcal{D}^\Gamma_{\mathrm{loc}}(\delta h)ds \lesssim_M \kappa^{-1}\sqrt{T_\kappa}\left(\mathcal{E}^\Gamma_{\mathrm{loc}}(\delta\bar{h}^\kappa) + \int_0^t \mathcal{D}^\Gamma_{\mathrm{loc}}(\delta\bar{h}^\kappa)ds\right).$$

We see that Φ_κ is a contraction for T_κ sufficiently small and the theorem follows from the contraction mapping theorem. ∎

3.3. Definition of the energy functionals. The key ingredient to the proof of the main theorems is the introduction of the higher-order *weighted* energy $\mathcal{E}_\kappa(t)$, which will be shown to control the norm $\mathcal{S}$ evaluated on the solutions of the regularized problem (3.17), which we define as

$$\mathcal{S}_\kappa(t) := \mathcal{S}\left(q^{\pm}, h^\kappa\right). \tag{3.35}$$

Note that $(q^{\pm}, h^\kappa)$ is the solution to the regularized problem (3.4) obtained in Theorem 3.4, and therefore $\mathcal{S}_\kappa(t)$ is bounded for all $t \in [0, T_\kappa]$.

The weight functions $W^{\pm}$. To define the energy associated with the two-phase Stefan problem, we will introduce *weight* functions $W^{\pm}(t,x)$, that will allow us to successfully include the nondegeneracy condition (1.25) in our theory. Let

$W^\pm : \Omega^\pm \to \mathbb{R}$ be a solution to the following Dirichlet problem:

$$\Delta W^\pm = 0, \; x \in \Omega^\pm, \tag{3.36a}$$

$$W^\pm = \frac{e^{(-\lambda_1+\eta)t}}{\partial_N q^\pm}, \; x \in \Gamma, \tag{3.36b}$$

$$W^+ \equiv \frac{e^{(-\lambda_1+\lambda_1^+ +\eta)t}}{|c_1^+|}, \quad x \in \partial\Omega, \tag{3.36c}$$

where $c_1^+ := (q_0^+, \varphi_1^+)_{L^2}$, defined in (1.28) and $\lambda_1 = \min\{\lambda_1^+, \lambda_1^-\}$. Note that $W^\pm > 0$ in $\Omega^\pm$ by the maximum principle and the Rayleigh–Taylor assumption (1.25), which, by continuity, guarantees that $\partial_N q^\pm > 0$ at least for short times. The long-time behavior of $W^\pm$ is very important for the proof of global stability and it depends on the difference between the first eigenvalues of the Dirichlet–Laplacian in the regions Ω^+ and Ω^-.

3.3.1. The choice of weights—heuristic. Our choice of weights (3.36a)–(3.36c) bears an important role for the global-in-time stability problem and we will now provide a heuristic motivation.

If we were to set $W^\pm = 1$ and thereby not include any weights in the definition of our norm and energy, then (see section 3.5) our strategy would fail as, roughly speaking, the boundary terms

$$\int_\Gamma \left(\left(\partial_N q^+\right) \bar\partial^k h \bar\partial^k \left(\partial_N q^+\right) - \left(\partial_N q^+\right) \bar\partial^k h \bar\partial^k \left(\partial_N q^+\right)\right) dx' \tag{3.37}$$

would both appear and it is not clear how to extract a positive definite "energy" contribution. By introducing a weight, we instead produce a term of the form

$$\begin{aligned}
&\int_\Gamma \left(\left(\partial_N q^+\right) \bar\partial^k h \bar\partial^k \left(\partial_N q^+\right) W^+ - \left(\partial_N q^-\right) \bar\partial^k h \bar\partial^k \left(\partial_N q^-\right) W^-\right) dx' \\
&\quad = \int_\Gamma \left(e^{(-\lambda_1+\eta)t} \bar\partial^k h \bar\partial^k \left(\partial_N q^+\right) - e^{(-\lambda_1+\eta)t} \bar\partial^k h \bar\partial^k (\partial_N q^-)\right) dx' \\
&\quad = \frac{1}{2} \int_\Gamma e^{(-\lambda_1+\eta)t} \bar\partial^k h \, \bar\partial^k \left[\partial_N q\right]_-^+ \approx \frac{1}{2} \frac{d}{dt} \int_\Gamma e^{(-\lambda_1+\eta)t} |\bar\partial^k h|^2 \, dx' + \text{a positive term.}
\end{aligned} \tag{3.38}$$

As is evident from (3.37)–(3.38) one can choose any positive weights $W^\pm$ with the property

$$\frac{W^+}{W^-} = \frac{\partial_N q^-}{\partial_N q^+} \quad \text{on } \Gamma \tag{3.39}$$

to obtain a positive definite energy contribution. The point is that the two phases "communicate" via (3.39) and therefore the choices of the weights reflect the long-time decay properties of $\partial_n q^+$ and $\partial_N q^-$.

To understand the role of the first Dirichlet–Laplace eigenvalues $\lambda_1^\pm$ in our weights, we must observe that, in the case of the linear heat equation on a given bounded domain, the decay rate of the temperature is precisely given by a *constant* $\times\, e^{-\lambda_1^\pm t}$. In the case of the Stefan problem, we *anticipate* the free boundary to settle back to some fixed domain Ω_∞ which is close to the initial domain. This suggests that the temperatures $q^\pm$ as well as $\partial_N q^\pm$ will decay to 0 at a rate which is *approximately* equal to $e^{-\lambda_1^{\pm} t}$. To handle the fact that these rates are not exactly the same, we introduce the parameter $0 < \eta \ll 1$ thus providing a little bit of "wiggle" room.

By means of the maximum principle and Sobolev embedding we will show that, roughly speaking,

$$e^{-(\lambda_1^{\pm}+\eta)t} \lesssim \partial_N q^{\pm}(t,x) \lesssim e^{-(\lambda_1^{\pm}-\eta)t}, \quad x \in \Gamma,$$

for the solutions $q^{\pm}$ of the nonlinear flow. This translates into the statement that $W^{\pm} \lesssim e^{-\lambda_1+\lambda_1^{\pm}+2\eta t}$ on Γ, thus providing an approximate "normalization" condition with $W^{\pm}$ allowed to grow at most like $e^{|\lambda_1-\lambda_2|t}$ times a mild correction $e^{2\eta t}$. The crux of our energy estimates, related to an idea which appeared first in the earlier works [31, 32] is to show that despite the possibility of such a growth, the anticipated exponential decay of $q^{\pm}$ and its derivatives is sufficient to close the estimates.

Finally, since $\lambda_1 = \min\{\lambda_1^+, \lambda_1^-\}$, condition (3.36c) merely ensures that the $\partial\Omega$ growth-in-time of W^+ is not worse than $e^{|\lambda_1-\lambda_2|t+\eta t}$ so that it remains consistent with the worst possible growth occurring on Γ (explained above).

3.3.2. The short-time behavior of the weights. On the other hand, the short-time behavior of $W^{\pm}$ is easily characterized in the following lemma.

LEMMA 3.6 (local estimates for $W^{\pm}$). *Suppose the Taylor sign condition* (1.25) *holds for some $\delta > 0$, and assume there exists a constant $M > 0$ such that $S(t) \leq M$. Then there exist positive constants $c_{\delta,M} > 0$, $C_{\delta,M} > 0$, such that the solution $W^{\pm}$ to* (3.36) *satisfy,*

$$c_{\delta,M} \leq W^{\pm} \leq C_{\delta,M}.$$

Proof. Notice that for short time, the Rayleigh–Taylor condition (1.25) gives us the following *upper bound on $\partial_N q^{\pm}$*,

$$\left|\partial_N q^{\pm}\right| \leq C \left\|q^{\pm}\right\|_{2.25} \leq CS(t) \leq C_M.$$

Similarly, by the fundamental theorem of calculus

$$\left|\partial_N q^{\pm}(t)\right| \geq \left|\partial_N q_0^{\pm}\right| - \left|\int_0^t \partial_s \left(\partial_N q^{\pm}\right)(s)\, ds\right| \geq \left|\partial_N q_0^{\pm}\right| - C\sqrt{t}\|q_t\|_2 \geq \delta - C\sqrt{t}M.$$

Therefore, for small times we have the lower bound

$$\left|\partial_N q^{\pm}\right| \geq c_{\delta,M} > 0.$$

By the maximum principle we conclude that

$$c_{\delta,M} e^{(-\lambda_1+\eta)t} \leq \min_{x\in\Gamma} W^{\pm}(t,x) \leq \max_{x\in\Gamma} W^{\pm}(t) \leq C_{\delta,M} e^{(-\lambda_1+\eta)t}.$$

Taking t so small that $1/2 \leq e^{(-\lambda_1+\eta)t} \leq 1$, we obtain the result. ∎

The natural energy $\mathcal{E}_\kappa(t)$. The following definition of the "natural" energy is seemingly technical, but as it will become apparent in section 3.10 it is precisely the natural higher-order positive definite quantity arising from an integration-by-parts argument.

DEFINITION 3.7 (higher-order weighted energy $\mathcal{E}_\kappa(t)$ and dissipation functional $\mathcal{D}_\kappa(t)$). *Let $q^{\pm} : \Omega^{\pm} \to \mathbb{R}$, $h : \Gamma \to \mathbb{R}$, and recall the cutoff function μ from* (1.6).

We set

$$
\begin{aligned}
&\mathcal{E}^{\pm}_{\kappa}(t)\\
&:= \frac{1}{2}\sum_{a+2b\le 5}\Big(\|\mu^{1/2}\bar\partial^a\partial_t^b v^{\pm}\|^2_{L^{2,W^{\pm}}} + \kappa^2 e^{(-\lambda_1+\eta)t}|\sqrt{r^{\pm}_{\kappa}}\bar\partial^a\partial_t^b v^{\pm}\cdot n^{\kappa}|^2_{L^2(\Gamma)}\Big)\\
&+ \frac{1}{2}\sum_{a+2b\le 6}\Big(\|\mu^{1/2}(\bar\partial^a\partial_t^b q^{\pm} + \bar\partial^a\partial_t^{b\,\kappa}\Psi^{\pm}\cdot v^{\pm})\|^2_{L^{2,W^{\pm}}} + e^{(-\lambda_1+\eta)t}|a_{\kappa}\bar\partial^a\partial_t^b\Lambda_{\kappa}h|^2_{L^2(\Gamma)}\Big)\\
&+ \frac{1}{2}\sum_{|a|+2b\le 5}\|(1-\mu)^{1/2}\partial^a\partial_t^b v^{\pm}\|^2_{L^2(\Omega^{\pm})}\\
&+ \frac{1}{2}\sum_{|a|+2b\le 6}\|(1-\mu)^{1/2}(\partial^a\partial_t^b q^{\pm} + \partial^a\partial_t^{b\,\kappa}\Psi^{\pm}\cdot v^{\pm})\|^2_{L^2(\Omega^{\pm})},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{D}^{\pm}_{\kappa}(t)\\
&:= \sum_{a+2b\le 6}\Big(\|\mu^{1/2}\bar\partial^a\partial_t^b v^{\pm}\|^2_{L^{2,W^{\pm}}} + \kappa^2 e^{(-\lambda_1+\eta)t}|\sqrt{r^{\pm}_{\kappa}}\bar\partial^a\partial_t^b v^{\pm}\cdot n^{\kappa}|^2_{L^2(\Gamma)}\Big)\\
&+ \sum_{a+2b\le 5}\Big(\|\mu^{1/2}(\bar\partial^a\partial_t^{b+1} q^{\pm} + \bar\partial^a\partial_t^{b+1\,\kappa}\Psi^{\pm}\cdot v^{\pm})\|^2_{L^{2,W^{\pm}}} + e^{(-\lambda_1+\eta)t}|a_{\kappa}\bar\partial^a\partial_t^{b+1}\Lambda_{\kappa}h|^2_{L^2(\Gamma)}\Big)\\
&+ \sum_{|a|+2b\le 6}\|(1-\mu)^{1/2}\partial^a\partial_t^b v^{\pm}\|^2_{L^2(\Omega^{\pm})}\\
&+ \sum_{|a|+2b\le 5}\|(1-\mu)^{1/2}(\partial^a\partial_t^{b+1} q^{\pm} + \partial^a\partial_t^{b+1\,\kappa}\Psi^{\pm}\cdot v^{\pm})\|^2_{L^2(\Omega^{\pm})},
\end{aligned}
$$

where $J_{\kappa} := \det \nabla^{\kappa}\Psi$ *is the determinant of the Jacobian,* g_{κ} *is defined by* $g_{\kappa} := (\bar\partial h^k)^2 + (1+H(x)h^{\kappa})^2$*, and the coefficients* $r^{\pm}_{\kappa}(t,x) := (\partial_N q^{\pm})^{-1}J_{\kappa}^{-2}g_{\kappa}$ *and* $a_{\kappa}(t,x) := J_{\kappa}^{-1}(1+Hh^{\kappa})$*.*

We remind the reader that the horizontal derivatives $\bar\partial$ are defined in section 1.5. We introduce the *total energy*

$$
\mathcal{E}_{\kappa}(t) := \sup_{0\le s\le t}\mathcal{E}^{+}_{\kappa}(s) + \sup_{0\le s\le t}\mathcal{E}^{-}_{\kappa}(s) + \int_0^t(\mathcal{D}^{+}_{\kappa}(s) + \mathcal{D}^{-}_{\kappa}(s))ds. \tag{3.40}
$$

Remark 3.8. For the proof of the *local well-posedness theorem*, we will show that the following a priori energy estimate holds,

$$
\mathcal{E}_{\kappa}(t) \le \mathcal{E}_{\kappa}(0) + C\sqrt{t}P(\mathcal{E}_{\kappa}(t)), \tag{3.41}
$$

where $P(\cdot)$ is some polynomial of degree greater than or equal to one, but that it does not depend on κ. A simple continuity argument then yields Theorem 2.1. A more careful energy estimate combined with a maximum principle argument gives us the *global stability result*, which is explained in section 4.

3.4. Local-in-time energy control. Assuming that the Rayleigh–Taylor condition (1.25) holds, we shall prove in this section that the control over the derivatives of $q^{\pm}$ and h^{κ} provided by the norm $\mathcal{S}_{\kappa}(t)$ is dominated by the energy $\mathcal{E}_{\kappa}(t)$ defined by (3.40).

PROPOSITION 3.9. *Suppose the Taylor sign condition* (1.25) *holds for some* $\delta > 0$, *then the norm* $\mathcal{S}_\kappa(t)$ *is equivalent to* $\mathcal{E}_\kappa(t)$ *in the sense that*

$$\mathcal{S}_\kappa(t) \le P(\mathcal{E}_\kappa(t)) \tag{3.42}$$

for any t *on the interval of definition of* $\mathcal{E}_\kappa(t)$ *and* $\mathcal{S}_\kappa(t)$ *and for* P *a universal polynomial as described in* 1.3.

Proof. The proof of this proposition follows exactly as the proof of [30, Proposition 2.4], but with the weights $W^\pm$. First, the contribution from the boundary terms is easy to bound, since, instead of having the weight $\partial_N q$ in our energy, we have the coefficient $e^{(-\lambda_1+\eta)t}$. Also we have the estimate

$$J^{-1}(1 + Hh^\kappa) = 1 + O(|h^\kappa|) \ge \frac{1}{2},$$

where we used the characterization of J from subsection A.3 of the appendix, the bound for the curvature H, and that for short time h^κ is small. Recall that H is the curvature of the smooth reference curve, thus it is *σ-close* to the original initial interface. Therefore

$$\mathcal{E}^\Gamma_{\mathrm{loc}}(t) \le C \sum_{a+2b\le 6} \left| J^{-1}(1 + Hh^\kappa)\bar\partial^a \partial_t^b \Lambda_\kappa h\right|^2_{L^\infty_t L^2(\Gamma)},$$

$$\mathcal{D}^\Gamma_{\mathrm{loc}}(t) \le C \sum_{a+2b\le 5} \left| J^{-1}(1 + Hh^\kappa)\bar\partial^a \partial_t^{b+1} \Lambda_\kappa h\right|^2_{L^2(\Gamma)}.$$

Second, note that both $\mathcal{S}_\kappa(t)$ and $\mathcal{E}_\kappa(t)$ have terms of the form $\|\mu^{1/2}\bar\partial^a\partial_t^b v\|_0^2$ with the difference that in $\mathcal{E}_\kappa(t)$, the norm has the weights $W^\pm$. The upper and lower bounds for $W^\pm$ in Lemma 3.6, gives us that these terms satisfy inequality (3.42) directly.

Finally, we need to show that we can control all derivatives in the interior by controlling only the tangential ones that appear on $\mathcal{E}_\kappa(t)$, but this process is analogous to the proof of estimate (a) from [30, Proposition 2.4], multiplying and dividing by the coefficient $W^\pm$ on the corresponding integrals over $\Omega^\pm$. We obtain

$$\sum_{|a|+2b\le 6} \left\|\partial^a\partial_t^b q^\pm\right\|_{L^\infty_t L^2(\Omega^\pm)} \le P\left(\frac{\mathcal{E}_\kappa(t)}{\inf W^\pm}\right) \le P(\mathcal{E}_\kappa(t)),$$

$$\sum_{|a|+2b\le 6} \left\|\partial^a\partial_t^b q^\pm\right\|_{L^2_t H^{0.5}(\Omega^\pm)} \le P\left(\frac{\mathcal{E}_\kappa(t)}{\inf W^\pm}\right) \le P(\mathcal{E}_\kappa(t)),$$

where the last inequalities follow again from the local estimates Lemma 3.6 for $W^\pm$. This concludes the proof. ∎

3.5. Derivation of the energy identities. For the various notations used in this section we encourage the reader to consult subsections 1.3–1.5.

LEMMA 3.10. *Let* $(q^\pm, h^\kappa)$ *be a smooth solution to the two-phase Stefan problem given by Theorem* 3.4 *on the time interval* $[0, T_\kappa]$. *Then the following energy identity holds:*

$$\frac{d}{dt}\mathcal{E}_\kappa(t) = \frac{d}{dt}(\mathcal{E}^+_\kappa(t) + \mathcal{E}^-_\kappa(t)) + \mathcal{D}^+_\kappa(t) + \mathcal{D}^-_\kappa(t) = \mathcal{R}(t), \tag{3.43}$$

where the right-hand term $\mathcal{R}(t)$ *is an error given explicitly in Lemma* A.8 *of the appendix.*

Proof. Apply the operator $\bar\partial^a\partial_t^b$ to (3.4b) for $0 \le a + 2b \le 6$, multiply by $\bar\partial^a\partial_t^b v W\mu$, and integrate over $\Omega^\pm$, respectively. The inclusion of the factor $W^\pm$ is very important as it allows us to recover the positive definite boundary energy by providing a common factor on both regions to form the difference $h_t = v^+ \cdot \tilde{n}^\kappa - v^- \cdot \tilde{n}^\kappa$. We obtain the identity

$$\left(\bar\partial^a \partial_t^b v + \bar\partial^a \partial_t^b \ {}^\kappa\!A^i_j q,_i + {}^\kappa\!A^i_j \bar\partial^a \partial_t^b q,_i \ , \ \bar\partial^a \partial_t^b v W \mu\right)_{L^2(\Omega^\pm)} = \mathcal{R}_1^\pm, \tag{3.44}$$

where $\mathcal{R}_1^\pm$ is the error term that contains the lower-order derivatives arising from the application of the product rule to ${}^\kappa\!A^i_j q,_i$, integrated over the regions $\Omega^\pm$, respectively. The process then follows the same methodology as in the proof of [31, Proposition 3.1], or the proofs of [30, Lemmas 2.2 and 2.3], with the added weights $W^\pm$. A few new error terms appear while integrating by parts, as some derivatives fall on $W^\pm$. Additional new terms appear to the fixed exterior boundary $\partial\Omega$, but there are no new ideas in the process. We describe the effect of this weight on the boundary terms, but omit the rest for brevity.

Analogously to the appendix of [31, (A.1)], when the derivatives $\bar\partial^a \partial_t^b$ hit the matrix A we have the following identity,

$$\bar\partial^a \partial_t^b \ {}^\kappa\!A^i_j = -{}^\kappa\!A^i_l \bar\partial^a \partial_t^b \Psi^l_\kappa,_m \ {}^\kappa\!A^m_j - \sum_{\substack{1 \le s \le a \\ 1 \le l \le b}} c_{sl} \bar\partial^{a-s} \partial_t^{b-l} \ {}^\kappa\!A^i_k \bar\partial^s \partial_t^l \Psi^k_\kappa,_m \ {}^\kappa\!A^m_j .$$

Therefore, the second term of (3.44) becomes

$$\left(\bar\partial^a \partial_t^b \ {}^\kappa\!A^T \nabla q, \ \bar\partial^a \partial_t^b v \mu W\right)_{L^2(\Omega^\pm)} = -\int_{\Omega^\pm} {}^\kappa\!A \bar\partial^a \partial_t^b \nabla^\kappa\Psi {}^\kappa\!A \nabla q \bar\partial^a \partial_t^b v W \mu - \mathcal{R}_2^\pm ,$$

where $\mathcal{R}_2^\pm$ is the error term containing the lower-order derivatives hitting $\nabla^\kappa\Psi$. We will specify briefly a computation over the two regions $\Omega^\pm$ as there is a small difference when integrating by parts over the region Ω^+ since it has an exterior fixed boundary $\partial\Omega$. We have, after integrating by parts,

$$\begin{aligned}
&-\int_{\Omega^+} {}^\kappa\!A \bar\partial^a \partial_t^b \nabla^\kappa\Psi {}^\kappa\!A \nabla q \bar\partial^a \partial_t^b v W \mu \\
&\quad = -\int_\Gamma {}^\kappa\!A^l_i \bar\partial^a \partial_t^b \Psi^i_\kappa {}^\kappa\!A^k_j q,_l \, \bar\partial^a \partial_t^b v^j W \left(-N^k\right) + \int_{\partial\Omega} \left(\bar\partial^a \partial_t^{b\,\kappa}\Psi \cdot v\right) \left(\bar\partial^a \partial_t^b v \cdot \mathbf{N}^+\right) W \\
&\qquad + \int_{\Omega^+} \bar\partial^a \partial_t^b \Psi^i_\kappa {}^\kappa\!A^k_j \left({}^\kappa\!A^l_i q,_l \, \bar\partial^a \partial_t^b v W \mu\right),_k .
\end{aligned}$$

Where the boundary condition (3.17d) implies that the term in $\partial\Omega$ is an error term, that we call $\mathcal{R}^{a,b}_{\partial\Omega_1}$, and, by the definition of W^+ on Γ (3.36b), the integral on Γ becomes

$$\begin{aligned}
&-\int_\Gamma {}^\kappa\!A^l_i \bar\partial^a \partial_t^b \Psi^i_\kappa {}^\kappa\!A^k_j q,_l \, \bar\partial^a \partial_t^b v^j W^+ \left(-N^k\right) \\
&\quad = \int_\Gamma \bar\partial^a \partial_t^b \Psi^i_\kappa {}^\kappa\!A^l_i q,_l \, \bar\partial^a \partial_t^b v^j {}^\kappa\!A^k_j N^k \frac{e^{(-\lambda_1+\eta)t}}{\partial_N q^+} \\
&\quad = \int_\Gamma \bar\partial^a \partial_t^b \Psi^i_\kappa {}^\kappa\!A^l_i \left(\left(\nabla q^+ \cdot \tau\right) \tau^l + \left(\nabla q^+ \cdot N\right) N^l\right) \bar\partial^a \partial_t^b v^j {}^\kappa\!A^k_j N^k \frac{e^{(-\lambda_1+\eta)t}}{\partial_N q^+} \\
&\quad = e^{(-\lambda_1+\eta)t} \int_\Gamma \left(\bar\partial^a \partial_t^{b\,\kappa}\Psi \cdot {}^\kappa\!A^T N\right) \left(\bar\partial^a \partial_t^b v^+ \cdot {}^\kappa\!A^T N\right) d\sigma + \mathcal{R}^{a,b}_{\Gamma_1} \\
&\quad = e^{(-\lambda_1+\eta)t} \int_\Gamma J_\kappa^{-2} g_\kappa \left(\bar\partial^a \partial_t^{b\,\kappa}\Psi \cdot n^\kappa\right) \left(\bar\partial^a \partial_t^b v^+ \cdot n^\kappa\right) d\sigma + \mathcal{R}^{a,b+}_{\Gamma_1},
\end{aligned}$$

where

$$\mathcal{R}^{a,b+}_{\Gamma_1} := e^{(-\lambda_1+\eta)t}\int_\Gamma \bar\partial^a\partial_t^b {}^\kappa\Psi^i {}^\kappa A^l_i \tau^l \bar\partial\left(-\kappa^2 v^+\cdot {}^\kappa A^\top N + \kappa^2\beta^+\right)\left(\bar\partial^a\partial_t^b v^+\cdot n^\kappa\right)\frac{J_\kappa^{-1}\sqrt{g_\kappa}}{\partial_N q^+}d\sigma,$$

since the term $\nabla q^+\cdot N = \partial_N q^+$, and q^+ along Γ is given by (3.4c). The last equalities follows from the geometric identity ${}^\kappa A^T N = J_\kappa^{-1}\sqrt{g_\kappa}n^\kappa$, where n^κ is the normal vector to the moving domain.

We obtain therefore,

$$\begin{aligned}
&\|\mu^{1/2}\bar\partial^a\partial_t^b v^+\Big\|^2_{L^{2,W^+}} + \frac{1}{2}\frac{d}{dt}\|\mu^{1/2}\left(\bar\partial^a\partial_t^b q + \bar\partial^a\partial_t^b {}^\kappa\Psi\cdot v^+\right)\Big\|^2_{L^{2,W^+}}\\
&\quad + e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}g_\kappa\left(\bar\partial^a\partial_t^b {}^\kappa\Psi\cdot n^\kappa\right)\left(\bar\partial^a\partial_t^b v^+\cdot n^\kappa\right)d\sigma\\
&\quad + \kappa^2 e^{(-\lambda_1+\eta)t}\int_\Gamma r_\kappa^+\left|\bar\partial^a\partial_t^b v^+\cdot n^\kappa\right|^2 d\sigma\\
&= \mathcal{R}^+_{a,b} + \mathcal{R}^{a,b+}_{\Gamma_1} + \mathcal{R}^{a,b\ +}_{\partial\Omega_1},
\end{aligned}$$

where we have gathered all the residue terms of the interior into $\mathcal{R}^+_{a,b}$. An analogous process now with W^- in the region Ω^- gives

$$\begin{aligned}
&\left\|\mu^{1/2}\bar\partial^a\partial_t^b v^-\right\|^2_{L^{2,W^-}} + \frac{1}{2}\frac{d}{dt}\left\|\mu^{1/2}\left(\bar\partial^a\partial_t^b q^- + \bar\partial^a\partial_t^b {}^\kappa\Psi\cdot v^-\right)\right\|^2_{L^{2,W^-}}\\
&\quad - e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}g_\kappa\left(\bar\partial^a\partial_t^b {}^\kappa\Psi\cdot n^\kappa\right)\left(\bar\partial^a\partial_t^b v^-\cdot n^\kappa\right)d\sigma\\
&\quad + \kappa^2 e^{(-\lambda_1+\eta)t}\int_\Gamma r_\kappa^-\left|\bar\partial^a\partial_t^b v^-\cdot n^\kappa\right|^2 d\sigma\\
&= \mathcal{R}^-_{a,b} + \mathcal{R}^{a,b-}_{\Gamma_1}.
\end{aligned}$$

Hence, adding together the terms from both regions we obtain

$$\begin{aligned}
&\left\|\mu^{1/2}\bar\partial^a\partial_t^b v^+\right\|^2_{L^{2,W^+}} + \frac{1}{2}\frac{d}{dt}\left\|\mu^{1/2}\left(\bar\partial^a\partial_t^b q^+ + \bar\partial^a\partial_t^b \Psi^+\cdot v^+\right)\right\|^2_{L^{2,W^+}}\\
&\quad + \left\|\mu^{1/2}\bar\partial^a\partial_t^b v^-\right\|^2_{L^{2,W^-}} + \frac{1}{2}\frac{d}{dt}\left\|\mu^{1/2}\left(\bar\partial^a\partial_t^b q^- + \bar\partial^a\partial_t^b \Psi^-\cdot v^-\right)\right\|^2_{L^{2,W^-}}\\
&\quad + \kappa^2 e^{(-\lambda_1+\eta)t}\left(\left|\sqrt{r_\kappa^+}\left(\bar\partial^a\partial_t^b v^+\cdot n^\kappa\right)\right|^2_{L^2(\Gamma)} + \left|\sqrt{r_\kappa^-}\left(\bar\partial^a\partial_t^b v^-\cdot n^\kappa\right)\right|^2_{L^2(\Gamma)}\right)\\
&\quad + e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}g_\kappa\left(\bar\partial^a\partial_t^b {}^\kappa\Psi^+\cdot n^\kappa\ \bar\partial^a\partial_t^b v^+\cdot n^\kappa - \bar\partial^a\partial_t^b {}^\kappa\Psi^-\cdot n^\kappa\ \bar\partial^a\partial_t^b v^-\cdot n^\kappa\right)d\sigma\\
&= \mathcal{R}^+_{a,b} + \mathcal{R}^-_{a,b} + \mathcal{R}^{a,b+}_{\Gamma_1} + \mathcal{R}^{a,b-}_{\Gamma_1} + \mathcal{R}^{a,b\ +}_{\partial\Omega_1}.
\end{aligned} \tag{3.45}$$

On the last boundary term of (3.45) we can factor the $(\bar\partial^a\partial_t^b {}^\kappa\Psi^\pm\cdot n^\kappa)$ term, since on the boundary Γ, both ${}^\kappa\Psi^+$ and ${}^\kappa\Psi^-$ are the same, obtaining

$$\begin{aligned}
I_\Gamma &:= e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}g_\kappa(\bar\partial^a\partial_t^b {}^\kappa\Psi\cdot n^\kappa)\left(\bar\partial^a\partial_t^b v^+\cdot n^\kappa - \bar\partial^a\partial_t^b v^-\cdot n^\kappa\right)d\sigma\\
&= e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}g_\kappa(\bar\partial^a\partial_t^b {}^\kappa\Psi\cdot n^\kappa)[\bar\partial^a\partial_t^b v\cdot n^\kappa]^+_- d\sigma.
\end{aligned} \tag{3.46}$$

Recall that on the boundary Γ, ${}^{\kappa}\Psi(t,x) = x + h^{\kappa}(t,x)N(x)$, therefore,

$$\bar\partial^a \partial_t^b {}^{\kappa}\Psi = \bar\partial^a\partial_t^b x + \bar\partial^a\partial_t^b h^k N + h^\kappa \bar\partial^a \partial_t^b N + \sum_{s,l} \bar\partial^{a-l}\partial_t^{b-s} h^\kappa \bar\partial^l \partial_t^s N,$$

and the normal vector

$$n^\kappa = \frac{-\bar\partial h^\kappa \tau + (1+H(x)h^\kappa)N}{\sqrt{(\bar\partial h^k)^2 + (1+H(x)h^\kappa)^2}} = \left(-\bar\partial h^\kappa \tau + (1+H(x)h^\kappa)N\right)\frac{1}{\sqrt{g_\kappa}}.$$

Then the boundary term (3.46) can be rewritten as

$$I_\Gamma = e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}\bar\partial^a\partial_t^b h^k (1+Hh^\kappa)^2\left[\bar\partial^a\partial_t^b v\cdot\left(N - \frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau\right)\right]_-^+ - \mathcal{R}_{\Gamma_2}^{a,b},$$

where

$$\begin{aligned}\mathcal{R}_{\Gamma_2}^{a,b} = -e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}(1+Hh^\kappa)&\left(\left(h^\kappa\bar\partial^a\partial_t^b N + \bar\partial^a\partial_t^b x + \sum_{s,l}\bar\partial^{a-l}\partial_t^{b-s}h^\kappa\bar\partial^l\partial_t^s N\right)\right.\\ &\qquad\qquad\left.\cdot\left(-\bar\partial h^\kappa\tau + (1+Hh^\kappa)N\right)\right)\\ &\cdot\left[\bar\partial^a\partial_t^b v\cdot(N - \frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau)\right]_-^+.\end{aligned}$$

Recall from (3.28),

$$h_t = \left[v\cdot\left(N - \frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau\right)\right]_-^+;$$

then we have

$$I_\Gamma = e^{(-\lambda_1+\eta)t}\int_\Gamma a_\kappa^2\left(\bar\partial^a\partial_t^b h^\kappa\right)\left(\bar\partial^a\partial_t^b h_t\right) - \mathcal{R}_{\Gamma_2}^{a,b} - \mathcal{R}_{\Gamma_3}^{a,b},$$

where

$$\begin{aligned}\mathcal{R}_{\Gamma_3}^{a,b} := e^{(-\lambda_1+\eta)t}\int_\Gamma a_\kappa^2\bar\partial^a\partial_t^b h^\kappa&\left(\left[v\cdot\bar\partial^a\partial_t^b(N - \frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau\right]_-^+\right.\\ &\left.+\sum_{s,l}c_{sl}\left[\bar\partial^{a-l}\partial_t^{b-s}v\cdot\bar\partial^l\partial_t^s\left(N - \frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau\right)\right]_-^+\right).\end{aligned}$$

First, observe that in this error there is a higher-order term hidden when the highest-order derivatives hit the tangential derivative of h^κ, i.e., a term of the form $\bar\partial^{a+1}\partial_t^b h^\kappa$, and must be considered carefully when we prove the energy estimates. Second, notice that one of the factors has the regularized h^κ, but the other factor is not regularized, since it comes from (3.28). Therefore, we must commute the smoothing operator Λ_κ from h^κ to h_t, to form the quadratic energy term. Recall that this operator commutes with the tangential derivatives and therefore

$$\begin{aligned} I_\Gamma &= e^{(-\lambda_1+\eta)t} \int_\Gamma a_\kappa^2 \left(\bar{\partial}^a \partial_t^b \Lambda_\kappa \Lambda_k h\right) \left(\bar{\partial}^a \partial_t^b h_t\right) - \mathcal{R}_{\Gamma_2}^{a,b} - \mathcal{R}_{\Gamma_3}^{a,b} \\ &= e^{(-\lambda_1+\eta)t} \int_\Gamma a_\kappa^2 \left(\bar{\partial}^a \partial_t^b \Lambda_k h\right) \left(\bar{\partial}^a \partial_t^b \Lambda_k h_t\right) - \mathcal{R}_{comm}^{a,b} - \mathcal{R}_{\Gamma_2}^{a,b} - \mathcal{R}_{\Gamma_3}^{a,b}, \end{aligned}$$

where $\mathcal{R}_{comm}^{a,b}$ is a commutation error given by

$$\mathcal{R}_{comm}^{a,b} = -e^{(-\lambda_1+\eta)t} \int_\Gamma \bar{\partial}^a \partial_t^b (\Lambda_\kappa h) \left[\Lambda_\kappa, a_\kappa^2 \bar{\partial}^a \partial_t^b\right] h_t.$$

Pulling out a time derivative and grouping the error terms, we recover the positive definite energy term,

$$I_\Gamma = \frac{1}{2}\frac{d}{dt}\left(e^{(-\lambda_1+\eta)t} \int_\Gamma a_\kappa^2 \left(\bar{\partial}^a \partial_t^b \Lambda_\kappa h\right)^2\right) - \mathcal{R}_{\Gamma_4}^{a,b},$$

where $\mathcal{R}_{\Gamma_4}^{a,b}$ is given by

$$\mathcal{R}_{\Gamma_4}^{a,b} = \frac{1}{2}\int_\Gamma \partial_t \left(e^{(-\lambda_1+\eta)t} a_\kappa^2\right) \left(\bar{\partial}^a \partial_t^b \ \Lambda_\kappa h\right)^2 + \mathcal{R}_{comm}^{a,b} + \mathcal{R}_{\Gamma_2}^{a,b} + \mathcal{R}_{\Gamma_3}^{a,b}.$$

Collecting all together we have

$$\begin{aligned} &\left\|\mu^{1/2} \bar{\partial}^a \partial_t^b v^+\right\|_{L^{2,W^+}}^2 + \frac{1}{2}\frac{d}{dt}\left\|\mu^{1/2}\left(\bar{\partial}^a \partial_t^b q^+ + \bar{\partial}^a \partial_t^{b\kappa}\Psi^+ \cdot v^+\right)\right\|_{L^{2,W^+}}^2 \\ &\quad + \left\|\mu^{1/2} \bar{\partial}^a \partial_t^b v^-\right\|_{L^{2,W^-}}^2 + \frac{1}{2}\frac{d}{dt}\left\|\mu^{1/2}\left(\bar{\partial}^a \partial_t^b q^- + \bar{\partial}^a \partial_t^{b\kappa}\Psi^- \cdot v^-\right)\right\|_{L^{2,W^-}}^2 \\ &\quad + \kappa^2 e^{(-\lambda_1+\eta)t}\left(\left|r_\kappa^+ \left(\bar{\partial}^a \partial_t^b v^+ \cdot n^\kappa\right)\right|_{L^2(\Gamma)}^2 + \left|r_\kappa^- \left(\bar{\partial}^a \partial_t^b v^- \cdot n^\kappa\right)\right|_{L^2(\Gamma)}^2\right) \\ &\quad + \frac{1}{2}\frac{d}{dt}\left(e^{(-\lambda_1+\eta)t}\int_\Gamma a_\kappa^2 \left(\bar{\partial}^a \partial_t^b \Lambda_\kappa h\right)^2 d\sigma\right) \\ &= \mathcal{R}_{a,b}^+ + \mathcal{R}_{a,b}^- + \mathcal{R}_\Gamma^{a,b} + \mathcal{R}_{\partial\Omega}^{a,b}, \end{aligned}$$

where $\mathcal{R}_\Gamma^{a,b}$ contains all the error terms in the boundary Γ and $\mathcal{R}_{\partial\Omega}^{a,b}$ the errors in $\partial\Omega$. An analogous analysis can be done to obtain energy identities of the second type by considering the differential operator $\bar{\partial}^a \partial_t^{b+1}$ to (3.17b), and multiplying by $\bar{\partial}^a \partial_t^b v^\pm W^\pm \mu$. For the interior derivatives we consider the differential operator $\partial^a \partial_t^b$, where a is now a multi-index, and $\partial^a = \partial_{x_1}^{a_1} \partial_{x_2}^{a_2}$ is a combination of derivatives in all Cartesian directions. The result follows by summing over the corresponding values of a, b. See [31] for more details. ∎

Remark 3.11. In contrast to [31], when the time derivative is applied to the weight $e^{(-\lambda_1+\eta)t}$, we obtain the obvious inequality

$$(-\lambda_1+\eta)e^{(-\lambda_1+\eta)t}\int_\Gamma a_\kappa^2 \left(\bar{\partial}^a \partial_t^b \ \Lambda_\kappa h\right)^2 < 0 \quad \text{for } \eta < \lambda_1.$$

Therefore, we do not need to prove estimates for this energy-critical term as it is sign definite with a favorable sign. In particular, many of the technical complications from [31] are eliminated.

3.6. Energy estimates for the local theory. In this section we prove energy estimates for the solutions of the regularized problem (3.17). The aim is to obtain κ-independent estimates, and therefore a uniform-in-κ time of existence for our family of regularized solutions. We will accomplish this by using the energy identity (3.43), and bounding the error terms in $\mathcal{R}(t)$. As a first step, we prove short time a priori bounds for $\nabla W^{\pm}$ and $W_t^{\pm}$.

LEMMA 3.12. *Under the assumptions of Theorem* 2.1*, the derivatives of the weight functions* $W^{\pm}$ *satisfy the following bounds,*

$$\left\|\nabla W^{\pm}\right\|_{L^\infty(\Omega^\pm)} + \left\|W_t^{\pm}\right\|_{L^\infty(\Omega^\pm)} \leq C(1 + P(\mathcal{E}_\kappa(t))).$$

Proof. Without loss of generality we will show only the estimates in Ω^+. Since W^+ satisfies (3.36), standard elliptic estimates and the Sobolev inequality give

$$\begin{aligned}\left\|\nabla W^{+}\right\|_{L^\infty} &\leq C\left\|W^+\right\|_{2.25} \leq Ce^{(-\lambda_1+\eta)t}\left|\frac{1}{\partial_N q^+}\right|_2 + Ce^{(-\lambda_1+\lambda_1^++\eta)t}\\ &\leq Ce^{(-\lambda_1+\lambda_1^++\eta)t}\left(1 + |\frac{1}{\partial_N q^+}|_2\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|\bar\partial^2\left(\frac{1}{\partial_N q^+}\right)\right|_{L^2} &\leq \frac{|\bar\partial^2\partial_N q^+|_{L^2}}{\delta^2} + \frac{|\bar\partial\partial_N q^+|^2_{L^4}}{\delta^3}\\ &\leq \frac{\|q^+\|_{3.5}}{\delta^2} + \frac{\|q^+\|_3^2}{\delta^3} \leq CP(\mathcal{E}_\kappa^+(t)),\end{aligned}$$

where we used the lower bound for $\partial_N q^+$ from (1.25). The other components of the $H^2(\Gamma)$ norm follow similarly, therefore,

$$\left\|\nabla W^+\right\|_{L^\infty} \leq Ce^{(-\lambda_1+\lambda_1^++\eta)t}\left(1 + P(\mathcal{E}_\kappa^+(t))\right).$$

Taking time short enough so that $e^{(-\lambda_1+\lambda_1^++\eta)t} \leq 2$ gives the desired bound. Next, W_t^+ satisfies the following problem,

$$\Delta W_t^+ = 0 \quad \text{in } \Omega^+, \tag{3.47a}$$

$$W_t^+ = e^{(-\lambda_1+\eta)t}\left(\frac{(-\lambda_1+\eta)}{\partial_N q^+} - \frac{\partial_N q_t^+}{(\partial_N q^+)^2}\right) \quad \text{on } \Gamma, \tag{3.47b}$$

$$W_t^+ = \left(-\lambda_1 + \lambda_1^+ + \eta\right)e^{(-\lambda_1+\lambda_1^++\eta)t} \quad \text{on } \partial\Omega^+_{\text{fixed}}. \tag{3.47c}$$

On the interface Γ,

$$\begin{aligned}|W_t^+|_{L^\infty(\Gamma)} &\leq Ce^{(-\lambda_1+\eta)t}\left(\frac{|-\lambda_1+\eta|}{\delta} + \frac{|\partial_N q_t^+|}{\delta^2}\right)\\ &\leq Ce^{(-\lambda_1+\eta)t}\left(1 + \|q_t^+\|_{2.75}\right) \leq Ce^{(-\lambda_1+\eta)t}\left(1 + P(\mathcal{E}_\kappa^+(t))\right).\end{aligned}$$

Therefore, by the maximum principle,

$$\|W_t^+\|_{L^\infty} \leq Ce^{(-\lambda_1+\lambda_1^++\eta)t}(1 + P(\mathcal{E}_\kappa^+(t))),$$

which again, for a sufficiently short time, gives the result. $\square$

LEMMA 3.13 (higher regularity for q). *We have the following inequality,*

$$\int_0^t \left\|q^{\pm}(s)\right\|_7^2 + \left\|{}^{\kappa}\Psi^{\pm}(s)\right\|_7^2 ds \leq C\mathcal{E}_\kappa(t). \tag{3.48}$$

Proof. The proof of this lemma follows the same argument detailed in the proof of [30, Lemma 2.4], so we will omit it for economy. ∎

PROPOSITION 3.14. *For each $\kappa > 0$, the energy function $\mathcal{E}_\kappa(t)$ is continuous in $[0, T_\kappa]$, and there exists a constant C and a polynomial P, both independent of κ, such that the following bound holds:*

$$\mathcal{E}_\kappa(t) \leq C\mathcal{E}_\kappa(0) + C\sqrt{t}P(\mathcal{E}_\kappa(t)) \text{ for all } \ t \in [0, T_\kappa]. \tag{3.49}$$

Proof. First we show that the map $t \to \mathcal{E}_\kappa(t)$ is continuous on $[0, T_\kappa]$. The continuity of the terms of the type $L^2([0,t]; H^s)$ follows from the fundamental theorem of calculus, and, for the norms $\|\partial_t^l q^{\pm}(t)\|_{H^{6-2l}}$, continuity follows from the parabolic regularity estimates that we carried out in section 3.2.2 for the solution of the regularized equation (3.4).

Using the definition of h_t (3.4d), we can obtain the continuity of $h_t \in C([0, T_\kappa]; H^4(\Gamma))$ and $h_{tt} \in C([0, T_\kappa]; H^2(\Gamma))$, from the trace estimates for $v^{\pm}$ and $v_t^{\pm}$, respectively, combined with the already known continuity of q and q_t. Moreover, the continuity of $h_{ttt} \in C([0, T_\kappa]; L^2(\Gamma))$ can be obtained from the three-time-differentiated definition of h_t in (3.4d), since, for a fixed $\kappa > 0$, we have control of $\kappa|v_{ttt} \cdot n^\kappa|_{L_t^2 L^2(\Gamma)}$, so the fundamental theorem of calculus gives us the desired continuity.

Finally, from the higher regularity estimate (3.48), we have the norm $h \in L^2([0, T_\kappa]; H^{6.5}(\Gamma))$. This estimate, combined with the control of $\kappa|\bar{\partial}^6 v \cdot n^\kappa|_{L_t^2 L^2}$, which implies that $h_t \in L^2([0, T_\kappa]; H^6(\Gamma))$, gives us by interpolation that $h \in C([0, T_\kappa]; H^6(\Gamma))$.

Now we will prove the estimate (3.49). The argument consists of carefully bounding the error terms from the energy identity (3.43). Most of these estimates are done analogously to the proof of [30, Proposition 2.5], so we will address first the new error terms that appear as a consequence of having two interacting phases. These terms appear in the last line of the definitions of $\mathcal{R}_{a,b}^{\pm}$, $\widetilde{\mathcal{R}}_{a,b}^{\pm}$ with derivatives of the weight function $W^{\pm}$. Consider, for example, on the positive region Ω^+ the errors

$$\mathcal{R}_5^{a,b+} := \int_0^t \int_{\Omega^+} \left(\bar{\partial}^a \partial_t^b q^+ + \bar{\partial}^a \partial_t^b \Psi^+ \cdot v^+\right) A^+ \bar{\partial}^a \partial_t^b v^+ \nabla W^+ \mu,$$
$$\mathcal{R}_6^{a,b+} := \frac{1}{2} \int_0^t \int_{\Omega^+} \left(\bar{\partial}^a \partial_t^b q^+ + \bar{\partial}^a \partial_t^b \Psi^+ \cdot v^+\right)^2 W_t^+ \mu.$$

By Lemma 3.12, $\mathcal{R}_5^{a,b+}$ can be easily bounded as follows:

$$\begin{aligned}
&\left|\mathcal{R}_5^{a,b+}\right| \\
&\leq \int_0^t \left\|\mu^{1/2} \left(\bar{\partial}^a \partial_t^b q^+ + \bar{\partial}^a \partial_t^b \Psi^+ \cdot v^+\right)\right\|_{L^{2,W^+}} \left\|A^+\right\|_{L^\infty} \left\|\mu^{1/2} \bar{\partial}^a \partial_t^b v^+\right\|_{L^{2,W^+}} \left\|\frac{\nabla W^+}{W^+}\right\|_{L^\infty} \\
&\leq C \int_0^t \mathcal{E}_\kappa^+(s)^{1/2} \mathcal{D}_\kappa^+(s)^{1/2} \left(1 + P\left(\mathcal{E}_\kappa^+(s)\right)\right) ds \\
&\leq P\left(\sup_{0 \leq s \leq t} \mathcal{E}_\kappa^+(s)\right) \int_0^t \mathcal{D}_\kappa^+(s)^{1/2} ds \leq \sqrt{t}P(\mathcal{E}_\kappa(t)),
\end{aligned}$$

where we used the bounds for A^+ from Lemma A.1, Hölder's inequality, and the definition of the energy $\mathcal{E}_\kappa(t)$. In the same way we bound $\mathcal{R}_6^{a,b+}$:

$$\left|\mathcal{R}_6^{a,b+}\right| \leq \frac{1}{2}\int_0^t \mathcal{E}_\kappa^+(s)\left\|\frac{W_t^+}{W^+}\right\|_{L^\infty} ds \leq \ t\ P\left(\sup_{0\leq s\leq t}\mathcal{E}_\kappa^+(s)\right).$$

The remaining error terms are dealt with in the same way as in the proof of [30, Proposition 2.5], since the only difference, the weight $W^\pm$, can be bounded in L^∞ using Lemma 3.6.

Boundary estimates. We write the boundary error $\mathcal{R}_\Gamma$ as a sum of its integral terms:

$$\mathcal{R}_\Gamma = \mathcal{R}_{\Gamma_1}^{a,b} + \mathcal{R}_{\Gamma_2}^{a,b} + \mathcal{R}_{\Gamma_3}^{a,b} + \mathcal{R}_{\Gamma_4}^{a,b} + \mathcal{R}_{comm}^{a,b}.$$

Estimates for $\mathcal{R}_{\Gamma_4}^{a,b}$. The third term, as we have mentioned previously, is very distinctive, since it has the same order as the energy. We will see now that, because of our choice of $W^\pm$, this is not a problem.

$$\begin{aligned}\int_0^t \mathcal{R}_{\Gamma_4}^{a,b}(s)ds &= \frac{1}{2}\int_0^t\int_\Gamma \partial_t\left(e^{(-\lambda_1+\eta)t}a_\kappa^2\right)\left(\bar\partial^a\partial_t^b\ \Lambda_\kappa h\right)^2 \\ &= (-\lambda_1+\eta)\int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}a_\kappa^2\left(\bar\partial^a\partial_t^b\Lambda_\kappa h\right)^2 \\ &\quad + \int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}\partial_t(a_\kappa^2)\left(\bar\partial^a\partial_t^b\Lambda_\kappa h\right)^2,\end{aligned}$$

where we recall the meaning of H from section 1.7, and a_κ from Definition 3.7. The first term is negative since $-\lambda_1+\eta<0$ for η small enough, so we can eliminate it from the estimates, and the second can be bounded by

$$\begin{aligned}\int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}\partial_t\left(a_\kappa^2\right)\left(\bar\partial^a\partial_t^b\Lambda_\kappa h\right)^2 &\leq \int_0^t \mathcal{E}_\kappa(s)\left|\frac{\partial_t(a_\kappa^2)}{a_\kappa^2}\right|_{L^\infty} \\ &\leq \int_0^t \mathcal{E}_\kappa(s)2\left|\frac{\partial_t J_\kappa}{J_\kappa}+\frac{Hh_t^\kappa}{(1+Hh^\kappa)}\right|_{L^\infty} \leq t\ P(\mathcal{E}_\kappa(t)),\end{aligned}$$

where we used the estimates for J_κ from Lemma A.3, the Sobolev embedding, and the evolution equation (3.4d) for h_t^κ.

Estimates for $\mathcal{R}_{\Gamma_2}^{a,b}$ and $\mathcal{R}_{\Gamma_3}^{a,b}$. In these errors there are *problematic* terms that contain higher-order derivatives of the normal vector to the reference curve. If the reference curve were the initial domain (which is H^6), we would have at most 6 tangential derivatives in L^2, which is not enough to bound $\bar\partial^6 N$, since N contains one derivative of the parametrization. Instead, the reference domain is a C^∞ curve, which is σ-close to the initial domain, described by a height function $h_0\in H^6$, and σ is a fixed, but small parameter. Our energy estimates will therefore depend on the parameter σ as well as the time of existence, but that will not interfere with the proof.

We write the first error terms as

(3.50)

$$
\begin{aligned}
&\int_0^t \mathcal{R}^{a,b}_{\Gamma_2}(s)ds \\
&\quad = \int_0^t \int_\Gamma e^{(-\lambda_1+\eta)s} J_\kappa^{-2}(1+Hh^\kappa)h^\kappa \left(\bar\partial^a\partial_t^b N\cdot\tau\right)\bar\partial h^\kappa \bar\partial^a\partial_t^b h_t \\
&\quad\quad - \int_0^t \int_\Gamma e^{(-\lambda_1+\eta)s} J_\kappa^{-2}(1+Hh^\kappa)h^\kappa \left(\bar\partial^a\partial_t^b N\cdot\tau\right)\bar\partial h^\kappa \left[v\cdot\bar\partial^a\partial_t^b\left(N-\frac{\bar\partial h^\kappa}{1+Hh^\kappa}\tau\right)\right]_-^+ \\
&\quad\quad + \text{ l.o.t.},
\end{aligned}
$$

where "l.o.t." stands for a collection of integral terms of the form

$$
\int_0^t \int_\Gamma f(x,t)dxdt, \quad \text{or} \quad \int_0^t \int_{\Omega^\pm} g(x,t)dxdt, \tag{3.51}
$$

that can be bounded to obtain estimate (3.49) using Hölder's inequality in a straightforward way. The first term of (3.50), after taking one copy of the smoothing operator Λ_κ from $h^\kappa = \Lambda_\kappa\Lambda_\kappa h$ and applying it to h_t, gives

(3.52)

$$
I_{\Gamma_2} := \int_0^t \int_\Gamma e^{(-\lambda_1+\eta)s} J_\kappa^{-2}\left(1+Hh^\kappa\right)\left(\Lambda_\kappa h\right)\left(\bar\partial^a\partial_t^b N\cdot\tau\right)\bar\partial h^\kappa\left(\bar\partial^a\partial_t^b\Lambda_\kappa h_t\right) + I_{\text{comm}},
$$

where I_{comm} is a lower-order commutator error given by

$$
I_{\text{comm}} = \int_0^t \int_\Gamma \Lambda_\kappa h\left[\Lambda_\kappa,\ e^{(-\lambda_1+\eta)s} J_\kappa^{-2}\left(1+Hh^\kappa\right)\left(\bar\partial^a\partial_t^b N\cdot\tau\right)\bar\partial h^\kappa\bar\partial^a\partial_t^b\right] h_t.
$$

The first term in (3.52) has too many tangential derivatives over $\Lambda_\kappa h_t$ since the index $a\le 6$, and the natural energy $\mathcal{E}_\kappa(t)$ can control at most 5 tangential derivatives of $\Lambda_\kappa h_t$. We integrate by parts to pass one derivative to the other terms which are lower order,

$$
\begin{aligned}
&I_{\Gamma_2} \\
&= -\int_0^t \int_\Gamma \bar\partial\left(e^{(-\lambda_1+\eta)s} J_\kappa^{-2}(1+Hh^\kappa)(\Lambda_\kappa h)(\bar\partial^a\partial_t^b N\cdot\tau)\bar\partial h^\kappa\right)(\bar\partial^{a-1}\partial_t^b\Lambda_\kappa h_t) + I_{\text{comm}} \\
&\le C\sqrt{t}|e^{(-\lambda_1+\eta)t/2}\sqrt{a_\kappa}\bar\partial^{a-1}\partial_t^b\Lambda_\kappa h_t|_{L^2_tL^2}\left|\frac{\bar\partial\left(J_\kappa^{-2}(1+Hh^\kappa)(\Lambda_\kappa h)(\bar\partial^a\partial_t^b N\cdot\tau)\bar\partial h^\kappa\right)}{J_\kappa^{-1}(1+Hh^\kappa)}\right|_{L^\infty_tL^2} \\
&\quad + \text{l.o.t.} \\
&\le \sqrt{t}\ \mathcal{E}_\kappa(t)^{1/2}P(\mathcal{E}_\kappa(t)).
\end{aligned}
$$

For the second integral term of (3.50), we notice that there is a higher-order term when the operator $\bar\partial^a\partial_t^b$ inside the braket hits the $\bar\partial h^\kappa$ coefficient. Integrating by parts we get

$$
\begin{aligned}
&I_{\Gamma_3}\\
&:= \int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}J_\kappa^{-2}(1+Hh^\kappa)h^\kappa(\bar\partial^a\partial_t^b N\cdot\tau)\bar\partial h^\kappa\left[(v\cdot\tau)\frac{\bar\partial^{a+1}\partial_t^b h^\kappa}{1+Hh^\kappa}\right]_-^+ + \text{l.o.t.}\\
&\;= -\int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}\bar\partial\left(J_\kappa^{-2}(1+Hh^\kappa)h^\kappa(\bar\partial^a\partial_t^b N\cdot\tau)\bar\partial h^\kappa\frac{[(v\cdot\tau)]_-^+}{(1+Hh^\kappa)}\right)\bar\partial^a\partial_t^b h^\kappa + \text{l.o.t.}\\
&\le Ct\,\left|e^{(-\lambda_1+\eta)t/2}J_\kappa^{-1}(1+Hh^\kappa)\bar\partial^a\partial_t^b h^\kappa\right|_{L_t^\infty L^2}P(\mathcal{E}_\kappa(t)) + \text{l.o.t.}\\
&\le C\;t\;\mathcal{E}_\kappa(t)^{1/2}P(\mathcal{E}_\kappa(t)).
\end{aligned}
$$

We now examine the error $\mathcal{R}^{a,b}_{\Gamma_3}$. Notice that a similar situation occurs when the derivatives $\bar\partial^a\partial_t^b$ inside the bracket hits the term $\bar\partial h^\kappa$, but observe that now we cannot just integrate by parts since there is another higher-order factor $\bar\partial^{a+1}\partial_t^b h^\kappa$ inside the integral. We extract a full derivative instead as follows:

$$
\begin{aligned}
&\int_0^t \mathcal{R}^{a,b}_{\Gamma_3}(s)ds\\
&\quad= \int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}J_\kappa^{-2}(1+Hh^\kappa)^2\bar\partial^a\partial_t^b h^\kappa\frac{[v\cdot\tau]_-^+}{(1+Hh^\kappa)}\bar\partial^{a+1}\partial_t^b h^\kappa\\
&\quad= \int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}J_\kappa^{-2}(1+Hh^\kappa)[v\cdot\tau]_-^+\frac{1}{2}\bar\partial(\bar\partial^a\partial_t^b h^\kappa)^2\\
&\quad= -\frac{1}{2}\int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}\bar\partial\left(J_\kappa^{-2}(1+Hh^\kappa)[v\cdot\tau]_-^+\right)(\bar\partial^a\partial_t^b h^\kappa)^2\\
&\quad\le C\;t\;|e^{(-\lambda_1+\eta)t/2}J_\kappa^{-1}(1+Hh^\kappa)\bar\partial^a\partial_t^b h^\kappa|^2_{L_t^\infty L^2}\left|\frac{\bar\partial\left(J_\kappa^{-2}(1+Hh^\kappa)[v\cdot\tau]_-^+\right)}{J_\kappa^{-2}(1+Hh^\kappa)^2}\right|_{L_t^\infty L^\infty}\\
&\quad\le\; t\;\mathcal{E}_\kappa(t)P(\mathcal{E}_\kappa(t)).
\end{aligned}
$$

Estimates for the commutation error $\mathcal{R}^{a,b}_{comm}$.

$$
\int_0^t \mathcal{R}^{a,b}_{comm}(s)ds = -\int_0^t\int_\Gamma e^{(-\lambda_1+\eta)s}\bar\partial^a\partial_t^b(\Lambda_\kappa h)\left[\Lambda_\kappa, a_\kappa^2\bar\partial^a\partial_t^b\right]h_t.
$$

To bound this term, we will use the commutation estimate described in [17, Lemma 5.1],

$$
\left|\Lambda_\kappa(f\bar\partial g) - f\Lambda_\kappa\bar\partial g\right|_0 \le C|f|_{W^{1,\infty}(\Gamma)}|g|_0,
$$

where the constant C does not depend on κ. Therefore,

$$
\begin{aligned}
\int_0^t \mathcal{R}^{a,b}_{comm}(s)ds &\le \int_0^t e^{(-\lambda_1+\eta)s}\left|\bar\partial^a\partial_t^b(\Lambda_\kappa h)\right|_0\left|\left[\Lambda_\kappa,\; a_\kappa^2\bar\partial^a\partial_t^b\right]h_t\right|_0\\
&\le P(\mathcal{E}_\kappa(t))\int_0^t e^{(-\lambda_1+\eta)s/2}\left|\left[\Lambda_\kappa,\; a_\kappa^2\bar\partial^a\partial_t^b\right]h_t\right|_0
\end{aligned}
$$

and

$$
\left|\left[\Lambda_\kappa,\; a_\kappa^2\bar\partial^a\partial_t^b\right]h_t\right|_0 \le \left|a_\kappa^2\right|_{W^{1,\infty}(\Gamma)}\left|\bar\partial^{a-1}\partial_t^b h_t\right|_0.
$$

Notice that, since there is not a smoothing operator Λ_κ on h_t, we cannot bound it directly by the dissipation term $\mathcal{D}_\kappa(t)$. Instead, we use the definition of h_t from (3.28). We have

$$\left|\bar\partial^{a-1}\partial_t^b h_t\right|_0 \le \left|\left[\bar\partial^{a-1}\partial_t^b v\cdot\left(N-\frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau\right)\right]_-^+\right|_0 + \left|\frac{[v\cdot\tau]_-^+}{1+Hh^\kappa}\right|_{L^\infty}\left|\bar\partial^a\partial_t^b h^\kappa\right|_0+\text{l.o.t.}$$

We notice that the second term, multiplied by $e^{(-\lambda_1+\eta)t/2}$, is an energy term that is bounded by the L^∞-norm in time. The first term now has the same number of derivatives on v as the dissipation term $\mathcal{D}_\kappa(t)$, yet it cannot be bounded by it as it is a boundary norm. We have instead the following bounds using the control of the norm $\mathcal{S}_\kappa(t)$ with $\mathcal{E}_\kappa(t)$,

$$\begin{aligned}&\left|\left[\bar\partial^{a-1}\partial_t^b v\cdot\left(N-\frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau\right)\right]_-^+\right|_0\\ &\le\left(\left|\left[\bar\partial^{a-1}\partial_t^b A^T\nabla q\right]_-^+\right|_0+\left|\left[A^T\bar\partial^{a-1}\partial_t^b\nabla q\right]_-^+\right|_0\right)\left|\left(N-\frac{\bar\partial h^\kappa}{(1+Hh^\kappa)}\tau\right)\right|_{L^\infty}+\text{l.o.t.}\\ &\le P(\mathcal{E}_\kappa(t))\left(\left|\bar\partial^a\partial_t^b h^\kappa\right|_0+\left\|\partial_t^b q\right\|_{H^{6.5-2b}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^t\mathcal{R}^{a,b}_{comm}(s)ds&\le P(\mathcal{E}_\kappa(t))\int_0^t e^{(-\lambda_1+\eta)s/2}\left(|\bar\partial^a\partial_t^b h^\kappa|_0+\|\partial_t^b q\|_{H^{6.5-2b}}\right)ds+\text{l.o.t.}\\ &\le P(\mathcal{E}_\kappa(t))(t\mathcal{E}_\kappa(t)^{1/2}+\sqrt{t}P(\mathcal{E}_\kappa(t)))\le\sqrt{t}\;P(\mathcal{E}_\kappa(t)).\end{aligned}$$

Estimates for $\mathcal{R}^{a,b}_{\Gamma_1}$ are straightforward, since it has a factor κ^2, and therefore we can bound the higher-order factors inside of the integral in $L^2(\Gamma)$ and use the extra terms in the energy with the κ-coefficients to obtain the desired estimate. The error $\widetilde{\mathcal{R}}_\Gamma(t)$ can be studied similarly as $\mathcal{R}_\Gamma$, but it is actually simpler, since $\widetilde{\mathcal{R}}_\Gamma$ does not contain the conflicting term $\mathcal{R}^{a,b}_{\Gamma_4}$. Thus, simple integration by parts and Sobolev estimates suffice for obtaining the desired bounds. For the outside boundary error terms $\mathcal{R}^+_{\partial\Omega}$ and $\widetilde{\mathcal{R}}^+_{\partial\Omega}$, we capitalize on the boundary condition (3.17d), and the smoothness of $\partial\Omega$, which allows us to remove derivatives from the critical terms, and put them in the smooth normal vector. The estimates for the interior errors $\mathring{\mathcal{R}}^\pm$ follow the same methodology as the terms in $\mathcal{R}^\pm$. We will omit these estimates as there is no new ideas involved in their bounds.

3.6.1. Estimates for the errors with α and γ.

$$\begin{aligned}\mathcal{R}^{a,b}_\alpha&:=\int_{\Omega^\pm}(\bar\partial^a\partial_t^b q+\bar\partial^a\partial_t^{b\kappa}\Psi\cdot v)(\bar\partial^a\partial_t^b\alpha)W\mu\text{ for }a+2b\le 6,\\ \widetilde{\mathcal{R}}^{a,b}_\alpha&:=\int_{\Omega^\pm}(\bar\partial^a\partial_t^{b+1}q+\bar\partial^a\partial_t^{b+1\kappa}\Psi\cdot v)(\bar\partial^a\partial_t^b\alpha)W\mu\text{ for }a+2b\le 5.\end{aligned}$$

Let us look at the case when $a = 6,\ b = 0$,

$$\mathcal{R}^{6,0}_{\alpha} \le \int_{\Omega^\pm} (\bar\partial^7 q + \bar\partial^7 {}^\kappa\Psi \cdot v)\bar\partial^5 \alpha W\mu + \text{l.o.t.} \le C(\|q\|_7 + \|{}^\kappa\Psi\|_7)\|\alpha\|_5 + \text{l.o.t.}$$
$$\le \mathcal{E}_\kappa(t)^{1/2}\|\alpha\|_5 + \text{l.o.t.}$$

Now, recall that from the definition of α in (3.5) we get

$$\|\alpha\|_5 \le \|\alpha_0\|_5 + \int_0^t \|r(s)\|_5 ds \lesssim \|q_0\|_6 + \sqrt{T_\kappa}\|r\|_{L^2_t H^5} \lesssim \|q_0\|_6 + C\sqrt{T_\kappa}.$$

Therefore the bound follows directly. Similarly for the other terms a, b, and for the error terms involving γ,

$$\mathcal{R}^{a,b}_{\partial\Omega} = -\int_{\partial\Omega} (\bar\partial^a \partial_t^b q^+ + \bar\partial^a \partial_t^b {}^\kappa\Psi \cdot v)\bar\partial^a \partial_t^b \gamma\ W + \text{ l.o.t.},$$
$$\widetilde{\mathcal{R}}^{a,b}_{\partial\Omega} = \int_{\partial\Omega} (\bar\partial^a \partial_t^{b+1} q^+ + \bar\partial^a \partial_t^{b+1} {}^\kappa\Psi \cdot v)\bar\partial^a \partial_t^b \gamma\ W + \text{l.o.t.},$$

since, after integrating by parts a tangential derivative, control for $|\bar\partial^{a-1}\partial_t^b \gamma|_{L^2(\Gamma)}$ follows from the definition of γ in (3.7). Consider for example $a = 6,\ b = 0$, then,

$$|\bar\partial^5 \gamma|_{L^\infty_t L^2} \lesssim \|\tilde{\mathcal{G}}^0\|_{L^\infty_t H^{5.5}} + \sqrt{t}\|\tilde{\mathcal{G}}^1\|_{L^2_t H^{5.5}} ds + t^{3/2}\|\tilde{\mathcal{G}}^2\|_{L^2_t H^{5.5}}$$
$$\le C\left(1 + \sqrt{T_\kappa}\right),$$

where we have used the bound (3.10). Therefore, for T_κ small enough,

$$\int_0^t \mathcal{R}^{6,0}_{\partial\Omega} ds \lesssim\ t\ \mathcal{E}_\kappa(t)^{1/2}(1 + T_\kappa) \lesssim\ t\ \mathcal{E}_\kappa(t)^{1/2}.$$

The other terms of $\mathcal{R}^{a,b}_{\partial\Omega}$ and $\widetilde{\mathcal{R}}^{a,b}_{\partial\Omega}$ follow directly using the same ideas, so we will omit them for brevity. ∎

3.7. Proof of Theorem 2.1.

Existence of solutions to the two-phase Stefan problem. The energy estimates of Proposition 3.14, and a continuation argument like the one described in [15, section 9], gives us that there exists a κ-independent time $T > 0$ such that the following bound holds,

$$\mathcal{E}_\kappa(t) \le C\mathcal{E}_\kappa(0) \quad \text{for all } t \in [0, T], \tag{3.53}$$

where C is independent of κ. The energy control of Proposition 3.9, and the definition of the initial data ${}^\kappa Q_0$, gives us the bound,

$$\mathcal{S}_\kappa(t) \le P(\mathcal{E}_\kappa(t)) \le P(\mathcal{E}_\kappa(0)) \le C\mathcal{S}(0) \quad \text{for all } t \in [0, T],$$

where we note that the polynomial P is also independent of κ. We conclude then that for all $\kappa > 0$, the solutions to the κ-problem exist at least until a κ-independent time $T > 0$, and they are bounded by

$$\mathcal{S}_\kappa(t) \le C\mathcal{S}(0) \quad \forall\ t \in [0, T]. \tag{3.54}$$

Consider now $\kappa = \frac{1}{n}$ and let $n \to \infty$. Let us define the reflexive Hilbert space

$$X(T) := \Big\{ (\tilde{q}^{\pm}, \tilde{h}) |\ \partial_t^l \tilde{q}^{\pm} \in L^2\left([0,T]; H^{6.5-2l}\left(\Omega^{\pm}\right)\right) \text{ for } l = 0,1,2,3, \\ \partial_t^s \tilde{h}_t \in L^2\left([0,T]; H^{5-2l}(\Gamma)\right) \text{ for } s = 0,1,2 \Big\},$$

then the uniform bound (3.54) implies that there exists a subsequence that converges weakly in $X(T)$ to a limit which we call $(q^{\pm}, h)$, and that $\mathcal{S}(t) = \mathcal{S}(q^{\pm}, h)$ also satisfies the estimate

$$\mathcal{S}(t) \leq C\mathcal{S}(0) \quad \forall\ t \in [0,T]. \tag{3.55}$$

Since the space of functions such that $\mathcal{S}(T)$ is bounded imbeds compactly into

$$C^{1,2} := \left\{ (q^{\pm}, h) :\ q \in C^1\left([0,T]; C^0\left(\Omega^{\pm}\right)\right) \cap C\left([0,T]; C^2\left(\Omega^{\pm}\right)\right), \\ h \in C^1\left([0,T]; C^0(\Gamma)\right) \cap C\left([0,T]; C^2(\Gamma)\right) \right\},$$

we have that the subsequence in fact converges strongly to a solution $(q^{\pm}, h) \in C^{1,2}$ of (1.10). This finishes the existence part of Theorem 2.1.

Continuity of $\mathcal{S}(t)$. An application of the fundamental theorem of calculus and the bound (3.55) gives us continuity of the lower-order norms. Namely,

$$\partial_t^l q^{\pm} \in C\left([0,T]; H^{5-2l}\left(\Omega^{\pm}\right)\right),\ \partial_t^l h \in C\left([0,T]; H^{5-2l}(\Gamma)\right),\ \text{for } l = 0, \dots, 2. \tag{3.56}$$

Moreover, since we have the higher regularity estimate (3.48), then $q^{\pm} \in L^2([0,T]; H^7(\Omega^{\pm}))$ and $\partial_t^l q_t^{\pm} \in L^2([0,T]; H^{5-2l}(\Omega^{\pm}))$, so we can use interpolation estimates to obtain that in fact $q^{\pm} \in C([0,T]; H^6(\Omega^{\pm}))$, $q_t^{\pm} \in C([0,T]; H^4(\Omega^{\pm}))$, and $q_{tt}^{\pm} \in C([0,T]; H^2(\Omega^{\pm}))$. The same estimate (3.48) gives us that $h \in L^2([0,T]; H^{6.5}(\Gamma))$, and from the definition of h_t in (3.28), we can actually recover a higher regularity than $L^2([0,T], H^5(\Gamma))$. Indeed, notice that

$$|h_t|^2_{L^2_t H^{5.5}} \leq \int_0^t |[v \cdot \tilde{n}]^+_-|^2_{5.5} ds,$$

where the right-hand side is easily bounded using the known higher-order estimates (3.48). Therefore we have instead that $h_t \in L^2([0,T]; H^{5.5}(\Gamma))$ and, consequently, via interpolation, we have that $h \in C([0,T]; H^6(\Gamma))$, $h_t \in C([0,T], H^4(\Gamma))$ and $h_{tt} \in C([0,T], H^2(\Gamma))$. It is left to prove then that the norms with the higher-order time derivatives are continuous functions of time, i.e., $q_{ttt}^{\pm} \in C([0,T]; L^2(\Omega^{\pm}))$ and $h_{ttt} \in C([0,T]; L^2(\Gamma))$.

The continuity of $h_{ttt}(t)$ in $L^2(\Gamma)$ follows from the definition of h_t in (1.10d) time differentiated two times,

$$h_{ttt} = [v_{tt} \cdot \tilde{n} + 2v_t \cdot \tilde{n}_t + v \cdot \tilde{n}_{tt}]^+_-.$$

The continuity of $q_{tt}^{\pm} \in C([0,T], H^2(\Omega^{\pm}))$, and the continuity of $\partial_t^l h$ in the lower-order norms, gives us then that $v_{tt}^{\pm} \in C([0,T], H^1(\Omega^{\pm}))$ and thus the continuity of h_{ttt} follows. Finally, from the triple time differentiated problem (1.10a), we have that $q_{ttt}^{\pm}$ satisfies,

$$q_{ttt}^{\pm} = \partial_t^2\left(\Delta_{\Psi^{\pm}} q^{\pm} - \Psi_t^{\pm} \cdot v^{\pm}\right),$$

where we see that all the terms of the right-hand side are continuous in $L^2(\Omega)$, including the higher-order term $\Psi_{ttt} \cdot v$, since $h_{ttt} \in C([0,T], L^2(\Gamma))$. This concludes the proof of the continuity in time of $\mathcal{S}(t)$.

Uniqueness. The uniqueness can be derived from the energy estimates in a straightforward way. A brief sketch of the argument is presented in [30], so we omit the analogous proof in this article.

4. Global well-posedness. In sections 4.1–4.6, we shall collect all the necessary ingredients for the proof of Theorem 2.2, which is presented in section 4.7. We shall consider small initial data satisfying the hypothesis (2.2), which implies the bound

$$\mathcal{E}^{\pm}\left(q_0^{\pm}, h_0\right) \leq \frac{C}{|c_1^{\pm}|}\frac{\epsilon_0^2}{F(K)}, \tag{4.1}$$

where the denominator $|c_1^{\pm}|$ comes from (3.36b) and (1.27).

4.1. Bootstrap assumptions. As guaranteed by the local well-posedness Theorem 2.1, we assume that the solution $(q^{\pm}, h)$ to the two-phase Stefan problem (1.10) exists on a time interval $[0, T]$ for some $T > 0$. For $\epsilon_0 < \epsilon \ll 1$ to be specified later, we make the following bootstrap assumptions:

$$\frac{\mathcal{E}^{\pm}(t)}{|c_1^{\pm}|} + \mathcal{E}^{\Gamma}(t) + \int_0^t \left(\frac{\mathcal{D}^{\pm}(s)}{|c_1^{\pm}|} + \mathcal{D}^{\Gamma}(s)\right) ds \leq \frac{\epsilon^2}{|c_1^{\pm}|}, \tag{4.2a}$$

$$\sup_{0\leq s\leq T} E_{\beta}^{\pm}(s) \leq \tilde{C} E_{\beta}^{\pm}(0), \tag{4.2b}$$

$$\mathcal{X}^{\pm}(t) := \inf_{x\in\Gamma} \partial_N q^{\pm} \geq C|c_1^{\pm}| e^{-(\lambda_1^{\pm}+\eta/2)t}, \tag{4.2c}$$

where we recall the definitions from section 1.8.

Remark 4.1. Note that the bounds for the boundary norms $\mathcal{E}^{\Gamma}$, $\mathcal{D}^{\Gamma}$ satisfy (4.2a) in both the $+$ and the $-$ case and, therefore, with $c_1 = \max\{|c_1^+|, |c_1^-|\}$ we have the bound

$$\mathcal{E}^{\Gamma}(t) + \int_0^t \mathcal{D}^{\Gamma}(s) ds \leq \frac{\epsilon^2}{c_1}.$$

As in [31], we will prove first that if $\mathcal{T}$ is the maximal time for which the solution exists and satisfies the bootstrap assumptions, we actually improve upon the smallness assumptions (4.2a), (4.2b), and the lower bounds (4.2c). A standard continuity argument then leads to the proof of global existence.

The following technical lemma will be fundamental for our analysis and it is a direct consequence of the bootstrap assumptions. Intuitively, the lower-order norms of the temperature have strong enough decay to counter the decay of the weight $e^{(-\lambda_1+\eta)t}$ of the boundary norms. This will be used when proving the energy estimates of Lemma 4.15, as we will be able to bound products of boundary and interior terms by multiplying and dividing by the weight $e^{(-\lambda_1+\eta)t}$, while still maintaining control of the decay.

LEMMA 4.2. *If the bootstrap assumptions* (4.2) *hold, we have that*

$$\frac{E_{\beta}(t)^{\pm} e^{-\beta^{\pm} t}}{e^{(-\lambda_1+\eta)t}} \leq C\frac{\epsilon_0^2}{F(K)} e^{-\gamma^{\pm} t} \tag{4.3}$$

with

$$\gamma^{\pm} = 2\lambda_1^{\pm} - \lambda_1 > 0.$$

Proof. Using the bootstrap assumptions (4.2b) and (4.2c) we arrive at

$$\frac{E^\pm_\beta(t)e^{-\beta^\pm t}}{e^{(-\lambda_1+\eta)t}} \le \frac{\tilde{C}E^\pm_\beta(0)e^{-\beta^\pm t}}{e^{(-\lambda_1+\eta)t}} \le CE^\pm_\beta(0)e^{-\gamma^\pm t}.$$

Since by definition $E^\pm_\beta(0) = \sum_{b=0}^2 \|\partial_t^b q^\pm(0)\|^2_{H^{4-2b}}$, it follows from the compatibility assumptions (1.17a), (1.17b) that $E^\pm_\beta(0) \le C\|q_0^\pm\|_4^2$. Therefore

$$\frac{E^+_\beta(t)e^{-\beta^+ t}}{e^{(-\lambda_1+\eta)t}} \le C\|q_0^\pm\|_4^2 e^{-\gamma^\pm t} \le C\frac{\epsilon_0^2}{F(K)}e^{-\gamma^\pm t}.$$

where the last inequality follows from the smallness of the initial data (2.2). ∎

Remark 4.3. Using a higher-order Hardy inequality as in [31, Lemma 2.1], we obtain as well the following bound for the initial lower-order energy:

$$E^\pm_\beta(0) \le C\left(K^\pm\right)^4 |c_1^\pm|^2, \tag{4.4}$$

which, by inspecting the proof of the lemma leads to the following alternative bound:

$$\frac{E^\pm_\beta(t)e^{-\beta^\pm t}}{e^{(-\lambda_1+\eta)t}} \le C\left(K^\pm\right)^4 \left|c_1^\pm\right|^2 e^{-\gamma^\pm t}. \tag{4.5}$$

4.2. Global estimates for $W^\pm$ and energy equivalence.

LEMMA 4.4 (global estimates for $W^\pm$). *Let the bootstrap assumptions* (4.2) *hold. Then $W^\pm$ satisfy the following bounds,*

$$\frac{Ce^{\sigma^\pm t}}{(K^\pm)^2|c_1^\pm|} \le \min_{x\in\Omega^\pm} W^\pm(t,x) \le \max_{x\in\Omega^\pm} W^\pm(t,x) \le \frac{Ce^{(\sigma^\pm+\eta)t}}{|c_1^\pm|}, \tag{4.6}$$

where

$$\sigma^\pm := \lambda_1^\pm - \lambda_1 + \frac{\eta}{2} > 0. \tag{4.7}$$

Proof. We use the bootstrap assumptions (4.2) to obtain the following bounds for $\partial_N q^\pm$:

$$\begin{aligned} C|c_1^\pm|e^{(-\lambda_1^\pm-\eta/2)t} &\le \left|\partial_N q^\pm\right| \le \|q^\pm\|_{2.25} \\ &\le CE^\pm_\beta(t)^{1/2}e^{-\beta^\pm t/2} \le C\left(K^\pm\right)^2 |c_1^\pm|e^{(-\lambda_1^\pm+\eta/2)t}, \end{aligned}$$

where we used the Sobolev embedding $H^{1.25}(\Omega^\pm) \hookrightarrow L^\infty(\Omega^\pm)$ in the second inequality. Now the proof follows analogously to the proof of Lemma 3.6 but using the above bounds instead. ∎

The natural energy $\mathcal{E}$ is defined by setting

$$\mathcal{E}(t) := \sup_{0\le s\le t} \mathcal{E}^+(s) + \sup_{0\le s\le t} \mathcal{E}^-(s) + \int_0^t (\mathcal{D}^+(s) + \mathcal{D}^-(s))ds, \tag{4.8}$$

where $\mathcal{E}^\pm$ and $\mathcal{D}^\pm$ are defined by setting $\kappa = 0$ in the definition (3.7) of $\mathcal{E}^\pm_\kappa$ and $\mathcal{D}^\pm_\kappa$, respectively. Note that the natural energy $\mathcal{E}$ contains the $W^\pm$-weights for the interior norms and an exponential-in-time weight for the boundary norms. Keeping track of the weights allows us to prove a more precise lower bound on the energy $\mathcal{E}$ in terms of (differently weighted-in-time) components of the total norm $S(t)$, which is defined in (1.13).

PROPOSITION 4.5. *With the bootstrap assumptions* (4.2) *holding, and with* $\epsilon > 0$ *sufficiently small, there exists a constant* C *and a* $\gamma > 0$ *such that*

$$\begin{aligned}\sup_{0\le s\le t}\frac{e^{\sigma^\pm s}\mathcal{E}^\pm(s)}{(K^\pm)^2|c_1^\pm|}+\sup_{0\le s\le t}\mathcal{E}^\Gamma(s)+\int_0^t\left(\frac{e^{\sigma^\pm s}\mathcal{D}^\pm(s)}{(K^\pm)^2|c_1^\pm|}+\mathcal{D}^\Gamma(s)\right)ds\\ \le C\sup_{0\le s\le t}\mathcal{E}^\pm(s)+C\int_0^t \epsilon e^{-\gamma^\pm s}\mathcal{E}^\pm(s)ds+C\int_0^t\mathcal{D}^\pm(s)ds\\ \le C\mathcal{E}(t).\end{aligned} \tag{4.9}$$

Proof. The proof of this proposition follows the same steps as the proofs of [31, Lemmas 2.9 and 2.10 and Corollary 2.11], with the caveat of adding the weights $W^\pm$. We will show the highest-order estimate for ilustration purposes, assuming that we have shown suitable estimates up to $\|q_t\|_{4.5}$, using the time differentiated problems. We will omit the upper indices $\pm$ since both regions follow the same argument. Consider (1.10a),

$$\Delta q = q_t + \Psi_t\cdot v + \left(\left(\delta_l^k - A_j^k A_j^l\right) q_{,l}\right)_{,k},$$

therefore,

$$\|\bar\partial^6\nabla q\|_0 \le \|q_t\|_5 + (\|\bar\partial^6\nabla\Psi\|_0 + \|\bar\partial^5\Psi_t\|_0 + \|\bar\partial^5 v\|_0)\epsilon^{1/2}e^{-\beta t/2} + C\epsilon^{1/2}\mathcal{E}(t)^{1/2}e^{-\beta t/2},$$
$$\|\bar\partial^5\nabla q\|_0 \le \|q_t\|_4 + (\|\bar\partial^5\nabla\Psi\|_0 + \|\bar\partial^4\Psi_t\|_0 + \|\bar\partial^4 v\|_0)\epsilon^{1/2}e^{-\beta t/2} + C\epsilon^{1/2}\mathcal{E}(t)^{1/2}e^{-\beta t/2}.$$

Thus, using the estimates for Ψ and Ψ_t in terms of their boundary value, interpolating, and using that control of the *ψ-divergence* of ∇q gives us control of the normal derivatives. We obtain the bound

$$\|q\|_{6.5} \le \|q_t\|_{4.5} + (|h|_6 + |h_t|_4 + \|\bar\partial^5 v\|_0)\epsilon^{1/2}e^{-\beta t/2} + C\epsilon^{1/2}\mathcal{E}(t)^{1/2}e^{-\beta t/2}.$$

Recall that the boundary norms in the definition (3.7) of the natural energy $\mathcal{E}$ have the weight $e^{(-\lambda_1+\eta)t}$, and the interior norm of $\bar\partial^5 v$ has the weights W as the summand $\|\bar\partial^5 v\|_{L^2,W^\pm}$ appears in the definition. Therefore,

$$\begin{aligned}\|q\|_{6.5} &\le \|q_t\|_{4.5} + C\left(\mathcal{E}(t)^{1/2}e^{(\lambda_1-\eta)t/2} + \frac{\mathcal{E}(t)^{1/2}}{(\inf W)^{1/2}}\right)\epsilon^{1/2}e^{-\beta t/2} + C\epsilon^{1/2}\mathcal{E}(t)^{1/2}e^{-\beta t/2}\\ &\le \|q_t\|_{4.5} + C\epsilon^{1/2}\mathcal{E}(t)^{1/2}e^{-\gamma^\pm t} + \frac{\mathcal{E}(t)^{1/2}}{(\inf W)^{1/2}}\epsilon^{1/2}e^{-\beta t/2}\end{aligned}$$

with $\gamma^\pm = \lambda_1^\pm - \lambda_1/2$. The result then follows using the estimates for $\|q_t\|_{4.5}$. ∎

Remark 4.6. We will use this relationship in section 4.7 to prove the global stability theorem.

Remark 4.7. Notice that, just as in the one-phase problem, the exponential growth introduced by bounding the norms of h with the natural energy is counterbalanced by the decay of the lower order norms.

4.3. A priori bounds on h.

LEMMA 4.8 (suboptimal decay bound for h_t). *Under the bootstrap assumptions* (4.2)*, the following decay bound holds:*

$$|h_t|_{2.5} \le C\epsilon\left(e^{-\gamma^+ t/2} + e^{-\gamma^- t/2}\right) \le C\epsilon e^{-\lambda_1 t/2}. \tag{4.10}$$

Proof. The proof of this lemma follows exactly as in [31, Lemma 2.4], but instead of multiplying and dividing by $\sqrt{\mathcal{X}}$ to obtain a boundary energy term, we multiply by $e^{(-\lambda_1+\eta)t}$. Since the boundary condition for h_t depends on the jump of the temperature gradients from both regions, we obtain the sum of the two exponential decays. The last inequality follows since we defined $\lambda_1 = \min\{\lambda_1^+, \lambda_1^-\}$. Notice that the weights $W^\pm$ do not show up in this proof, as this estimate only uses the lower-order energy $E_\beta(t)$, which has no weights. ∎

Remark 4.9. A more precise statement can be achieved by following the bootstrap regularity arguments of [31, Lemma 2.6], wherein we keep track of constant c_1 in our estimates. In particular we have

$$|h_t|_{2.5} \leq C c_1 e^{-\lambda_1 t},$$

which will be useful in Lemma 4.10.

LEMMA 4.10 (smallness of the height function). *Suppose that the bootstrap assumptions* (4.2) *hold. Then, for $\epsilon > 0$ taken sufficiently small,*

$$\sup_{0\leq s\leq t} |h(s)|_{4.5} \leq C\sqrt{\epsilon}, \tag{4.11}$$

while for the lower-order norms,

$$\sup_{0\leq s\leq t} |h(s)|_{2.5} \leq C c_1 \quad \textit{and} \quad \sup_{0\leq s\leq t} |h(s)|_4 \leq C\epsilon^{1/2} c_1^{1/4}. \tag{4.12}$$

Proof. The proof follows the same argument as in the proof of [31, Lemma 2.6], but again with the same modification as before, when instead of multiplying by $\sqrt{\mathcal{X}}$ we use $e^{(-\lambda_1+\eta)t}$. We will show only the higher-order estimate for brevity. Recall the interpolation estimate,

$$|h|_{4.5}^2 \leq C \int_0^t |h|_6 |h_t|_3 ds \leq C \int_0^t |h|_6 |h_t|_5^{1/5} |h_t|_{2.5}^{4/5}.$$

Then, using an improved bound for $|h_t|_{2.5}$, we have

$$\begin{aligned}
|h|_{4.5}^2 &\leq C \int_0^t |h|_6 |h_t|_{2.5}^{4/5} |h_t|_5^{1/5} \leq C c_1^{4/5} \int_0^t |h|_6 e^{-4\lambda_1 t/5} |h_t|_5^{1/5}\, ds \\
&\leq C c_1^{4/5} \int_0^t e^{(-\lambda_1+\eta)s/2} |h|_6 |h_t|_5^{1/5} e^{(-\frac{3\lambda_1}{10} - \frac{1}{2}\eta)s}\, ds.
\end{aligned}$$

Let $\bar\gamma := \frac{3\lambda_1}{10} + \frac{1}{2}\eta > 0$. Then,

$$\begin{aligned}
|h|_{4.5}^2 &\leq C\epsilon c_1^{3/10} \int_0^t e^{-\bar\gamma s} |h_t|_5^{1/5} ds \\
&\leq C\epsilon c_1^{3/10} \left(\int_0^t (e^{-\bar\gamma s/2})^{10/9} ds \right)^{9/10} \left(\int_0^t (e^{-\bar\gamma s/2} |h_t|_5^{1/5})^{10} \right)^{1/10} \\
&\leq C\epsilon c_1^{3/10} \left(\int_0^t e^{(-\frac{\lambda_1}{2} + \frac{3}{2}\eta)s} e^{(-\lambda_1+\eta)s} |h_t|_5^2 \right)^{1/10} \\
&\leq C\epsilon c_1^{3/10} \left(\int_0^t \mathcal{D}^\Gamma(s) \right)^{1/10} \\
&\leq C\epsilon^{6/5} c_1^{1/5},
\end{aligned}$$

and the claim easily follows. Note that we have used Hölder's inequality, the definition of $\mathcal{D}^\Gamma$ from section 1.8, and the bootstrap assumption (4.2a). ◻

4.4. Lower bounds on $\partial_N q^\pm$. The main objective of this section is to prove a lower bound which improves the bootstrap assumption (4.2c). The detailed study of the decay rates of $\partial_N q^\pm$ is of fundamental importance as it also determines the growth and decay-in-time properties of the weights $W^\pm$. We present the following,

LEMMA 4.11 (lower bound for $\mathcal{X}^\pm(t)$). *Assuming the bootstrap assumptions* (4.2) *with ϵ small enough, there exists a universal constant $C > 0$ such that*

$$\mathcal{X}^\pm(t) \geq C|c_1^\pm| e^{-(\lambda_1^\pm + \tilde{\lambda}^\pm(t))t}. \tag{4.13}$$

Moreover, $\tilde{\lambda}^\pm \geq 0$ satisfies $\tilde{\lambda}^\pm \leq C\sqrt{\epsilon}$ for some positive constant C. In particular, with $\epsilon > 0$ sufficiently small so that $C\sqrt{\epsilon} < \eta/4$, we obtain the improvement of the bootstrap bound (4.2c) *given by*

$$\mathcal{X}^\pm(t) \geq C|c_1^\pm| e^{-(\lambda_1^\pm + \eta/4)t}.$$

We will omit the proof as it is detailed in [32, section 2.6], where a comparison function is constructed using the so-called *demieigenvalues and demieigenfunctions* of the maximal Pucci operators detailed in [42].

4.5. Improved bounds for the lower-order terms of the energy $S(t)$. In this section we prove the improvement of the bootstrap bounds for the terms $E_\beta^\pm$, responsible for the decay of the below-top-order energy terms.

LEMMA 4.12. *There exists a constant $\tilde{C}$ and $\epsilon > 0$ sufficiently small, such that if the bootstrap assumptions* (4.2) *hold with such ϵ and $\tilde{C}$, then the following improved bound holds:*

$$E_\beta^\pm(t) \leq \frac{\tilde{C}}{2} E_\beta^\pm(0).$$

Proof. Notice that the lower-order norms $E_\beta^\pm$ do not contain weights $W^\pm$ in their definitions (see section 1.8). Therefore the proof of the lemma is analogous to the proof of [31, Lemma 4.1] and we omit the details of the proof for brevity purposes. In [31] the authors used elliptic estimates and a Poincaré inequality. Note that the Poincaré constant is given precisely by the first eigenvalue of the Dirichlet–Laplacian in each region $\Omega^\pm$, which gives us the different decay rates $\lambda_1^\pm$ in each respective region. The basic mechanism for the decay of the H^4-norm of the temperature q is exactly the same as in the standard heat equation. We cannot, however, propagate this nearly optimal decay rate to the top-order norms of q as that would require a stronger control on the top-order derivatives of h than the one dictated by the natural energy $\mathcal{E}$. ◻

4.6. Improved bounds for the energy $\mathcal{E}(t)$. In this section we prove the higher-order energy estimates to be used in the proof of Theorem 2.2 in section 4.7. Most of the energy terms are dealt with using the same techniques as in [31], but all the new terms that arise from the interaction of the two phases via the weights $W^\pm$ are presented in detail. This is the price we must pay for eliminating the *critical term* of [31], but we will see it is a low price to pay. As a starting point we first prove sharp upper bounds on the space-time derivatives of the weights $W^\pm$ as they will be used crucially in dealing with the above mentioned critical terms.

LEMMA 4.13 (global estimates for $\nabla W^\pm$ and $W_t^\pm$). *Under the bootstrap assumptions* (4.2)*, the derivatives of the weight functions* $W^\pm$ *satisfy the following bounds:*

$$\|W_t^\pm\|_{L^\infty(\Omega^\pm)} \le P(K)\frac{e^{\sigma^\pm t}}{|c_1^\pm|}e^{2\eta t} \quad \text{and} \quad \|\nabla W^\pm\|_{L^\infty(\Omega^\pm)} \le P(K)\frac{e^{\sigma^\pm t}}{|c_1^\pm|}e^{3\eta t}$$

where we recall from (4.7) $\sigma^\pm = \lambda_1^\pm - \lambda_1 + \frac{\eta}{2}$.

Proof. Recall that $W_t^\pm$ solves the elliptic boundary value problem (3.47). Therefore

$$\begin{aligned}
|W_t^\pm|_{L^\infty(\Gamma)} &\le Ce^{(-\lambda_1+\eta)t}\left(\frac{1}{\mathcal{X}^\pm(t)} + \frac{|\partial_N q_t^\pm|_{L^\infty}}{\mathcal{X}^\pm(t)^2}\right)\\
&\le Ce^{(-\lambda_1+\eta)t}\left(\frac{e^{(\lambda_1^\pm+\eta/2)t}}{|c_1^\pm|} + \frac{(K^\pm)^2|c_1^\pm|e^{-\beta^\pm t/2}}{|c_1^\pm|^2}\right)\\
&\le \frac{e^{(-\lambda_1+\lambda_1^\pm+3\eta/2)t}}{|c_1^\pm|}\left(1+(K^\pm)^2e^{\eta t}\right)\\
&\le C(K^\pm)^2\frac{e^{(-\lambda_1+\lambda_1^\pm+5\eta/2)t}}{|c_1^\pm|}\\
&\le P(K)\frac{e^{\sigma^\pm t}}{|c_1^\pm|}e^{2\eta t}.
\end{aligned}$$

And so the result follows from the maximum principle applied to $W_t^\pm$. To obtain the bound on $\|\nabla W^\pm\|_{L^\infty}$, we first use the Sobolev inequality and then the elliptic theory to infer that

$$\begin{aligned}
\|\nabla W^-\|_{L^\infty} &\le C\|\nabla W^-\|_{1.25} \le Ce^{(-\lambda_1+\eta)t}\left|\frac{1}{\partial_N q^-}\right|_{1.75},\\
\|\nabla W^+\|_{L^\infty} &\le C\|\nabla W^+\|_{1.25} \le Ce^{(-\lambda_1+\eta)t}\left|\frac{1}{\partial_N q^+}\right|_{H^{1.75}(\Gamma)} + C\frac{e^{(-\lambda_1+\lambda_1^\pm+\eta)t}}{|c_1^+|}.
\end{aligned}$$

To estimate the right-hand sides, let us first estimate the $L^2(\Gamma)$-norm of two tangential derivatives applied to $\frac{1}{\partial_N q^\pm}$:

$$\begin{aligned}
\left|\bar\partial^2\left(\frac{1}{\partial_N q^\pm}\right)\right|_0 &\le C\frac{|\bar\partial\partial_N q^\pm|_{L^4}^2}{\mathcal{X}^\pm(t)^3} + C\frac{|\bar\partial^2\partial_N q^\pm|_0}{\mathcal{X}^\pm(t)^2}\\
&\le \frac{\|q^\pm\|_3^2}{|c_1^\pm|^3}e^{3(\lambda_1^\pm+\eta/2)t} + \frac{\|q^\pm\|_{3.5}}{|c_1^\pm|^2}e^{(2\lambda_1^\pm+\eta)t}\\
&\le \frac{(K^\pm)^4}{|c_1^\pm|}e^{(-\beta^\pm+3\lambda_1^\pm+3\eta/2)t} + \frac{(K^\pm)^2}{|c_1^\pm|}e^{(-\beta^\pm/2+2\lambda_1^\pm+\eta)t}\\
&\le \frac{e^{(\lambda_1^\pm+5\eta/2)t}}{|c_1^\pm|}\left((K^\pm)^4+(K^\pm)^2e^{-\eta t}\right)\\
&\le P(K)\frac{e^{(\lambda_1^\pm+5\eta/2)t}}{|c_1^\pm|},
\end{aligned}$$

where we used the continuous embedding $H^{1/2}\hookrightarrow L^4$, the trace estimates, and the bound (4.5). Recall as well the definition (1.14) of $\beta^\pm = 2\lambda_1^\pm - \eta$. The same bound for $|\bar\partial(\frac{1}{\partial_N q^\pm})|_0$ follows analogously. As a result, we obtain that

$$\|\nabla W^\pm\|_{L^\infty} \le P(K)\frac{e^{(-\lambda_1+\lambda_1^\pm+\eta/2)t}}{|c_1^\pm|}e^{3\eta t} = P(K)\frac{e^{\sigma^\pm t}}{|c_1^\pm|}e^{3\eta t}$$

for some universal polynomial P. This concludes the proof of the lemma. ∎

Remark 4.14. The proof of the lemma depends on the boundary estimates for $W_t^\pm$ and $\nabla W^\pm$ on Γ. This inevitably leads us to study sharp upper bounds for the reciprocals $\frac{1}{\partial_N q^\pm}$, which in turn demands a very good understanding of the lower bounds on the decay rate of the Neumann derivative $\partial_N q^\pm$. These ingredients are provided by Lemma 4.11.

We can now prove the improvement on the higher-order bootstrap assumption (4.2a). Notice that, because of the way in which the norm and the natural energy are related in (4.9), we need to build the estimates with the appropriate coefficients $|c_1^\pm|$, $K^\pm$, and exponential growth $e^{\sigma^\pm t}$.

LEMMA 4.15 (higher-order energy estimates). *Suppose that the bootstrap assumptions* (4.2) *hold with $\epsilon > 0$ and $\eta > 0$ sufficiently small. Then the following bound holds:*

$$\begin{aligned}\mathcal{E}(t) \le\ & C\mathcal{E}(0)\\ &+ \frac{P(K)\epsilon_0^2}{|c_1^\pm|F(K)} + O(\epsilon+\delta)\Bigg[\sup_{0\le s\le t}\frac{\varepsilon^+(s)e^{\sigma^+ s}}{(K^+)^2|c_1^+|} + \sup_{0\le s\le t}\frac{\varepsilon^-(s)e^{\sigma^- s}}{(K^-)^2|c_1^-|} + \sup_{0\le s\le t}\varepsilon^\Gamma(s)\\ &+ \int_0^t\left(\frac{\mathcal{D}^+(s)e^{\sigma^+ s}}{(K^+)^2|c_1^+|} + \frac{\mathcal{D}^-(s)e^{\sigma^- s}}{(K^-)^2|c_1^-|} + \mathcal{D}^\Gamma(s)\right)ds\Bigg] \quad \textit{for all } t \in [0,\mathcal{T}]\end{aligned} \tag{4.14}$$

where $P(K)$ is some universal polynomial, $0 \le \delta \ll 1$ is sufficiently small, and $\sigma^\pm = \lambda_1^\pm - \lambda_1 + \eta/2 > 0$ has already been introduced in (4.7).

Proof. Recall the energy identity (3.43) from Lemma 3.10, and notice that we need only to prove bounds for the error terms $\mathcal{R}$. We will exemplify this by showing the estimates for $\mathcal{R}^\pm_{a,b}$, defined in the statement of Lemma 3.10.

Bounds for $\boldsymbol{\mathcal{R}^\pm_{a,b}}$. Let us rewrite $\mathcal{R}^\pm_{a,b}$ as a sum:

$$\int_0^t \mathcal{R}^\pm_{a,b} = \sum_{a,b}\mathcal{R}_1^{a,b\pm} + \mathcal{R}_2^{a,b\pm} + \mathcal{R}_3^{a,b\pm} + \mathcal{R}_4^{a,b\pm} + \mathcal{R}_5^{a,b\pm} + \mathcal{R}_6^{a,b\pm},$$

where $\mathcal{R}_i^{a,b\pm}$ are the integrated terms written in the definition of $\mathcal{R}^\pm_{a,b}$; see Lemma 3.10. The estimates for $\mathcal{R}_1^{a,b\pm}, \mathcal{R}_2^{a,b\pm}, \mathcal{R}_3^{a,b\pm}, \mathcal{R}_4^{a,b\pm}$ follow the same strategy as in [31, section 3.2]. The only addition is the presence of the weights $W^\pm$, which we bound in L^∞ yielding an additional exponentially growing term with a rate $\sigma^\pm + \eta$. It is therefore left to show that in every such error term, there exists a below-top-order energy term which decays sufficiently fast to counteract the potential exponential growth stemming from $W^\pm$. We illustrate this by estimating the term $\mathcal{R}_1^{a,b\pm}$:

$$\begin{aligned}\left|\int_{\Omega^\pm}\bar\partial^5 A\bar\partial\nabla q\bar\partial^6 vW\right| &\le \|\bar\partial^5 A\|_{L^4}\|\bar\partial\nabla q\|_{L^4}\|\bar\partial^6 v\|_0\|W\|_{L^\infty}\\ &\le C|h|_6\|q\|_{2.5}\|\bar\partial^6 v\|_0\frac{C}{|c_1^\pm|}e^{(\sigma^\pm+\eta)t}\\ &\lesssim \frac{1}{|c_1^\pm|}e^{(-\lambda_1+\eta)t/2}|h|_6\frac{E_\beta(t)^{1/2}e^{-\beta t/2}}{e^{(-\lambda_1+\eta)t/2}}\mathcal{D}^{1/2}(t)e^{(\sigma^\pm+\eta)t}\\ &\lesssim K^\pm|c_1^\pm|^{-1/2}\frac{\sqrt{\epsilon_0}}{\sqrt{F(K)}}\mathcal{E}^\Gamma(t)^{1/2}\mathcal{D}(t)^{1/2}e^{(\sigma^\pm+\eta-\gamma^\pm/2)t}\\ &\le \delta\mathcal{E}^\Gamma(t)+C_\delta(K^\pm)^2|c_1^\pm|^{-1}\frac{\epsilon_0}{F(K)}\mathcal{D}^\pm(t)e^{\sigma^\pm t}e^{(\sigma^\pm-\gamma^\pm+2\eta)t}\\ &\le \delta\mathcal{E}^\Gamma(t)+C_\delta(K^\pm)^4\frac{\epsilon_0}{F(K^\pm)^{1/2}}\frac{\mathcal{D}^\pm(t)e^{\sigma^\pm t}}{(K^\pm)^2|c_1^\pm|},\end{aligned}$$

where $\sigma^\pm$ is given by (4.7), $\gamma^\pm = 2\lambda_1^\pm-\lambda_1$, and, therefore, $\sigma^\pm-\gamma^\pm+\eta = -\frac{1}{2}\lambda_1+\frac{3}{2}\eta<0$ for η sufficiently small. Note that we used (4.6) in the second line, the definition of E_β in the third line, and the estimate (4.5) in the fourth line to bound the ratio $\frac{E_\beta(t)^{1/2}e^{-\beta t/2}}{e^{(-\lambda_1+\eta)t/2}}$ by $CK^\pm|c_1^\pm|^{1/2}\frac{\sqrt{\epsilon_0}}{\sqrt{F(K)}}e^{-\gamma^\pm t/2}$. In the fifth line we used Young's inequality and in the last line the negativity of $\sigma^\pm-\gamma^\pm+\eta$. Considering ϵ_0 small enough so that $C_\delta(K^\pm)^4\frac{\epsilon_0}{F(K^\pm)^{1/2}}<\epsilon$, gives us the desired inequality.

We can apply an entirely analogous reasoning to bound the terms $\mathcal{R}_j^{a,b\pm}$, $j=2,3,4$. Integrating in time we therefore obtain that

$$\begin{aligned}&\mathcal{R}_1^{a,b\pm}+\mathcal{R}_2^{a,b\pm}+\mathcal{R}_3^{a,b\pm}+\mathcal{R}_4^{a,b\pm}\\ &\quad\le O(\epsilon+\delta)\left[\sup_{0\le s\le t}\left(\frac{\mathcal{E}^+(s)e^{\sigma^+ s}}{(K^+)^2|c_1^+|}+\frac{\mathcal{E}^-(s)e^{\sigma^- s}}{(K^-)^2|c_1^-|}+\mathcal{E}^\Gamma(s)\right)\right.\\ &\qquad\qquad\left.+\int_0^t\left(\frac{\mathcal{D}^+(s)e^{\sigma^+ s}}{(K^+)^2|c_1^+|}+\frac{\mathcal{D}^-(s)e^{\sigma^- s}}{(K^-)^2|c_1^-|}+\mathcal{D}^\Gamma(s)\right)ds\right].\end{aligned}$$

The estimates for the terms $\tilde{\mathcal{R}}^\pm$, $\mathcal{R}_\Gamma$, $\mathcal{R}_\Gamma^+$, and $\mathring{\mathcal{R}}^\pm$ follow the same methodology and we omit the details.

The remaining integrals $\mathcal{R}_5^{a,b\pm}$ and $\mathcal{R}_6^{a,b\pm}$ in the definition of $\mathcal{R}_{a,b}^\pm$ (see Lemma 3.10) are *new* error terms with respect to [31, 32] and they involve derivatives of the weights $W^\pm$:

$$\begin{aligned}\mathcal{R}_5^{a,b\pm} &:= \int_0^t\int_{\Omega^\pm}(\bar\partial^a\partial_t^b q^\pm+\bar\partial^a\partial_t^b\Psi^\pm\cdot v^\pm)A\bar\partial^a\partial_t^b v^\pm\nabla W^\pm,\\ \mathcal{R}_6^{a,b\pm} &:= \frac{1}{2}\int_0^t\int_{\Omega^\pm}(\bar\partial^a\partial_t^b q^\pm+\bar\partial^a\partial_t^b\Psi^\pm\cdot v^\pm)^2 W_t^\pm.\end{aligned}$$

To bound $\mathcal{R}_5^{a,b\pm}$ and $\mathcal{R}_6^{a,b\pm}$ we need upper bounds for $\nabla W^\pm$ and $W_t^\pm$ in L^∞ provided by Lemma 4.13. We will show that even for these terms, the additional exponential growth is also counterbalanced by the decay of the lower-order norms. In the following we omit the upper index $\pm$ for simplicity, and consider only the hardest case $a=6,\ b=$

0, as the argument remains the same for the other cases.

(4.15)

$$
\begin{aligned}
&\mathcal{R}_5^{a=6,b=0,\,\pm} \\
&\quad\le \int_0^t \|\bar\partial^6 q + \bar\partial^6\Psi\cdot v\|_0 \|\bar\partial^6 v\|_0 \|\nabla W^\pm\|_{L^\infty} \\
&\quad\le \frac{P(K)}{|c_1^\pm|}\int_0^t (\|q\|_6 + \|\bar\partial^6\Psi\|_0\|v\|_{L^\infty})\|\bar\partial^6 v\|_0 e^{\sigma^\pm s}e^{3\eta s}ds \\
&\quad\le \frac{P(K)}{|c_1^\pm|}\int_0^t \|q\|_6\|\bar\partial^6 v\|_0 e^{\sigma^\pm s}e^{3\eta s}ds + \frac{P(K)}{|c_1^\pm|}\int_0^t \|\bar\partial^6\Psi\|_0\|v\|_{L^\infty}\|\bar\partial^6 v\|_0 e^{\sigma^\pm s}e^{3\eta s}ds,
\end{aligned}
$$

where we used Lemma 4.13 in the second line. The first term of the rightmost side is estimated as follows:

$$
\begin{aligned}
&\frac{P(K)}{|c_1^\pm|}\int_0^t \|q\|_6\|\bar\partial^6 v\|_0 e^{\sigma^\pm s}e^{3\eta s}ds \le \frac{P(K)}{|c_1^\pm|}\int_0^t \|q\|_4^{1/5}\|q\|_{6.5}^{4/5}\mathcal{D}^\pm(s)^{1/2}e^{\sigma^\pm s}e^{3\eta s}ds \\
&\quad\le \frac{P(K)}{|c_1^\pm|}\int_0^t E_\beta^\pm(s)^{1/10}e^{-\beta^\pm s/10}\mathcal{D}^\pm(s)^{9/10}e^{\sigma^\pm s}e^{3\eta s}ds \\
&\quad\le \frac{C}{|c_1^\pm|}\int_0^t \left(C_\delta P(K)E_\beta^\pm(s)e^{(-\beta^\pm+30\eta)s} + \frac{\delta}{(K^\pm)^2}\mathcal{D}^\pm(s)\right)e^{\sigma^\pm s}ds \\
&\quad\le \frac{C_\delta P(K)E_\beta^\pm(0)}{|c_1^\pm|}\int_0^t e^{-\bar\gamma^\pm s}ds + \delta\int_0^t \frac{\mathcal{D}^\pm(s)e^{\sigma^\pm s}}{(K^\pm)^2|c_1^\pm|}ds \\
&\quad\le \frac{P(K)\epsilon_0^2}{|c_1^\pm|F(K)} + \delta\int_0^t \frac{\mathcal{D}^\pm(s)e^{\sigma^\pm s}}{(K^\pm)^2|c_1^\pm|}ds,
\end{aligned}
$$

where we note that $\bar\gamma^\pm = \lambda_1^\pm + \lambda_1 - 63\eta/2 > 0$ for η small enough. We used an interpolation estimate and the definition of the norm $\mathcal{D}^\pm$ in the first line, and Young's inequality in the third. Similarly, the second integral on the rightmost side of (4.15) satisfies:

$$
\begin{aligned}
&\frac{P(K)}{|c_1^\pm|}\int_0^t \|\bar\partial^6\Psi\|_0\|v\|_{L^\infty}\|\bar\partial^6 v\|_0 e^{\sigma^\pm s}e^{3\eta s}ds \\
&\quad\le \frac{P(K)}{|c_1^\pm|}\int_0^t |h|_{5.5}E_\beta^\pm(s)^{1/2}e^{(-\lambda_1^\pm+\frac{\eta}{2})s}\mathcal{D}(s)^{1/2}e^{\sigma^\pm s}e^{3\eta s}ds \\
&\quad\le P(K)\int_0^t e^{(-\lambda_1+\eta)s/2}|h|_{5.5}\mathcal{D}(s)^{1/2}e^{\sigma^\pm s}e^{(-\lambda_1^\pm+\lambda_1/2+3\eta)s}ds \\
&\quad\le P(K)\int_0^t \mathcal{E}^\Gamma(s)^{1/2}\mathcal{D}(s)^{1/2}e^{(-\lambda_1/2+3\eta)s}ds \\
&\quad\le \delta\sup_{0\le s\le t}\mathcal{E}^\Gamma(s) + C_\delta\frac{P(K)\epsilon_0}{F(K)^{1/2}}\int_0^t \frac{\mathcal{D}^\pm(s)}{(K^\pm)^2|c_1^\pm|}e^{(-\lambda_1/2+3\eta)s}ds.
\end{aligned}
$$

Taking ϵ_0 so small that $\frac{P(K)\epsilon_0}{F(K)^{1/2}} \le \epsilon$, we obtain the desired inequality. For the error term $\mathcal{R}_6^{a,b\,\pm}$ in the case $a=6, b=0$ we follow the same idea, but instead use the bound on $W_t^{\pm}$ from Lemma 4.13:

$$
\begin{aligned}
\mathcal{R}_6^{a=6,b=0,\pm} &\le \frac{1}{2}\int_0^t \|\bar\partial^6 q + \bar\partial^6\Psi\cdot v\|_0^2 \|W_t\|_{L^\infty} ds \\
&\le \frac{P(K)}{|c_1^{\pm}|}\int_0^t (\|q\|_6^2 + \|\bar\partial^6\Psi\|_0^2\|v\|_{L^\infty}^2) e^{\sigma^{\pm} t} e^{2\eta t} ds \\
&\le \frac{P(K)}{|c_1^{\pm}|}\int_0^t \|q\|_4^{2/5}\|q\|_{6.5}^{8/5} e^{\sigma^{\pm} t} e^{2\eta t} ds + \frac{P(K)}{|c_1^{\pm}|}\int_0^t |h|_{5.5}^2 \|q^+\|_{2.75}^2 e^{\sigma^{\pm} t} e^{2\eta t} ds \\
&\le \frac{P(K)}{|c_1^{\pm}|}\int_0^t E_\beta^{\pm}(0)^{1/5} e^{(\frac{\sigma^{\pm}}{5} - \beta^{\pm}/5 + 2\eta)s} \mathcal{D}(s)^{4/5} e^{4\sigma^{\pm} s/5} ds \\
&\quad + \frac{P(K)\epsilon_0}{F(K)^{1/2}}\int_0^t \mathcal{E}^{\Gamma}(s) e^{(-\lambda_1^{\pm} + 5\eta/2)s} ds \\
&\le C_\delta \frac{P(K)}{|c_1^{\pm}|}\int_0^t E_\beta(0) e^{-\tilde\gamma s} ds + \delta\int_0^t \frac{\mathcal{D}^{\pm}(s) e^{\sigma^{\pm} s}}{(K^{\pm})^2 |c_1^{\pm}|} ds + \epsilon \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s) \\
&\le C_\delta \frac{P(K)\epsilon_0^2}{|c_1^{\pm}| F(K)} + \delta\int_0^t \frac{\mathcal{D}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|} e^{\sigma^{\pm} s} ds + \epsilon \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s), \qquad (4.16)
\end{aligned}
$$

where $\tilde\gamma^{\pm} = \lambda_1^{\pm} + \lambda_1 - 19\eta/2 > 0$ for a sufficiently small $\eta > 0$. Note that we used norm interpolation in the third line and the exponential decay in the fifth line to infer that $\frac{P(K)\epsilon_0}{F(K)^{1/2}}\int_0^t \mathcal{E}^{\Gamma}(s) e^{(-\lambda_1^{\pm}+5\eta/2)s} ds \le \epsilon \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s)$. This completes the proof of the higher-order energy estimate. ∎

4.7. Proof of Theorem 2.2. To finish the proof we consider $\mathcal{T} > 0$, to be the maximal time at which the solution exists and satisfies the bootstrap assumptions (4.2). We will assume by means of a contradiction argument that $\mathcal{T}$ is finite, and will obtain an improved estimate for the higher energy bootstrap up to $\mathcal{T}$, which, via the local well-posedness theorem, will give us a contradiction to the maximality of $\mathcal{T}$.

From the global energy equivalence relation (4.9), we have that

$$
\sup_{0\le s\le t}\frac{\mathcal{E}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|} e^{\sigma^{\pm} s} + \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s) + \int_0^t \left(\frac{\mathcal{D}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|} e^{\sigma^{\pm} s} + \mathcal{D}^{\Gamma}(s)\right) ds \le C\mathcal{E}(t).
$$

Therefore, using Lemma 4.15, we have that

$$
\begin{aligned}
&\sup_{0\le s\le t}\frac{\mathcal{E}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|} e^{\sigma^{\pm} s} + \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s) + \int_0^t \left(\frac{\mathcal{D}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|} e^{\sigma^{\pm} s} + \mathcal{D}^{\Gamma}(s)\right) ds \le C\mathcal{E}(0) \\
&+ \frac{P(K)\epsilon_0^2}{F(K)|c_1^{\pm}|} + O(\epsilon+\delta)\left[\sup_{0\le s\le t}\frac{\mathcal{E}^{\pm}(s) e^{\sigma^{\pm} s}}{(K^{\pm})^2|c_1^{\pm}|} + \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s) + \int_0^t \left(\frac{\mathcal{D}^{\pm}(s) e^{\sigma^{\pm} s}}{(K^{\pm})^2|c_1^{\pm}|} + \mathcal{D}^{\Gamma}(s)\right) ds\right],
\end{aligned}
$$

Taking δ and ϵ small enough, we can absorb the term in the rectangular brackets to obtain

$$
\begin{aligned}
&\sup_{0\le s\le t}\frac{\mathcal{E}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|} e^{\sigma^{\pm} s} + \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s) \\
&\quad + \int_0^t \left(\frac{\mathcal{D}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|} e^{\sigma^{\pm} s} + \mathcal{D}^{\Gamma}(s)\right) ds \le C\mathcal{E}(0) + \frac{P(K)\epsilon_0^2}{F(K)|c_1^{\pm}|}.
\end{aligned}
$$

Using the smallness condition on the initial data (4.1),

$$\sup_{0\le s\le t} \frac{\mathcal{E}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|}e^{\sigma^{\pm}s} + \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s) + \int_0^t \left(\frac{\mathcal{D}^{\pm}(s)}{(K^{\pm})^2|c_1^{\pm}|}e^{\sigma^{\pm}s} + \mathcal{D}^{\Gamma}(s)\right) ds \le \frac{\epsilon_0^2}{|c_1^{\pm}|F(K)}(1+P(K)).$$

Taking ϵ_0 small enough so that

$$\frac{\epsilon_0^2(K^{\pm})^2(1+P(K))}{F(K)} \le \frac{\epsilon^2}{2},$$

we obtain an improvement on the bootstrap assumption (4.2a),

$$\sup_{0\le s\le t} \frac{\mathcal{E}^{\pm}(s)}{|c_1^{\pm}|}e^{\sigma^{\pm}s} + \sup_{0\le s\le t}\mathcal{E}^{\Gamma}(s) + \int_0^t \left(\frac{\mathcal{D}^{\pm}(s)}{|c_1^{\pm}|}e^{\sigma^{\pm}s} + \mathcal{D}^{\Gamma}(s)\right) ds \le \frac{\epsilon^2}{2|c_1^{\pm}|} \tag{4.17}$$

for all $t\in[0,\mathcal{T}]$. Therefore, by the continuity of the energy, we can extend the solution by the local well-posedness theorem to $[0,\mathcal{T}+T_0)$, for some small $T_0>0$, so that the bootstrap assumptions hold up until $\mathcal{T}+T_0$. This contradicts the maximality of $\mathcal{T}$, and therefore $\mathcal{T}=+\infty$, and the proof is complete.

Remark 4.16. Notice that not only have we obtained an improvement on the energy bootstrap, but from (4.17) we also obtained that the higher-order norms $\mathcal{E}^{\pm}$ and $\mathcal{D}^{\pm}$ decay at a rate $e^{-\sigma^{\pm}t}$ with $\sigma^{\pm}=\lambda_1^{\pm}-\lambda_1+\eta/2$. Recall that λ_1 is the minimum of the two eigenvalues, which means intuitively that the temperature in the region with smaller surface area decays faster to an equilibrium. Note however that the norm $\mathcal{E}^{\Gamma}$ measuring the size of the free boundary deviation from the reference domain *does not* exhibit any decay, which is consistent with the idea that the asymptotic equilibrium shape is selected from a continuum of possible nearby steady states.

Appendix A. Basic inequalities.

LEMMA A.1 (a priori bounds for A in terms of h). *Let $h\in H^3(\Gamma)$ such that $|h|_3$ is small, and consider $\Psi^{\pm}$ the solution to the elliptic problem* (1.7)*, and $A^{\pm}:=(\nabla\Psi^{\pm})^{-1}$, then,*

$$\|A^{\pm}\|_{L^\infty} \le 1 + C|h|_{2.25} + O(|h|_{2.25}^2). \tag{A.1}$$

Proof. We will omit the superindex $\pm$ throughout the proof for clarity. First, we bound the difference $\nabla\Psi - I$, using Sobolev embedding and elliptic estimates:

$$\|\nabla\Psi-\mathrm{Id}\|_{L^\infty} \le \|\nabla\Psi - I\|_{1.75} \le C|h|_{2.25}.$$

Moreover, for $0\le s\le 3$,

$$\|\nabla^2\Psi\|_s \le C|h|_{s+1.5}.$$

Using this we obtain

$$\|A-\mathrm{Id}\|_{L^\infty} \le \|A(\mathrm{Id}-\nabla\Psi)\|_{L^\infty} \le C\|A\|_{L^\infty}|h|_{2.25}.$$

Therefore,

$$\|A\|_{L^\infty} \le \|\mathrm{Id}\|_{L^\infty} + \|A-\mathrm{Id}\|_{L^\infty} \le 1 + C\|A\|_{L^\infty}|h|_{2.25},$$
$$\|A\|_{L^\infty} \le \frac{1}{1-C|h|_{2.25}} \le 1 + C|h|_{2.25} + O(|h|_{2.25}^2).$$

Moreover we have that for $0 \leq s \leq 3$,

$$\|\nabla A\|_s \leq C|h|_{s+1.5}. \qquad \square$$

LEMMA A.2. *Considering $\bar{h}_\alpha$ such that $S(\bar{q}_\alpha, \bar{h}_\alpha) \leq M$, $\alpha = 1, 2$, we have that there exists a time T_κ small enough so that the following bound for $\delta A = A_1 - A_2$ holds, for $s > 1.5$:*

$$\|\delta A\|_{L^\infty H^s} \leq \epsilon |\delta \bar{h}_t|_{L^2 H^{s+0.5}} \tag{A.2}$$

for any $\epsilon > 0$.

Proof.

$$A_1 - A_2 = \int_0^t A_{1t} - A_{2t} = -\int_0^t A_1 A_1 \nabla \delta \Psi_t^1 + (A_1(A_1 - A_2) + (A_1 - A_2)A_2)\nabla \Psi_t^2,$$

$$\|\delta A\|_s \leq \int_0^t \left(\|A_1\|_s^2 \|\delta \Psi_t\|_{s+1} + (\|A_1\|_s + \|A_2\|_s)\|\delta A\|_s \|\Psi_t^2\|_{s+1}\right),$$

(A.3)

$$\|\delta A\|_{L^\infty H^s} \leq \frac{\sqrt{T_\kappa}\|A_1\|^2_{L^\infty H^s}}{(1 - \sqrt{T_\kappa} C(M))} |\delta \bar{h}_t|_{L^2 H^{s+0.5}}.$$

Taking T_κ small enough yields the result. $\square$

LEMMA A.3 (bounds for J). *Under the hypothesis of Proposition* 3.9*, there exists a $\delta > 0$ such that if $t < \delta$, then the determinants $J^\pm = det(\nabla \Psi^\pm)$ satisfy the bound*

$$\frac{1}{2} \leq J \leq \frac{3}{2} \quad \text{on } \Gamma. \tag{A.4}$$

Proof. For any of the regions $\Omega^\pm$,

$$J = det(\nabla \Psi) = \Psi^1,_1 \Psi^2,_2 - \Psi^1,_2 \Psi^2,_1 = 1 + \left[\left(\Psi^1,_1 - 1\right)\Psi^2,_2 + \left(\Psi^2,_2 - 1\right) - \Psi^1,_2 \Psi^2,_1\right].$$

Now,

$$\begin{aligned}
&\left|\left(\Psi^1,_1 - 1\right)\Psi^2,_2 + \left(\Psi^2,_2 - 1\right) - \Psi^1,_2 \Psi^2,_1\right|_\infty \\
&\quad \leq \left|\Psi^1,_1 - 1\right|_\infty \left|\Psi^2,_2\right|_\infty + \left|\Psi^2,_2 - 1\right|_\infty + \left|\Psi^1,_2\right|_\infty \left|\Psi^2,_1\right|_\infty \\
&\quad \leq \|\nabla \Psi - I\|_{1.25}(\|\nabla \Psi\|_{1.25} + 1) + \|\nabla \Psi - I\|^2_{1.25} \\
&\quad \leq C|h|_{1.75} + C|h|^2_{1.75}.
\end{aligned}$$

But, recall that $h = h_0 + \int_0^t h_t(s)ds$, therefore, $|h|_s \leq |h_0|_s + \int_0^t |h_t(s)|_s ds \leq |h_0|_s + \sqrt{t}|h_t|_{L^2 H^s}$ with $|h_0|_s = O(\sigma)$ which we can make as small as we want. Therefore, taking t small enough, and since we are considering $|h_t|_{L^2 H^s} \leq M$, we obtain that $|h|_s$ can be made small for short time, and so

$$1 - C|h|_{1.75} - C|h|^2_{1.75} \leq J \leq 1 + C|h|_{1.75} + C|h|^2_{1.75},$$
$$\frac{1}{2} \leq J \leq \frac{3}{2} \quad \text{on } \Gamma. \qquad \square$$

Lemma A.4 (parabolic estimates for q^m). *Given q^m as in (3.19), there exists a small enough time T_κ independent of m, such that we have the following estimates:*

$$\|q^m(t)\|^2_{L^2(\Omega)} + \|q^m\|^2_{L^2_tH^1} + \kappa^{-2}|q^m|^2_{L^2_tL^2} \le C(M, q_0) \text{ for all } t \in [0, T_\kappa],$$

where $C(M, q_0)$ is a constant independent of m.

Proof. Indeed, considering $\phi = q^m$ on the weak formulation (3.18) for $l = 0$, and integrating in time, we obtain

$$\|\bar{J}_\kappa^{1/2} q^m(t)\|_0^2 + \int_0^t \|\bar{J}_\kappa^{1/2}\nabla_{^\kappa\bar{\Psi}} q^m(s)\|_0^2 + \kappa^{-2}|\bar{J}_\kappa^{1/2} q^m|_0^2 ds \le \|{}^\kappa Q_0{}^m\|_0^2 + \mathcal{R}, \tag{A.5}$$

where the error $\mathcal{R}$ is given by

$$\mathcal{R} := \int_0^t \left[\int_\Omega \partial_t(\bar{J}_\kappa)(q^m)^2 - \int_\Gamma \bar{J}_\kappa \beta^m q^m d\sigma + \int_\Omega {}^\kappa\bar{a}^i_j q^m{}_{,i}\, {}^\kappa\bar{\Psi}^j_t q^m + \int_\Omega \bar{J}_\kappa \alpha^m q^m \right] ds.$$

Using Sobolev and the Cauchy–Schwarz inequality for the terms of $\mathcal{R}$, we have that,

$$\begin{aligned}\mathcal{R} &\le \int_0^t \big[\|\bar{J}_{\kappa t}\|_{L^\infty}\|q^m\|_0^2 + |\beta^m|_0|q^m|_{L^2(\Gamma)} \\ &\qquad + \|\bar{J}_\kappa\|_{L^\infty}\|{}^\kappa\bar{\Psi}_t\|_{L^\infty}\|\nabla_{^\kappa\bar{\Psi}} q^m\|_0\|q^m\|_0 + \|\bar{J}_\kappa\|_{L^\infty}\|\alpha^m\|_0\|q^m\|_0 \big] ds \\ &\le C(M, q_0) + C\sqrt{T^m_\kappa}(\|q^m\|^2_{L^2_tH^1} + \|q^m\|^2_{L^\infty_tL^2}),\end{aligned} \tag{A.6}$$

where in the last inequality we used Young's inequality, trace estimates, and that by definition, α^m and β^m are bounded by a function of the initial data $C(M, q_0)$. Finally, estimates for ${}^\kappa\bar{A}$ from Lemma A.1, gives us that for small enough time (that does not depend on m),

$$\begin{aligned}\|q^m\|_1^2 &\le \|q^m\|_0^2 + \|\nabla_{^\kappa\bar{\Psi}} q^m\|_0^2 + \|A - I\|^2_{L^\infty}\|q^m\|_1^2 \\ &\le \|q^m\|_0^2 + \|\nabla_{^\kappa\bar{\Psi}} q^m\|_0^2 + \epsilon\|q^m\|_1^2,\end{aligned}$$

therefore, $\|q^m\|_1^2 \lesssim \|q^m\|_0^2 + \|\nabla_{^\kappa\bar{\Psi}} q^m\|_0^2$, and combining (A.6) with (A.5) along with the estimates for $\bar{J}_\kappa$ from Lemma A.3, we obtain that for a time $T^m_\kappa \le 1/4C^2$,

$$\|q^m\|^2_{L^\infty_tL^2} + \|q^m\|^2_{L^2_tH^1} + \kappa^{-2}|q^m|^2_{L^2_tL^2} \le C(M, q_0). \tag{A.7}$$

Note that since the previous estimate holds up to any $T_\kappa \le 1/4C^2$, if $T^m_k < T_k$, we can extend the solution $q^m(t)$ past T^m_κ all the way to T_κ, while still satisfying the bound (A.7). ∎

Lemma A.5. *For a small enough time $T_\kappa > 0$, the operator Φ_κ defined in (3.28) is a well-defined function from X^κ_M to itself.*

Proof. From the definition of Φ_κ (3.28), the definition of $v^\pm$ (3.17b), and the regularization of $\bar{h}^\kappa$ (3.15), we have that h satisfy the initial data of X^κ_M. The parabolic estimates obtained for the solution $q^\pm$ of the linear problem (3.17) in section 3.2.2 are the key ingredients to show that $\Phi_\kappa(\bar{h}) = h \in X^\kappa_M$. We have, for example, by the estimates for ${}^\kappa A^\pm$ from Lemma A.1,

$$\begin{aligned}|h_{ttt}|_1 = |\partial_t^2[v\cdot\tilde{n}]^+_-|_1 &\le |\nabla_{^\kappa\bar{\Psi}^+} q^+_{tt}\cdot\tilde{n}|_1 + |{}^\kappa A^\top_{tt}\nabla q^+\cdot\tilde{n}|_1 + \left|\nabla_{^\kappa\bar{\Psi}^+} q^+\cdot\tau\frac{\bar{\partial}\bar{h}^\kappa_{tt}}{(1+H\bar{h}^\kappa)}\right|_1 \\ &\quad + |\nabla_{^\kappa\bar{\Psi}^-} q^-_{tt}\cdot\tilde{n}|_1 + |{}^\kappa A^\top_{tt}\nabla q^-\cdot\tilde{n}|_1 + \left|\nabla_{^\kappa\bar{\Psi}^-} q^-\cdot\tau\frac{\bar{\partial}\bar{h}^\kappa_{tt}}{(1+H\bar{h}^\kappa)}\right|_1 + \text{l.o.t.} \\ &\lesssim \|q^+_{tt}\|_{2.5} + \|q^-_{tt}\|_{2.5} + |\bar{h}^\kappa_{tt}|_2(\|q^+\|_3 + \|q^-\|_3) + \text{l.o.t.}\end{aligned}$$

Therefore, interpolating: $\|q^{\pm}_{tt}\|_{2.5} \le \|q^{\pm}_{tt}\|_2^{1/2}\|q^{\pm}_{tt}\|_3^{1/2}$, and using the parabolic estimates for $q^{\pm}$, we obtain

$$\begin{aligned} |h_{ttt}|_{L^2_t H^1(\Gamma)} &\lesssim \sqrt{t}(\|q^+_{tt}\|^{1/2}_{L^\infty H^2}\|q^+_{tt}\|^{1/2}_{L^2H^3} + \|q^-_{tt}\|^{1/2}_{L^\infty H^2}\|q^-_{tt}\|^{1/2}_{L^2H^3}) \\ &\qquad + t|\bar h^\kappa_{tt}|_{L^\infty H^2}(\|q^+\|_{L^\infty H^3} + \|q^-\|_{L^\infty H^3}) + \text{l.o.t.} \\ &\lesssim \sqrt{t}C(M_0) + tM_0 M \\ &\le \sqrt{t}M, \end{aligned}$$

where $C(M_0)$ is a constant function of the initial data, and the last inequality follows by choosing M so that $M \ge M_0$ and T_κ small enough so that $\sqrt{t}M_0 \le 1$ $\forall t \le T_\kappa$. A similar estimate can be made for $h_t \in H^5(\Gamma)$,

$$\begin{aligned} |h_t|_5 &\le C(\|v^+\|_{5.5} + \|v^-\|_{5.5})(1 + |\bar h^\kappa|_6) \\ &\le C(\|q^+\|_{6.5} + \|q^-\|_{6.5})(1 + |\bar h^\kappa|_6)^2 \\ &\le C(\|q^+\|_{6.5} + \|q^-\|_{6.5})\left(1 + |h^\kappa_0|_6 + \sqrt{T_\kappa}|\bar h^\kappa_t|_{L^2H^6}\right)^2 \\ &\le C(\|q^+\|_{6.5} + \|q^-\|_{6.5})\left(1 + |h^\kappa_0|_6 + \kappa^{-1}\sqrt{T_\kappa}|\bar h^\kappa_t|_{L^2H^5}\right)^2, \end{aligned}$$

where on the third line we used the definition of $\bar h^\kappa$ and the Cauchy–Schwarz inequality to get the term $\sqrt{T_\kappa}|\bar h^\kappa_t|_{L^2H^6}$. The next line follows from absorbing one of the derivatives of the H^6 norm in exchange for the κ^{-1} coefficient due to the tangential convolution structure of $\bar h^\kappa_t$. Taking the L^2_t norm in time we obtain

$$\begin{aligned} |h_t|_{L^2H^5} &\le C(\|q^+\|_{L^2H^{6.5}} + \|q^-\|_{L^2H^{6.5}})\left(1 + |h^\kappa_0|_6 + \kappa^{-1}\sqrt{T_\kappa}|\bar h^\kappa_t|_{L^2H^5}\right)^2 \\ &\le C(M_0)\left(1 + |h^\kappa_0|_6 + \kappa^{-1}\sqrt{T_\kappa}|\bar h^\kappa_t|_{L^2H^5}\right)^2. \end{aligned}$$

Choosing $M \ge C(M_0)(2 + |h_0|_6)^2$ and T_κ small enough so that $\kappa^{-1}\sqrt{T_\kappa}M \le 1$ we obtain the desired inequality. The bounds for the other norms of h in the definition of X^κ_M follow in similar fashion, so we will omit their proof for brevity. ◻

Remark A.6. Notice that the time of existence T_κ depends on M, which is a function of the initial data.

Lemma A.7. *The source function f defined by* (3.32) *satisfies the bound*

$$\|f\|_{L^2H^{0.5}} \le C_M\kappa^{-1}\sqrt{T_\kappa}\ \mathcal{S}(\delta q, \delta\bar h^\kappa)^{1/2}, \tag{A.8}$$

where $\mathcal{S}(\delta q, \delta\bar h^\kappa)$ is the norm defined in (1.12) *evaluated in the differences δq and $\delta\bar h^\kappa$.*

Proof. First, let us identify the higher-order terms of f,

$$\begin{aligned} f = {}& {}^\kappa\!\bar A^i_{1\,j\,tt}({}^\kappa\!\bar A_1{}^k_j\delta q,_k),_i - \delta v_{tt}\cdot \bar w_{1\kappa} - \delta v\cdot \bar w_{1\kappa tt} \\ &- v_{2tt}\cdot\delta^\kappa\bar\Psi_t - v_2\cdot\delta^\kappa\bar\Psi_{ttt} + \delta^\kappa A_{tt}{}^i_j({}^\kappa\!\bar A_1{}^k_j q_2,_k),_i \\ &+ \delta^\kappa\!\bar A^i_j({}^\kappa\!\bar A_1{}^k_j q_{2tt},_k),_i + \text{l.o.t.}, \end{aligned} \tag{A.9}$$

where we group together in "l.o.t." all the lower-order terms that can be bounded as in (A.8) via a simple application of the Cauchy–Schwarz inequality and where the norms are directly bouded by the norm $\mathcal{S}(\delta q, \delta h)$. The first term in the definition (3.32) of f satisfies the bound

$$\begin{aligned}\|{}^\kappa\!A^i_{1jtt}({}^\kappa\!A_1{}^k_j \delta q,_k),_i\|_{L^2_t H^1} &\le C_M \|\nabla {}^\kappa\bar\Psi^1_{tt} \nabla^2 \delta q\|_{L^2_t H^1} + \text{l.o.t.}\\ &\le C_M |\bar h^{\kappa}_{tt}{}^1|_{L^2_t H^2} \|\delta q\|_{L^\infty_t H^{3.5}}\\ &\le C_M \sqrt{T_\kappa} |\bar h^{\kappa}_{tt}{}^1|_{L^\infty_t H^2} \|\delta q\|_{L^\infty_t H^{3.5}}\\ &\le C_M \sqrt{T_\kappa} \|\delta q\|_{L^\infty H^{3.5}},\end{aligned}$$

where we used the Cauchy–Schwarz inequality in the time integration to obtain the coefficient $\sqrt{T_\kappa}$. The second term can be bounded by

$$\begin{aligned}\|\delta v_{tt} \cdot \bar w_{1\kappa}\|_1 &\le \|\delta v_{tt}\|_1 \|{}^\kappa\bar\Psi^1_t\|_{L^\infty} + \|\delta v_{tt}\|_{L^4} \|\nabla {}^\kappa\bar\Psi^1_t\|_{L^4}\\ &\le C_M (\|\nabla {}^\kappa\bar\Psi^1_{tt}\|_1 \|\delta q\|_{2.5} + \|\delta q_{tt}\|_2) |\bar h^{\kappa 1}_t|_1 + \text{l.o.t.}\\ &\le C_M (|\bar h^{\kappa}_{tt}{}^1|_{1.5} \|\delta q\|_{2.5} + \|\delta q_{tt}\|_2) |\bar h^{\kappa 1}_t|_1 + \text{l.o.t.},\end{aligned}$$

therefore,

$$\|\delta v_{tt} \cdot \bar w_{1\kappa}\|_{L^2_t H^1} \le C_M \sqrt{T_\kappa} (\|\delta q\|_{L^\infty_t H^{2.5}} + \|\delta q_{tt}\|_{L^\infty_t H^2}).$$

The third term can be bounded by

$$\begin{aligned}\|\delta v \cdot {}^\kappa\bar\Psi^1_{ttt}\|_{L^2_t H^1} &\le C \|\delta v\|_{L^\infty_t H^{1.5}} |\bar h^{\kappa}_{ttt}{}^1|_{L^2 H^1}\\ &\le C\kappa^{-1} \|\delta v\|_{L^\infty_t H^{1.5}} |\bar h^{\kappa}_{ttt}{}^1|_{L^2_t L^2}\\ &\le C\kappa^{-1} \sqrt{T_\kappa} \|\delta v\|_{L^\infty_t H^{1.5}} |\bar h^{\kappa}_{ttt}{}^1|_{L^\infty_t L^2}\\ &\le C_M \kappa^{-1} \sqrt{T_\kappa} \|\delta v\|_{L^\infty_t H^{1.5}},\end{aligned}$$

where we used the smoothing of $\bar h^{\kappa}_{ttt}{}^1$ in line 2, to lower the Sobolev norm in exchange for the coefficient κ^{-1}. The fourth term can be bounded by

$$\begin{aligned}\|v_{2tt} \cdot \delta {}^\kappa\bar\Psi_t\|_{L^2_t H^1} &\le C \|v_{2tt}\|_{L^2 H^1} \|\delta {}^\kappa\bar\Psi_t\|_{1.5}\\ &\le C_M (\|{}^\kappa\bar\Psi^2_{tt}\|_{L^2_t H^2} \|q_2\|_{L^\infty_t H^{2.5}} + \|q_{2tt}\|_{L^2_t H^2}) |\delta \bar h^\kappa_t|_{L^\infty_t H^1} + \text{l.o.t.}\\ &\le C_M \sqrt{T_\kappa} (|\bar h^{\kappa}_{tt}{}^2|_{L^\infty_t H^{1.5}} \|q_2\|_{L^\infty_t H^{2.5}} + \|q_{2tt}\|_{L^\infty_t H^2}) |\delta \bar h^\kappa_t|_{L^\infty_t H^1} + \text{l.o.t.}\\ &\le C_M \sqrt{T_\kappa} |\delta \bar h^\kappa_t|_{L^\infty_t H^1}.\end{aligned}$$

The fifth term,

$$\begin{aligned}\|v_2 \cdot \delta {}^\kappa\bar\Psi_{ttt}\|_{L^2_t H^1} &\le C \|v_2\|_{L^\infty_t H^{1.5}} \|\delta {}^\kappa\bar\Psi_{ttt}\|_{L^2_t H^1}\\ &\le C \|v_2\|_{L^\infty_t H^{1.5}} |\delta \bar h^\kappa_{ttt}|_{L^2_t H^1}\\ &\le C\kappa^{-1} \|v_2\|_{L^\infty_t H^{1.5}} |\delta \bar h^\kappa_{ttt}|_{L^2_t L^2}\\ &\le C_M \kappa^{-1} \sqrt{T_\kappa} |\delta \bar h^\kappa_{ttt}|_{L^\infty_t L^2}.\end{aligned}$$

The sixth term,

$$\begin{aligned}\|\delta {}^\kappa\!A_{tt}{}^i_j ({}^\kappa\!A_1{}^k_j q_2,_k),_i\|_{L^2_t H^1} &\le \|\delta {}^\kappa\!A_{tt} {}^\kappa\!A_1 \nabla^2 q_2\|_{L^2_t H^1} + \text{l.o.t.}\\ &\le C_M \|\delta {}^\kappa\!A_{tt}\|_{L^2_t H^1} \|q_2\|_{L^\infty_t H^{3.5}} + \text{l.o.t.}\\ &\le C_M \sqrt{T_\kappa} |\delta \bar h^\kappa_{tt}|_{L^\infty_t H^{1.5}}.\end{aligned}$$

The last and most critical term needs to be analyzed in the actual $H^{0.5}$ Sobolev norm,

$$\begin{aligned}\|\delta^\kappa A^i_j({}^\kappa A_1{}^k_j q_{2tt},_k),_i\|_{L^2_t H^{0.5}} &\le C_M \|\delta^\kappa A\|_{L^\infty_t H^{1.5}} \|q_{2tt}\|_{L^2_t H^{2.5}} + \text{l.o.t.}\\ &\le C_M \sqrt{T_\kappa} |\delta \bar h^\kappa_t|_{L^2_t H^2} \|q_{2tt}\|_{L^2_t H^{2.5}}\\ &\le C_M \sqrt{T_\kappa} |\delta \bar h^\kappa_t|_{L^2_t H^2},\end{aligned}$$

where we used the bounds for $\delta^\kappa A$ from Lemma A.2, and that $\|q_{2tt}\|_{L^2_t H^{2.5}} \le \mathcal{S}(q_2) \le C_M$. The proof then follows from collecting all the terms together along with the straightforward estimates of the lower-order terms. ∎

Lemma A.8 (error terms). *The error terms from Lemma* 3.10 *are given by,*

$$\mathcal{R}(t) = \mathcal{R}^+(t) + \mathcal{R}^-(t) + \mathcal{R}_\Gamma(t) + \mathcal{R}^+_{\partial\Omega}(t) + \mathring{\mathcal{R}}^+(t) + \mathring{\mathcal{R}}^-(t),$$

where

$$\begin{aligned}\mathcal{R}^\pm &= \sum_{a+2b\le 6} \mathcal{R}^\pm_{a,b} + \sum_{a+2b\le 5} \widetilde{\mathcal{R}}^\pm_{a,b},\\ \mathcal{R}_\Gamma &= \sum_{a+2b\le 6} \mathcal{R}^{a,b}_\Gamma + \sum_{a+2b\le 5} \widetilde{\mathcal{R}}^{a,b}_\Gamma,\\ \mathcal{R}^+_{\partial\Omega} &= \sum_{a+2b\le 6} \mathcal{R}^{a,b}_{\partial\Omega} + \sum_{a+2b\le 5} \widetilde{\mathcal{R}}^{a,b}_{\partial\Omega},\\ \mathring{\mathcal{R}}^\pm &= \sum_{a+2b\le 6} \mathring{\mathcal{R}}^\pm_{a,b} + \sum_{a+2b\le 5} \widetilde{\mathcal{R}}^{\circ\pm}_{a,b}\end{aligned}$$

with

$$\begin{aligned}\mathcal{R}^\pm_{a,b} = &\sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} \int_{\Omega^\pm} c_{sl} \bar\partial^{a-s}\partial_t^{b-l} A^T \left(-\bar\partial^s\partial_t^l \nabla q + \bar\partial^s \partial_t^l \nabla\Psi A\nabla q\right) \bar\partial^a\partial_t^b v W\mu\\ &+ \int_{\Omega^\pm} \bar\partial^a\partial_t^b \Psi A \nabla v \bar\partial q \partial_t^b v W\mu\\ &- \int_{\Omega^\pm} (\bar\partial^a\partial_t^b q + \bar\partial^a\partial_t^b\Psi\cdot v)\left(\bar\partial^a\partial_t^b A\nabla v + \sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} c_{sl}\bar\partial^{a-s}\partial_t^{b-l} A \bar\partial^s\partial_t^l \nabla v\right) W\mu\\ &- \int_{\Omega^\pm} (\bar\partial^a\partial_t^b q + \bar\partial^a\partial_t^b\Psi\cdot v)\\ &\qquad \cdot\left(- \bar\partial^a\partial_t^b\Psi\cdot v_t + \Psi_t\cdot\bar\partial^a\partial_t^b v + \sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} c_{sl}\bar\partial^{a-s}\partial_t^{b-l}\Psi_t\cdot\bar\partial^s\partial_t^l v\right) W\mu\\ &+ \int_{\Omega^\pm} (\bar\partial^a\partial_t^b q + \bar\partial^a\partial_t^{b\kappa}\Psi\cdot v)(\bar\partial^a\partial_t^b\alpha) W\mu\\ &+ \frac{1}{2}\int_{\Omega^\pm} (\bar\partial^a\partial_t^b q + \bar\partial^a\partial_t^b\Psi\cdot v)^2 W_t + \int_{\Omega^\pm} (\bar\partial^a\partial_t^b q + \bar\partial^a\partial_t^b\Psi\cdot v) A\bar\partial^a\partial_t^b v \nabla(W\mu),\end{aligned}$$

$$
\begin{aligned}
\widetilde{\mathcal{R}}^{\pm}_{a,b} = &- \sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} c_{sl} \int_{\Omega^\pm} \bar\partial^{a-s}\partial_t^{b+1-l} A^i_j \bar\partial^s \partial_t^l q,_i \, \bar\partial^a\partial_t^b v^j W\mu \\
&+ \int_{\Omega^\pm} (\Psi^k_t,_l \, \bar\partial^a\partial_t^b (A^i_k A^l_j) + \sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} c_{sl} \bar\partial^{a-s}\partial_t^{b-l}\Psi^k_t,_l \, \bar\partial^s\partial_t^l (A^i_k A^l_j)) q,_i \, \bar\partial^a\partial_t^b v^j W\mu \\
&+ \int_{\Omega^\pm} \bar\partial^a\partial_t^{b+1}\Psi^k v^k,_l \, A^l_j \bar\partial^a\partial_t^b v^j W\mu \\
&- \int_{\Omega^\pm} (\bar\partial^a\partial_t^{b+1} q + \bar\partial^a\partial_t^{b+1}\Psi\cdot v)\Bigg(\Psi_t\cdot\bar\partial^a\partial_t^b v + \bar\partial^a\partial_t^b A^i_j v^j,_i \\
&\qquad\qquad + \sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} c_{sl} (\bar\partial^{a-s}\partial_t^{b-l}\Psi_t\cdot\bar\partial^s\partial_t^l v + \bar\partial^{a-s}\partial_t^{b-l} A^i_j \bar\partial^s\partial_t^l v^j,_i\,)\Bigg) W\mu \\
&+ \int_{\Omega^\pm} (\bar\partial^a\partial_t^{b+1} q + \bar\partial^a\partial_t^{b+1\kappa}\Psi\cdot v)(\bar\partial^a\partial_t^b \alpha) W\mu \\
&+ \frac{1}{2}\int_{\Omega^\pm} (\bar\partial^a\partial_t^b v)^2 W_t\mu + \int_{\Omega^\pm} (\bar\partial^a\partial_t^{b+1} q + \bar\partial^a\partial_t^{b+1}\Psi\cdot v)(A^i_j \bar\partial^a\partial_t^b v^j)(W\mu),_i \,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}^{a,b}_{\Gamma} = &\frac{1}{2}\int_\Gamma \partial_t \left(e^{(-\lambda_1+\eta)t} a_\kappa^2\right) (\bar\partial^a\partial_t^b \ \Lambda_\kappa h)^2 \\
&- e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}(1+Hh^\kappa)((h^\kappa\bar\partial^a\partial_t^b N + \bar\partial^a\partial_t^b x \\
&+ \sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} \bar\partial^{a-s}\partial_t^{b-l} h^\kappa \bar\partial^s\partial_t^l N)\cdot(-\bar\partial h^\kappa \tau \\
&+ (1+Hh^\kappa)N))[\bar\partial^a\partial_t^b v\cdot\tilde n^\kappa]^+_- \\
&+ e^{(-\lambda_1+\eta)t}\int_\Gamma a_\kappa^2 \bar\partial^a\partial_t^b h^\kappa \Bigg([v\cdot\bar\partial^a\partial_t^b\tilde n^\kappa]^+_- + \sum_{\substack{1\le s\le a-1\\ 1\le l\le b-1}} c_{sl}[\bar\partial^{a-s}\partial_t^{b-l} v\cdot\bar\partial^s\partial_t^l\tilde n^\kappa]^+_-\Bigg) \\
&- e^{(-\lambda_1+\eta)t}\int_\Gamma \bar\partial^a\partial_t^b(\Lambda_\kappa h)[\Lambda_\kappa, a_\kappa^2\bar\partial^a\partial_t^b] h_t \\
&- \kappa^2 e^{(-\lambda_1+\eta)t}\int_\Gamma \left(r_\kappa^+\bar\partial^a\partial_t^b\beta^+(\bar\partial^a\partial_t^b v^+\cdot n^\kappa) + r_\kappa^-\bar\partial^a\partial_t^b\beta^-(\bar\partial^a\partial_t^b v^-\cdot n^\kappa)\right) \\
&- \kappa^2 e^{(-\lambda_1+\eta)t}\sum_{\substack{1\le s\le a\\ 1\le l\le b}} c_{sl}\int_\Gamma \big((\partial_N q^+)^{-1}\bar\partial^{a-s}\partial_t^{b-l} v^+\cdot\bar\partial^s\partial_t^l(A^\top N)\bar\partial^a\partial_t^b v^+\cdot A^\top N \\
&+ (\partial_N q^-)^{-1}\bar\partial^{a-s}\partial_t^{b-l} v^-\cdot\bar\partial^s\partial_t^l(A^\top N)\bar\partial^a\partial_t^b v^-\cdot A^\top N\big) \\
&- \kappa^2 e^{(-\lambda_1+\eta)t}\int_\Gamma \Big(\bar\partial^a\partial_t^{b\kappa}\Psi^{i\kappa}A^l_i\tau^l\bar\partial(v^+\cdot{}^\kappa\!A^\top N)(\bar\partial^a\partial_t^b v^+\cdot n^\kappa)J_\kappa^{-1}\sqrt{g_\kappa}(\partial_N q^+)^{-1} \\
&\qquad + \bar\partial^a\partial_t^{b\kappa}\Psi^{i\kappa}A^l_i\tau^l\bar\partial(v^-\cdot{}^\kappa\!A^\top N)(\bar\partial^a\partial_t^b v^-\cdot n^\kappa)J_\kappa^{-1}\sqrt{g_\kappa}(\partial_N q^-)^{-1}\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathcal{R}}^{a,b}_{\Gamma} = & -e^{(-\lambda_1+\eta)t}\int_\Gamma J_\kappa^{-2}(1+Hh^\kappa)(h_t^\kappa\bar\partial^a\partial_t^b N + \sum_{\substack{1\le s\le a-1\\1\le l\le b-1}} c_{sl}\bar\partial^{a-s}\partial_t^{b-l}h_t^\kappa\bar\partial^s\partial_t^l N)\\
&\quad\cdot(-\bar\partial h^\kappa\tau + (1+Hh^\kappa)N)[\bar\partial^a\partial_t^{b+1}v\cdot n]^{\pm}_{-}\\
&\quad + e^{(-\lambda_1+\eta)t}\int_\Gamma a_\kappa^2\bar\partial^a\partial_t^{b+1}h^\kappa\left([v\cdot\bar\partial^a\partial_t^b\tilde n^\kappa]^{\pm}_{-} + \sum_{\substack{1\le s\le a-1\\1\le l\le b-1}} c_{sl}[\bar\partial^{a-s}\partial_t^{b-s}v\cdot\bar\partial^l\partial_t^s\tilde n^\kappa]^{\pm}_{-}\right)\\
& + \int_\Gamma(\partial_N q^-)(1+Hh^\kappa)^2 J_\kappa^{-2}[\Lambda_\kappa,\bar\partial^a]\partial_t^{b+1}\Lambda_\kappa h\bar\partial^a\partial_t^b h_t\\
&\quad + e^{(-\lambda_1+\eta)t}\int_\Gamma\bar\partial^a\partial_t^{b+1}\Lambda_\kappa h[a_\kappa^2\bar\partial^a,\Lambda_\kappa]\partial_t^{b+1}h\\
&\quad + \kappa^2 e^{(-\lambda_1+\eta)t}\int_\Gamma\big(\bar\partial^a\partial_t^{b+1\kappa}\Psi{}^\kappa\!A^i_k\tau^i\bar\partial(v^+\cdot{}^\kappa\!A^\top N)\bar\partial^a\partial_t^b v^+\cdot{}^\kappa\!A^\top N(\partial_N q^+)^{-1}\\
&\qquad\quad + \bar\partial^a\partial_t^{b+1\kappa}\Psi{}^\kappa\!A^i_k\tau^i\bar\partial(v^-\cdot{}^\kappa\!A^\top N)\bar\partial^a\partial_t^b v^-\cdot{}^\kappa\!A^\top N(\partial_N q^-)^{-1}\big)\\
&\quad - \kappa^2 e^{(-\lambda_1+\eta)t}\sum_{s,l} c_{sl}\int_\Gamma((\partial_t(\bar\partial^{a-l}\partial_t^{b-s}v^+\cdot\bar\partial^l\partial_t^s(A^\top N)) - \bar\partial^a\partial_t^{b+1}\beta^+)\\
&\qquad\qquad\qquad\qquad\cdot(\bar\partial^a\partial_t^b v^+\cdot A^\top N)(\partial_N q^+)^{-1}\\
&\qquad - (\partial_t(\bar\partial^{a-l}\partial_t^{b-s}v^-\cdot\bar\partial^l\partial_t^s(A^\top N)) - \bar\partial^a\partial_t^{b+1}\beta^-)(\bar\partial^a\partial_t^b v^-\cdot A^\top N)(\partial_N q^-)^{-1})\\
&\quad + \frac{\kappa^2}{2}\int_\Gamma\Big((\bar\partial^a\partial_t^b v^+\cdot A^\top N)^2\partial_t(e^{(-\lambda_1+\eta)t}(\partial_N q^+)^{-1})\\
&\qquad\quad + (\bar\partial^a\partial_t^b v^-\cdot A^\top N)^2\partial_t(e^{(-\lambda_1+\eta)t}(\partial_N q^-)^{-1})\Big),\\
\mathcal{R}^{a,b}_{\partial\Omega} = & -\int_{\partial\Omega}(\bar\partial^a\partial_t^b q^+ + \bar\partial^a\partial_t^{b\kappa}\Psi\cdot v)(\bar\partial^a\partial_t^b\gamma - \sum_{s,l}c_{sl}\bar\partial^{a-l}\partial_t^{b-s}v\cdot\bar\partial^l\partial_t^s\mathbf{N}^+)W,\\
\widetilde{\mathcal{R}}^{a,b}_{\partial\Omega} = & \int_{\partial\Omega}(\bar\partial^a\partial_t^{b+1}q^+ + \bar\partial^a\partial_t^{b+1}\Psi^+\cdot v^+)(\bar\partial^a\partial_t^b\gamma - \sum_{s,l}c_{sl}\bar\partial^{a-l}\partial_t^{b-s}v\cdot\bar\partial^l\partial_t^s\mathbf{N}^+)W,\\
\mathring{\mathcal{R}}^{\pm}_{a,b} = & \sum_{\substack{0\le s^i\le a^i\\1\le|s|\le|a|-1\\1\le l\le b-1}} c_{sl}\int_{\Omega^\pm}(1-\mu)\Big(-\partial^{a-s}\partial_t^{b-l}A^j_i\partial^s\partial_t^l q,_j + \partial^{a-s}\partial_t^{b-l}A^j_k\partial^s\partial_t^l\Psi^k,_l A^l_i q,_j\Big)\partial^a\partial_t^b vW\\
&\quad - \int_{\Omega^\pm}(1-\mu)\partial^a\partial_t^b\Psi^k v^k,_l A^l_i\partial^a\partial_t^b v^i W\\
&\quad + \int_{\Omega^\pm}(\partial^a\partial_t^b q + \partial^a\partial_t^b\Psi\cdot v)\Bigg[((1-\mu)W),_j A^j_i\partial^a\partial_t^b v^i + (1-\mu)W\Bigg((\partial^a\partial_t^b A^j_i v^i),_j\\
&\qquad\qquad\qquad\qquad + \sum_{\substack{0\le s^i\le a^i\\1\le|s|\le|a|-1\\1\le l\le b-1}} c_{sl}(\partial^{a-s}\partial_t^{b-l}A^j_i\partial^s\partial_t^l v^i),_j\Bigg)\Bigg]\\
&\quad + \int_{\Omega^\pm}(1-\mu)(\partial^a\partial_t^b q + \partial^a\partial_t^b\Psi\cdot v)(\partial^a\partial_t^b\Psi\cdot v_t)W\\
&\quad + \int_{\Omega^\pm}(\bar\partial^a\partial_t^b q + \bar\partial^a\partial_t^{b\kappa}\Psi\cdot v)(\bar\partial^a\partial_t^b\alpha)(1-\mu)W.
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathcal{R}}^{\circ\pm}_{a,b} = &- \sum_{\substack{0\le s^i\le a^i\\ 1\le|s|\le|a|-1\\ 1\le l\le b-1}} c_{sl} \int_{\Omega^\pm} (1-\mu)\partial^{a-s}\partial_t^{b-l} A^i_j \partial^s\partial_t^l q,_i \, \partial^a\partial_t^b v^j W \\
&+ \int_{\Omega^\pm} \left(\Psi_t^k,_l \, \partial^a\partial_t^b (A^i_k A^l_j) + \sum_{\substack{0\le s^i\le a^i\\ 1\le|s|\le|a|-1\\ 1\le l\le b-1}} c_{sl} \partial^{a-l}\partial_t^{b-s} \Psi_t^k,_l \, \partial^l\partial_t^s (A^i_k A^l_j) \right) \\
&\qquad \cdot q,_i \, \partial^a\partial_t^b v^j (1-\mu) W \\
&+ \int_{\Omega^\pm} \partial^a\partial_t^{b+1}\Psi^k v^k,_l \, A^l_j \partial^a\partial_t^b v^j (1-\mu) \\
&+ \int_{\Omega^\pm} (\partial^a\partial_t^{b+1} q + \partial^a\partial_t^{b+1}\Psi\cdot v)(A^i_j \partial^a\partial_t^b v^j)((1-\mu)W),_i \\
&- \int_{\Omega^\pm} (\partial^a\partial_t^{b+1} q + \partial^a\partial_t^{b+1}\Psi\cdot v) \\
&\qquad \cdot \Bigg(\Psi_t\cdot\partial^a\partial_t^b v + \partial^a\partial_t^b A^i_j v^j,_i + \sum_{\substack{0\le s^i\le a^i\\ 1\le|s|\le|a|-1\\ 1\le l\le b-1}} c_{sl} \big(\partial^{a-s}\partial_t^{b-l}\Psi_t \cdot \partial^s\partial_t^l v \\
&\qquad\qquad + \partial^{a-s}\partial_t^{b-l} A^i_j \partial^s\partial_t^l v^j,_i \Bigg) (1-\mu) W \\
&+ \int_{\Omega^\pm} (\partial^a\partial_t^{b+1} q + \partial^a\partial_t^{b+1}\Psi\cdot v)(\bar\partial^a\partial_t^b \alpha)(1-\mu) W,
\end{aligned}
$$

where we have omitted the upper indices $^\pm$ *inside the interior integrals for simplicity of notation, but it is assumed that the functions* $q^\pm$, $\Psi^\pm$, $A^\pm$, $v^\pm$, $W^\pm$ *are being integrated over the corresponding region* $\Omega^\pm$.

Acknowledgment. We thank the anonymous referees for many improvement suggestions.

REFERENCES

[1] I. Athanasopoulos, L. Caffarelli, and S. Salsa, *Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems*, Ann. of Math. (2), 143 (1996), pp. 413–434.

[2] I. Athanasopoulos, L. Caffarelli, and S. Salsa, *Regularity of the free boundary in parabolic phase-transition problems*, Acta Math., 176 (1996), pp. 245–282.

[3] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa, *Phase transition problems of parabolic type: Flat free boundaries are smooth*, Comm. Pure Appl. Math., 51 (1998), pp. 77–112.

[4] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa, *Stefan-like problems with curvature*, J. Geom. Anal., 13 (2003), pp. 21–27 .

[5] L. Caffarelli and S. Salsa, *A Geometric Approach to Free Boundary Problems*, Grad. Stud. Math. 68, AMS, Providence, RI, 2005.

[6] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math., 139 (1977), pp. 155–184.

[7] L. A. Caffarelli, *Some aspects of the one-phase Stefan problem*, Indiana Univ. Math. J., 27 (1978), pp. 73–77.

[8] L. A. Caffarelli and L. C. Evans, *Continuity of the temperature in the two-phase Stefan problem*, Arch. Ration. Mech. Anal., 81 (1983), pp. 199–220.

[9] L. A. Caffarelli and A. Friedman, *Continuity of the temperature in the Stefan problem*, Indiana Univ. Math. J., 28 (1979), pp. 53–70.

[10] C. H. A. Cheng, D. Coutand, and S. Shkoller, *Global existence and decay for solutions of the Hele-Shaw flow with injection*, Interfaces Free Bound., 16 (2014), pp. 297–338.

[11] C. H. A. Cheng, R. Granero-Belinchón, and S. Shkoller, *Well-posedness of the Muskat problem with H^2 initial data*, Adv. Math., 286 (2016), pp. 32–104.

[12] S. Choi and I. C. Kim, *Regularity of one-phase Stefan problem near Lipschitz initial data*, Amer. J. Math., 132 (2010), pp. 1693–1727.

[13] S. Choi and I. C. Kim, *The two-phase Stefan problem: Regularization near Lipschitz initial data by phase dynamics*, Anal. PDE, 5 (2012), pp. 1063–1103.

[14] A. Córdoba, D. Córdoba, and F. Gancedo, *Interface evolution: The Hele-Shaw and Muskat problems*, Ann. of Math. (2), 173 (2011), pp. 477–542.

[15] D. Coutand and S. Shkoller, *The interaction between quasilinear elastodynamics and the Navier-Stokes equations*, Arch. Ration. Mech. Anal., 179 (2006), pp. 303–352.

[16] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc., 20 (2007), pp. 829–930.

[17] D. Coutand and S. Shkoller, *A simple proof of well-posedness for the free-surface incompressible Euler equations*, Discrete Contin. Dyn. Syst. Ser. S, 3 (2010), pp. 429–449.

[18] D. Coutand and S. Shkoller, *Well-posedness in smooth function spaces for moving-boundary 1-D compressible Euler equations in physical vacuum*, Comm. Pure Appl. Math., 64 (2011), pp. 328–366.

[19] D. Coutand and S. Shkoller, *Well-posedness in smooth function spaces for moving-boundary 3-D compressible Euler equations in physical vacuum*, Arch. Ration. Mech. Anal., 2012.

[20] D. Coutand and S. Shkoller, *On the finite-time splash and splat singularities for the 3-D free-surface Euler equations*, Comm. Math. Phys., 325 (2014), pp. 143–183.

[21] D. Coutand and S. Shkoller, *On the impossibility of finite-time splash singularities for vortex sheets*, Arch. Ration. Mech. Anal., 221 (2016), pp. 987–1033.

[22] P. Daskalopoulos and K. Lee, *All time smooth solutions of the one-phase Stefan problem and the Hele-Shaw flow*, Comm. Partial Differential Equations, 29 (2004), pp. 71–89.

[23] D. G. Ebin, *The equations of motion of a perfect fluid with free boundary are not well posed*, Comm. Partial Differential Equations, 12 (1987), pp. 1175–1201.

[24] A. Friedman, *The Stefan problem in several space variables*, Trans. Amer. Math. Soc., 133 (1968), pp. 51–87.

[25] A. Friedman, *Variational Principles and Free-Boundary Problems*, 2nd ed., Robert E. Krieger, Malabar, FL, 1988.

[26] A. Friedman and D. Kinderlehrer, *A one phase Stefan problem*, Indiana Univ. Math. J., 24 (1975), pp. 1005–1035.

[27] E. V. Frolova and V. A. Solonnikov, *L_p-theory for the Stefan problem*, J. Math. Sci., 373 (2000), pp. 989–1006.

[28] M. Hadžić, *Orthogonality conditions and asymptotic stability in the Stefan problem with surface tension*, Arch. Ration. Mech. Anal., 203 (2012), pp. 719–745.

[29] M. Hadžić and Y. Guo, *Stability in the Stefan problem with surface tension* (I), Comm. Partial Differential Equations, 35 (2010), pp. 201–244.

[30] M. Hadžić and S. Shkoller, *Well-posedness for the classical Stefan problem and the zero surface tension limit*, Arch. Ration. Mech. Anal., 223 (2017), pp. 213–264.

[31] M. Hadžić and S. Shkoller, *Global stability and decay for the classical Stefan problem*, Comm. Pure Appl. Math., 68 (2015), pp. 689–757.

[32] M. Hadžić and S. Shkoller, *Global stability of steady states in the classical Stefan problem for general boundary shapes*, Philos. Trans. A, 373 (2015), 20140284.

[33] E. Hanzawa, *Classical solutions of the Stefan problem*, Tôhoku Math. J. (2), 33 (1981), pp. 297–335.

[34] S. L. Kamenomostskaja, *On Stefan's problem*, Mat. Sb. (N.S.), 53 (1961), pp. 489–514.

[35] I. C. Kim, *Uniqueness and existence results on the Hele-Shaw and the Stefan problems*, Arch. Ration. Mech. Anal., 168 (2003), pp. 299–328.

[36] I. C. Kim and N. Požár, *Viscosity solutions for the two-phase Stefan problem*, Comm. Partial Differential Equations, 36 (2011), pp. 42–66.

[37] D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4), 4 (1977), pp. 373–391.

[38] D. Kinderlehrer and L. Nirenberg, *The smoothness of the free boundary in the one phase Stefan problem*, Comm. Pure Appl. Math., 31 (1978), pp. 257–282.

[39] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Pure Appl. Math. 88, Academic Press, New York, 1980.
[40] H. Koch, *Classical solutions to phase transition problems are smooth*, Comm. Partial Differential Equations, 23 (1998), pp. 389–437.
[41] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and Quasilinear Equations of Parabolic Type*, Trans. Math. Monogr. 23. AMS, Providence, RI, 1968.
[42] P.-L. Lions, *Bifurcation and optimal stochastic control*, Nonlinear Anal., 7 (1983), pp. 177–207.
[43] A. M. Meirmanov, *The Stefan Problem*, De Gruyter Exp. Math. 3, Walter de Gruyter, Berlin, 1992.
[44] J. Prüss, J. Saal, and G. Simonett, *Existence of analytic solutions for the classical Stefan problem*, Math. Ann., 338 (2007), pp. 703–755.
[45] J. Prüss, G. Simonett, and R. Zacher, *Qualitative behavior of solutions for thermodynamically consistent Stefan problems with surface tension*, Arch. Ration. Mech. Anal., 207 (2013), pp. 611–667.
[46] F. Quirós and J. L. Vázquez, *Asymptotic convergence of the Stefan problem to Hele-Shaw*, Trans. Amer. Math. Soc., 353 (2001), pp. 609–634.
[47] L. I. Rubenstein, *The Stefan Problem*, Transl. Math. Monogr. 27, AMS, Providence, RI, 1971.
[48] J. Stefan, *Über die theorie der eisbildung, insbesondere über die eisbildung im polarmeere*, Sitzungs-berichteder Osterreichischen Akad. Wiss. Math. Naturwissen-schaftliche Klasse, Abteilung 2, Math. Astron. Phys. Meteorol. Tech., 98 (1889), pp. 965–983.
[49] J. Stefan, *Über die theorie der eisbildung, insbesondere über die eisbildung im polarmeere*, Ann. Phys, Chem., 42 (1891), pp. 269–286.
[50] A. Visintin, *Models of Phase Transitions*, Progr. Nonlinear Differential Equations Appl. 28. Birkhäuser, Boston, MA, 1996.
[51] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in* 3-*D*, J. Amer. Math. Soc., 12 (1990), pp. 445–495.