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THE VORTEX BLOB METHOD AS A SECOND-GRADE NON-NEWTONIAN FLUID

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Abstract. We show that a certain class of vortex blob approximations for ideal hydrodynamics in two dimensions can be rigorously understood as solutions to the equations of second-grade non-Newtonian fluids with zero viscosity, and initial data in the space of Radon measures $\mathcal{M}(\mathbb{R}^2)$. The solutions of this regularized PDE, also known as the averaged Euler or Euler-α equations, are geodesics on the volume preserving diffeomorphism group with respect to a new weak right invariant metric. We prove global existence of unique weak solutions (geodesics) for initial vorticity in $\mathcal{M}(\mathbb{R}^2)$ such as point-vortex data, and show that the associated coadjoint orbit is preserved by the flow. Moreover, solutions of this particular vortex blob method converge to solutions of the Euler equations with bounded initial vorticity, provided that the initial data is approximated weakly in measure, and the total variation of the approximation also converges. In particular, this includes grid-based approximation schemes of the type that are usually used for vortex methods.

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1. Introduction

The starting point of our investigation is the somewhat surprising fact that the equations of motion for an inviscid non-Newtonian fluid of second grade, and Chorin's vortex blob algorithm with a particular choice of cut-off or blob function are, at least formally, equivalent.

The velocity field $u = u(x,t)$ of a second grade fluid, under the assumptions of observer objectivity and material frame-indifference, satisfies the unique equation

$$(1-\alpha^2\Delta)\partial_t u + u\cdot\nabla(1-\alpha^2\Delta)u - \alpha^2\,(\nabla u)^t\cdot\Delta u = -\operatorname{grad} p\,, \tag{1.1a}$$

$$\operatorname{div} u = 0\,, \tag{1.1b}$$

$$u(0) = u_0\,, \tag{1.1c}$$

where $p = p(x,t)$ is the pressure function which is determined (modulo constants) by the velocity field. See [11] and references therein for a discussion of the constitutive theory of second grade fluids, and [9, 8] for well-posedness of the viscous second-grade fluid equations. In this context, the constant $\alpha > 0$ is a material parameter which represents the elastic response of the fluid.

In two dimensions, taking the curl of equation (1.1a) and setting $q = (1-\alpha^2\Delta)\operatorname{curl}_{2\mathrm{D}} u$ yields the vorticity form

$$\partial_t q + u\cdot\operatorname{grad} q = 0\,, \tag{1.2a}$$

$$u = K^\alpha * q\,, \tag{1.2b}$$

$$q(0) = q_0\,, \tag{1.2c}$$

where $q = q(x,t)$ is called the *potential vorticity*, and K^α is the integral kernel of the inverse of $(1-\alpha^2\Delta)\operatorname{curl}_{2\mathrm{D}}$, defined so that the divergence condition (1.1b) is satisfied.

When α is interpreted as a length scale, (1.1) or (1.2) are known as the averaged Euler or Euler-α equations [16] which model the large scale flow (spatial scales larger than α) of an ideal incompressible fluid. Their analysis and rich geometry has recently received much attention [21, 23, 24]. In particular, solutions of (1.1) on an n-dimensional Riemannian manifolds (M,g) arise as geodesic flow on the group of H^s-class volume preserving diffeomorphisms $\mathcal{D}^s_\mu$ provided $s > (n/2)+1$ with respect to a new weak right invariant metric, given at the identity element $e \in \mathcal{D}^s_\mu$ by

$$\langle u,v\rangle_e = (u,v)_{L^2} + 2\alpha^2(\operatorname{Def} u, \operatorname{Def} v)_{L^2} \tag{1.3}$$

where $\operatorname{Def} u = (\nabla u + \nabla u^t)/2$. Thus, following the program of Arnold [2] and Ebin and Marsden [13], local-in-time well-posedness for classical solutions is a direct consequence of the existence of C^∞ geodesics of $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathcal{D}^s_\mu$.

The vortex blob method was introduced by Chorin [6] as a regularization of the point vortex algorithm for ideal hydrodynamics, and can be understood as follows. Consider the vorticity form of the Euler equations on $\mathbb{R}^2$,

$$\partial_t \omega + u \cdot \operatorname{grad} \omega = 0\,, \tag{1.4a}$$

$$u = K * \omega, \tag{1.4b}$$

$$\omega(0) = \omega_0\,. \tag{1.4c}$$

Here $K(x,y) = -1/(2\pi)\,\nabla^\perp \log|x-y|$ and $\omega = \omega(x,t)$ is the physical vorticity of the flow. When the velocity field is sufficiently regular—u is at least continuous in t and quasi-Lipschitz in x, uniformly over finite intervals of time—we may define the Lagrangian flow map $\eta_t = \eta(\,\cdot\,,t)$ by

$$\partial_t \eta(x,t) = u(\eta(x,t),t)\,, \tag{1.5}$$

or equivalently by

$$\partial_t \eta_t = u_t \circ \eta_t\,. \tag{1.6}$$

For each t, the map $\eta(\,\cdot\,,t)$ is in $\mathcal{G}$, the group of all homeomorphisms ϕ of $\mathbb{R}^2$ which preserve the Lebesgue measure. The pointwise conservation of vorticity under the Euler flow is thus expressed by $\omega_t \circ \eta_t = \omega_0$; combining (1.5), (1.4b), and the initial condition $\eta(\,\cdot\,,0) = e$, we obtain the ODE

$$\partial_t \eta(x,t) = \int_{\mathbb{R}^2} K(\eta(x,t),\eta(y,t))\,\omega_0(y)\,\mathrm{d}y\,. \tag{1.7}$$

Letting δ denote the Dirac measure and substituting the point vortex ansatz

$$\omega(x,t) = \sum_{i=1}^N \Gamma_i\,\delta(x - x_i(t))\,, \tag{1.8}$$

into (1.7), we obtain a finite dimensional system of ordinary differential equations for the vortex centers $x_1,\ldots,x_N$. However, the induced velocity field has $1/|x|$-type singularities at the vortex centers. Hence, the point vortex system is neither numerically well-behaved (the exact solution of the point vortex system may even collapse in a finite time for small sets of initial data [20]), nor does it approximate physically relevant velocity fields very well.

The idea of the vortex blob method is to smooth the Dirac measure by a *cut-off* or *blob function* χ that decays at infinity and whose mass is mostly supported in a disc of diameter α. This leads to the following equation for the Lagrangian flow:

$$\partial_t \eta^\alpha(x,t) = \int_{\mathbb{R}^2} K^\alpha(\eta^\alpha(x,t), \eta^\alpha(y,t))\, \omega_0(y)\, \mathrm{d}y\,, \tag{1.9a}$$

where

$$K^\alpha = \nabla^\perp G^\alpha\,, \tag{1.9b}$$

$$-\Delta G^\alpha(x,y) = \chi^\alpha(|x-y|) \equiv \frac{1}{\alpha^2} \chi\left(\frac{|x-y|}{\alpha}\right). \tag{1.9c}$$

Many researchers have investigated the convergence properties of this scheme [3, 4, 7, 14, 19]. In particular, for certain smooth cut-off functions, such as Bessel functions, the order of accuracy with respect to the regularization parameter α depends only on the smoothness of the Euler flow ("infinite order accuracy").

It is now easy to see that the equation of a second grade fluid (1.2) and the vortex blob method coincide when $\chi(x) = -1/(2\pi)\, K_0(x)$, where K_0 is the modified Bessel function of the second kind which is the Green's kernel for the operator $(1-\Delta)$. Thus far, this relationship has only been formally established, as it remains to be proven that the point-vortex ansatz (1.8) makes sense as data for the PDE (1.2); moreover, it is not *a priori* clear if solutions to the vortex blob method converge to true Euler solutions (in the sense of PDE). Our results are the following.

We show that the Lagrangian flow formulation of the blob method (1.9) with K_0 cut-off function is well-posed for initial potential vorticities q_0 in $\mathcal{M}(\mathbb{R}^2)$, the space of Radon measures on $\mathbb{R}^2$. In particular, this includes point-vortex initial data. Such a result does not hold for the Euler equations, where the flow map of the point vortex system (1.7) is not known to be well-defined.

This result allows us to rigorously classify the co-adjoint orbits characterized by point-vortex initial data. Let us explain what we mean by this. The configuration space for ideal incompressible hydrodynamics is the volume-preserving diffeomorphism group, and for $s > (n/2)+1$, $\mathcal{D}^s_\mu$ is a C^∞ Hilbert manifold and a smooth topological group. While $\mathcal{D}^s_\mu$ is not a Lie group (left composition and inversion are only C^0 and the group exponential map does not cover a neighborhood of the identity), it behaves similar to a Lie group, because of the smooth properties of the Riemannian exponential map (see [13] and [23, 24]). The Eulerian phase space for the fluid motion is the single fiber $T_e\mathcal{D}^s_\mu$ consisting of H^s-class divergence free vector fields on the fluid container, and this vector space can be formally thought of as the "Lie algebra" of $\mathcal{D}^s_\mu$. The cotangent space at the identity is given by $H^s(\Lambda^1)/dH^s(\Lambda^0)$, the H^s-class differential 1-forms modulo exact 0-forms. Using the fact that the exterior derivative $d\colon T^*_e\mathcal{D}^s_\mu \to H^{s-1}(\Lambda^2)$ is an isomorphism, and the fact that we may identify $H^{s-1}(\Lambda^2)$ with $H^{s-1}(\Lambda^0)$, the role of the dual of the "Lie algebra" for 2D hydrodynamics is played by the H^{s-1}-class vorticity functions. The representation of $\mathcal{D}^s_\mu$ on this "Lie algebra" is provided by the co-Adjoint action, so that for $\eta \in \mathcal{D}^s_\mu$ and $\omega \in H^{s-1}(\Lambda^0)$, $\mathrm{Ad}^*_\eta(\omega) = \omega \circ \eta$, and the invariance of the co-Adjoint orbit is merely the pointwise conservation of vorticity which is fundamental to 2D hydrodynamics. If one temporarily ignores the topology and works formally, then it is possible to classify certain interesting and important co-Adjoint orbits. Specifically, it is a result of Marsden and Weinstein [22] that point-vortex initial data (1.8) define the co-Adjoint orbit on which point-vortex dynamics evolve. This is clearly a formal result as Dirac measures are not elements of H^{s-1} for $s > 2$; consequently, the problem is to supply a candidate topology for the "Lie algebra" which is general enough to contain the Dirac measures, and weaken the regularity of the configuration space so that its "representation" is well-defined. In doing so, one can establish a rigorous classification of the orbit. By using $\mathcal{G}$ for the configuration space and $\mathcal{M}(\mathbb{R}^2)$ for the "Lie algebra," and by defining a new notion of *weak* co-adjoint action which coincides with the notion of a weak solution, we are able to establish the orbit classification for point-vortex initial data, and prove that our particular vortex blob method leaves such weak co-adjoint orbits invariant.

Finally, we consider the matter of greatest practical importance: the convergence of solutions of the vortex blob method to solutions of the Euler equations as the blob diameter $\alpha \to 0$. We prove this convergence result under the rather mild assumption that the initial Euler vorticity field ω_0 is continuous with compact support and is approximated on its support by a sequence of weakly converging measures in $\mathcal{M}(\mathbb{R}^2)$ that have uniformly bounded total variation. (The restriction to compact support will be replaced by weaker assumptions on the decay at infinity.)

The precise statements of our results are as follows.

Theorem 1. *For initial data* $q_0 \in \mathcal{M}(\mathbb{R}^2)$*, there exists a unique global weak solution to* (1.2) *with*

$$\eta^\alpha \in C^1(\mathbb{R};\mathcal{G})\,, \quad u^\alpha \in C^0(\mathbb{R};C^0_{\mathrm{div}}(\mathbb{R}^2,\mathbb{R}^2))\,, \quad \textit{and} \quad q \in C^0(\mathbb{R};\mathcal{M}(\mathbb{R}^2))\,, \tag{1.10}$$

where the subscript div *denotes divergence-free. As a consequence, the co-Adjoint action* $\mathrm{Ad}^*_\eta(q)$ *and the weak co-adjoint action* $\text{w-ad}^*_u(q)$ *are conserved.*

Remark 1. The solution that we construct may not necessarily have finite energy, i.e., the velocity field u^α may not be in L^2. None of our results, however, relies on energy type estimates. Furthermore, as is the case for the Euler equations, the initial potential vorticity can be decomposed into a radially symmetric and a mean-zero part, with a corresponding velocity field u in the affine space $u_{\mathrm{stationary}} + L^2(\mathbb{R}^2,\mathbb{R}^2)$. For details see DiPerna and Majda [12].

Remark 2. An immediate consequence of the uniqueness of the solution and the time-reversibility of the equation is that the vortex blob system cannot collapse in finite time, i.e., two or more vortex centers cannot merge into one in finite time. For non-regularized Euler vortex dynamics, on the other hand, it is known that vortex collaps occurs on small sets of initial configurations [20].

Remark 3. The kernel K^α which corresponds to a second grade fluid is the least regular kernel (modulo possible sub-logarithmic corrections) for which uniqueness of point vortex solutions can be shown. An equivalent uniqueness result based on Sobolev space methods, and for bounded domains is given in [15].

Theorem 2. *Let* η *be the flow map of the Euler equation* (1.7) *with initial vorticity* $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$*. Suppose that* ω_0 *is approximated by a sequence of measures* q_0^n *in* $\mathcal{M}(\mathbb{R}^2)$ *such that* $q_0^n \rightharpoonup \omega_0$ *weakly in* $\mathcal{M}(\mathbb{R}^2)$ *and* $\|q_0^n\|_{\mathcal{M}} \to \|\omega_0\|_{L^1}$*. Then for every* $T > 0$*, there exists a sequence* $\{\alpha_n\}$ *converging to zero as* $n \to \infty$ *such that when* η^{α_n} *denotes the flow map of the vortex method with* $\alpha = \alpha_n$ *and initial data* q_0^n*,*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^2} \left|\eta^{\alpha_n}(x,t) - \eta(x,t)\right| = 0\,. \tag{1.11}$$

Remark 4. The idea of analyzing the vortex method as a PDE posed on some space of distributions was already used by Marchioro-Pulvirenti [19] and Cottet [7]. Cottet's result requires stronger assumptions on the cut-off function and hence smoothing kernel, stronger regularity assumptions on the underlying Euler flow, and his approximation of the Euler initial vorticity field required a uniform grid. The trade-off, however, is that these more stringent constraints give an improved (algebraic) convergence in α. There is, in general, a trade-off between the order of convergence on the one hand, and the assumptions placed on K^α, ω_0, and the approximation at time $t = 0$ on the other. The result which, to our knowledge, comes closest to Theorem 2 is given in Marchioro and Pulvirenti [19]. The authors, however, assume that K^α is Lipschitz, which again excludes kernels corresponding to the equations of second grade fluids.

Remark 5. Stronger results can be proved for kernels K^α with a higher degree of smoothing. For example, by replacing $1-\alpha^2\Delta$ with $(1-\alpha^2\Delta)^s$, one obtains a hierarchy of regularizations of the Euler equations which coincide with geodesic flow on the volume-preserving diffeomorphism group with respect to the H^s metric. Other choices of K^α may introduce nonlocal pseudo-differential operators into equation (1.2), but the analysis can still proceed as before.

Remark 6. In three dimensions, the formal connection between second grade fluids and particular vortex filament methods still holds and is the subject of a forthcoming article. In this setting, one looks at the set of vorticity distributions of the following form. Let γ be a curve in $\mathbb{R}^3$ extending to infinity in both directions, and let δ_γ be the Dirac distribution given by integration along γ with respect to arc length. Let ω_γ be the 2-form along γ defined by $i_T dx \wedge dz$, where T is the unit tangent vector to γ. Then if Γ is any constant, $\Gamma\omega_\gamma\delta_\gamma$ is the vorticity corresponding to γ with strength Γ. See [22].

Remark 7. As we described above, for $s > (n/2)+1$, local well-posedness follows form the existence of unique C^∞ geodesics $\dot\eta(t)$ on $\mathcal{D}^s_\mu$ with respect to the right invariant metric $\langle\cdot,\cdot\rangle$ defined in (1.3), with initial conditions $\eta(0) = e$ and $\dot\eta(0) = u_0$. In working with the geodesic flow $\dot\eta(t)$, one obtains C^∞ evolution in the tangent bundle $T\mathcal{D}^s_\mu$ and C^∞ dependence on initial data, while the projected evolution curve $u(t) = \dot\eta(t)\circ\eta(t)^{-1}$ in the single fiber of the tangent bundle $T_e\mathcal{D}^s_\mu$—which plays the role of the Eulerian phase space—has only C^0 smoothness, and C^0 dependence on the initial velocity field. In the case that the manifold has a smooth boundary, there are three new subgroups of $\mathcal{D}^s_\mu$ which are in one-to-one correspondence with the classical Dirichlet, Neumann, and mixed elliptic boundary value problems in the sense that elements of the "Lie algebras" of these three subgroups satisfy those boundary conditions. Hence, geodesic flow of $\langle \cdot, \cdot \rangle$ on these three subgroups gives the solutions of (1.1) with no-slip, free-slip, and mixed boundary conditions.

2. Kernel estimates

The crucial ingredients for the proof of our theorems are quasi-Lipschitz estimates on the Euler kernel K and the regularized kernel K^α. For $x \in \mathbb{R}^2$ we define the function

$$\varphi(x) = \begin{cases} |x| \, (1 - \ln |x|) & \text{for } |x| < 1 \,, \\ 1 & \text{for } |x| \geq 1 \,. \end{cases} \tag{2.1}$$

Lemma 3. *For* $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \bigl| K(x,y) - K(x',y) \bigr| \, |\omega(y)| \, \mathrm{d}y \leq c \, \varphi(x - x') \, (\|\omega\|_{L^1} + \|\omega\|_{L^\infty}) \,. \tag{2.2}$$

The proof is standard and can be found, for example, in McGrath [18]. Somewhat less standard is the following estimate, still for the Euler kernel, which is similar to estimates in Benedetto *et al.* [5].

Lemma 4. *Let* $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *and let* ϕ *be an area preserving measurable transformation on* $\mathbb{R}^2$. *Then*

$$\begin{aligned} \biggl| \int_{\mathbb{R}^2} \bigl(K(x,y) - K(x,\phi(y)) \bigr) \, \omega(y) \, \mathrm{d}y \biggr| & \\ \leq c \sup_{x \in \mathbb{R}^2} \varphi\bigl(x - \phi(x) \bigr) \, \bigl(\|\omega\|_{L^1} + \|\omega\|_{L^\infty} \bigr) \,. & \end{aligned} \tag{2.3}$$

Proof. Set $r = \sup_x |x - \phi(x)|$; as in the proof of Lemma 3, the interesting case is when $r < 1$. We split the integral in (2.3) into two parts. First, consider

$$\begin{aligned} \int_{|x-y| \leq 2r} & \bigl| K(x,y) - K(x,\phi(y)) \bigr| \, |\omega(y)| \, \mathrm{d}y \\ & \leq \frac{1}{2\pi} \int_{|x-y| \leq 2r} \frac{|\omega(y)|}{|x-y|} \, \mathrm{d}y + \int_{|x-y| \leq 2r} \frac{|\omega(y)|}{|x - \phi(y)|} \, \mathrm{d}y \\ & \leq \frac{1}{2\pi} \int_{|x-y| \leq 2r} \biggl(\frac{1}{|x-y|} + \frac{1}{|x - \phi(y)|} \biggr) \mathrm{d}y \, \|\omega\|_{L^\infty} \\ & \leq \frac{1}{\pi} \int_{|x-y| \leq 2r} \frac{1}{|x-y|} \, \mathrm{d}y \, \|\omega\|_{L^\infty} \equiv c \, r \, \|\omega\|_{L^\infty} \,. \end{aligned} \tag{2.4}$$

The last inequality above holds because ϕ is area preserving, and among all such transformations, the symmetric map $\phi = e$ maximizes the integral over $|x - \phi(y)|^{-1}$.

Next, we consider the case when $|x - y| \geq 2r$. Observe that

$$|x - \phi(y)| \geq |x - y| - |y - \phi(y)| \geq |x - y| - r \geq \tfrac{1}{2}\,|x - y|\,, \tag{2.5}$$

so that

$$\begin{aligned}
\int_{|x-y|\geq 2r} &\bigl|K(x,y) - K(x,\phi(y))\bigr|\,|\omega(y)|\,\mathrm{d}y \\
&\leq \frac{1}{2\pi}\int_{|x-y|\geq 2r} \frac{|y-\phi(y)|}{|x-y|\,|x-\phi(y)|}\,|\omega(y)|\,\mathrm{d}y \\
&\leq \frac{1}{\pi}\int_{|x-y|\geq 2r} \frac{r}{|x-y|^2}\,|\omega(y)|\,\mathrm{d}y \\
&= \frac{r}{\pi}\left(\int_{2r\leq|x-y|\leq 2} \frac{|\omega(y)|}{|x-y|^2}\,\mathrm{d}y + \int_{|x-y|\geq 2} \frac{|\omega(y)|}{|x-y|^2}\,\mathrm{d}y\right) \\
&\leq \frac{r}{\pi}\left(\int_{2r}^{2} \frac{\mathrm{d}\rho}{\rho}\,\|\omega\|_{L^\infty} + \frac{1}{4}\,\|\omega\|_{L^1}\right) \\
&\leq c\,\varphi(r)\,\bigl(\|\omega\|_{L^1} + \|\omega\|_{L^\infty}\bigr)\,.
\end{aligned} \tag{2.6}$$

By combining the two estimates we complete the proof. □

Finally, we give the corresponding result for the vortex method kernel.

Lemma 5. *There exists a constant c_2 which is independent of α, such that*

$$\sup_{y\in\mathbb{R}^2}\bigl|K^\alpha(x,y) - K^\alpha(x',y)\bigr| \leq \frac{c_2}{\alpha}\,\varphi\Bigl(\frac{x-x'}{\alpha}\Bigr)\,. \tag{2.7}$$

Proof. Note that on $\mathbb{R}^2$, $K^\alpha(x,y) = K^\alpha(|x-y|) = \nabla^\perp G^\alpha(|x-y|)$, where

$$G^\alpha(r) = -\frac{1}{2\pi}\,K_0\Bigl(\frac{r}{\alpha}\Bigr) - \frac{1}{2\pi}\,\ln r \tag{2.8}$$

and K_0 denotes the zero order modified Bessel function of the second kind [1]. For simplicity, we take $\alpha = 1$ and compute

$$\frac{\mathrm{d}G^\alpha}{\mathrm{d}r}(r) = \frac{1}{2\pi}\left(K_1(r) - \frac{1}{r}\right) = \frac{1}{4\pi}\,r\,\ln r + O(r)\,, \tag{2.9}$$

$$\frac{\mathrm{d}^2G^\alpha}{\mathrm{d}r^2}(r) = \frac{1}{2\pi}\left(\frac{1}{r^2} - K_0(r) - \frac{1}{r}\,K_1(r)\right) = \frac{1}{4\pi}\,\ln r + O(1)\,, \tag{2.10}$$

as $r \to 0$. Set $r \equiv |x - x'|$ and assume, without loss of generality as K^α is bounded, that $r < 1$.

If $|x - y| < 2r$, then $|x' - y| \leq |x' - x| + |x - y| < 3r$, so that

$$\begin{aligned} \left|K^\alpha(x, y) - K^\alpha(x', y)\right| &\leq \left|\nabla^\perp G^\alpha(|x - y|)\right| + \left|\nabla^\perp G^\alpha(|x - y|)\right| \\ &\leq \left|\frac{\mathrm{d}G^\alpha}{\mathrm{d}r}(|x - y|)\right| + \left|\frac{\mathrm{d}G^\alpha}{\mathrm{d}r}(|x - y|)\right| \\ &\leq \frac{2}{\pi}\, r\, |\ln r| + O(r)\,. \end{aligned} \tag{2.11}$$

Since $\mathrm{d}G^\alpha/\mathrm{d}r$ is continuous and decays at infinity, this implies a bound of the form

$$\left|K^\alpha(x, y) - K^\alpha(x', y)\right| \leq c\, \varphi(|x - x'|)\,. \tag{2.12}$$

If, on the other hand, $|x - y| \geq 2r$, we use the mean value theorem to estimate

$$\begin{aligned} \left|K^\alpha(x, y) - K^\alpha(x', y)\right| &\leq \sup_{x'' \in B(x,r)} |\nabla K^\alpha(x'', y)|\, |x - x'| \\ &\leq \sup_{x'' \in B(x,r)} \left|\frac{\mathrm{d}^2 G^\alpha}{\mathrm{d}r^2}(|x'' - y|)\right| r \\ &\leq \frac{1}{4\pi}\, r\, \ln r + O(r)\,, \end{aligned} \tag{2.13}$$

which again implies a bound of the form (2.12). In the last step we have used (2.10) in conjunction with $|x'' - y| > r$.

To recover the scaling of the estimate in α, divide (2.12) by α, rescale x, x', and y by α^{-1}, and note that $K^\alpha(r) = K^{\alpha=1}(r/\alpha)/\alpha$. □

Corollary 6. *For $q \in \mathcal{M}(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} \left|K^\alpha(x, y) - K^\alpha(x', y)\right| |q(y)|\, \mathrm{d}y \leq \frac{c_2}{\alpha}\, \varphi\Big(\frac{x - x'}{\alpha}\Big)\, \|q\|_{\mathcal{M}}\,. \tag{2.14}$$

3. Well-posedness

We can now prove the existence of unique, global, weak solutions to the Lagrangian flow equation (1.9).

Proof of Theorem 1. Due to the quasi-Lipschitz condition for K^α, we can adopt the method that Kato developed for the Euler equations in [17], by simply replacing the kernel estimates in L^1 by the corresponding estimates in L^∞. Our presentation follows to some extent that of Marchioro and Pulvirenti [20].

For simplicity, we assume $\alpha = 1$ throughout this proof. We introduce a sequence of approximate solutions

$$\partial_t \eta^n(x,t) = u^n(\eta^n(x,t),t)\,, \tag{3.1a}$$

$$\eta^n(x,0) = x\,, \tag{3.1b}$$

$$\eta^0(x,t) = x\,, \tag{3.1c}$$

$$q^n(\eta^n(x,t),t) = q_0(x)\,, \tag{3.1d}$$

$$u^n(x,t) = \int_{\mathbb{R}^2} K^\alpha(x,y)\, q^{n-1}(y,t)\,\mathrm{d}y\,, \tag{3.1e}$$

for $n \in \mathbb{N}$. The proof now proceeds in several steps.

Step 1. Prove that $\eta^n \in C^1([0,\infty);\mathcal{G})$ for every $n \in \mathbb{N}$.

We proceed inductively. Notice that for every n the vector field u^n is quasi-Lipschitz in space and continuous in time. This is a consequence of Lemma 5 as

$$\begin{aligned}
&\big|u^n(x,t) - u^n(x',t)\big| \\
&\qquad = \left| \int_{\mathbb{R}^2} \Big[K^\alpha(x,\eta^{n-1}(y,t)) - K^\alpha(x',\eta^{n-1}(y,t)) \Big]\, q_0(y)\,\mathrm{d}y \right| \\
&\qquad \le c\,\varphi(x-x')\,\|q_0\|_{\mathcal{M}}\,,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\big|u^n(x,t) - u^n(x,t')\big| \\
&\qquad = \left| \int_{\mathbb{R}^2} \Big[K^\alpha(x,\eta^{n-1}(y,t)) - K^\alpha(x,\eta^{n-1}(y,t')) \Big]\, q_0(y)\,\mathrm{d}y \right| \\
&\qquad \le \sup_{y\in\mathbb{R}^2} \big| K^\alpha(x,\eta^{n-1}(y,t)) - K^\alpha(x,\eta^{n-1}(y,t')) \big|\, \|q_0\|_{\mathcal{M}} \\
&\qquad \le c \sup_{y\in\mathbb{R}^2} \varphi\big(\eta^{n-1}(y,t) - \eta^{n-1}(y,t')\big)\, \|q_0\|_{\mathcal{M}} \\
&\qquad \le c \sup_{y\in\mathbb{R}^2} \sup_{x\in[t,t']} \varphi\big(|\dot\eta^{n-1}(y,s)|\,|t-t'|\big)\, \|q_0\|_{\mathcal{M}} \\
&\qquad = c \sup_{y\in\mathbb{R}^2} \sup_{x\in[t,t']} \varphi\big(|u^{n-1}(y,s)|\,|t-t'|\big)\, \|q_0\|_{\mathcal{M}}\,.
\end{aligned} \tag{3.3}$$

This implies uniform continuity in time, as u^n is bounded for every n:

$$\begin{aligned}
|u^n(x,t)| &= \left| \int_{\mathbb{R}^2} K^\alpha(x,\eta^{n-1}(y,t))\, q_0(y)\,\mathrm{d}y \right| \\
&\qquad\qquad \le \sup_{y\in\mathbb{R}^2} \big|K(x,y)\big|\, \|q_0\|_{\mathcal{M}} \equiv c\,\|q_0\|_{\mathcal{M}}\,.
\end{aligned} \tag{3.4}$$

Since u^n is continuous in time and quasi-Lipschitz in space, the vector field generates a local flow $\eta^n \in C^1([0,T);C(\mathbb{R}^2))$ for some $T>0$—see, e.g., Chapter 2, Lemma 3.2 in Marchioro and Pulvirenti [20]. Because of the global bound (3.4), the right side of (3.1a) is bounded and the flow exists globally in time.

Step 2. Show that there exists a limiting flow map $\eta \in C([0,\infty);\mathcal{G})$.

We first prove that the sequence η^n is Cauchy in $C([0,T];\mathcal{G})$ for some $T>0$. To simplify notation, we shall drop the explicit time dependence of u and η, and estimate

$$\begin{aligned}
&\big|\eta^n(x,t)-\eta^{n-1}(x,t)\big| \\
&\le \int_0^t \big|u^n(\eta^n)-u^{n-1}(\eta^{n-1})\big|\mathrm{d}s \\
&\le \int_0^t \left|\int_{\mathbb{R}^2}\Big[K^\alpha(\eta^n(x),\eta^{n-1}(y))-K^\alpha(\eta^{n-1}(x),\eta^{n-1}(y))\Big]q_0(y)\,\mathrm{d}y\right|\mathrm{d}s \\
&\quad + \int_0^t \left|\int_{\mathbb{R}^2}\Big[K^\alpha(\eta^{n-1}(x),\eta^{n-1}(y))-K^\alpha(\eta^{n-1}(x),\eta^{n-2}(y))\Big]q_0(y)\,\mathrm{d}y\right|\mathrm{d}s \\
&\le c\int_0^t \varphi\big(\eta^n(x)-\eta^{n-1}(x)\big)\,\mathrm{d}s\,\|q_0\|_{\mathcal{M}} \\
&\quad + c\int_0^t \varphi\big(\eta^{n-1}(x)-\eta^{n-2}(x)\big)\,\mathrm{d}s\,\|q_0\|_{\mathcal{M}}\,.
\end{aligned} \tag{3.5}$$

By taking the supremum over x on both sides, we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^2}\big|\eta^n(x,t)-\eta^{n-1}(x,t)\big| \\
&\qquad \le c\,\|q_0\|_{\mathcal{M}}\int_0^t\Bigg[\varphi\bigg(\sup_{x\in\mathbb{R}^2}\big|\eta^n(x)-\eta^{n-1}(x)\big|\bigg)+ \\
&\qquad\qquad \varphi\bigg(\sup_{x\in\mathbb{R}^2}\big|\eta^{n-1}(x)-\eta^{n-2}(x)\big|\bigg)\Bigg]\mathrm{d}s\,.
\end{aligned} \tag{3.6}$$

Defining

$$\rho^N(t)\equiv \sup_{n\ge N}\sup_{x\in\mathbb{R}^2}\big|\eta^n(x,t)-\eta^{n-1}(x,t)\big|\,, \tag{3.7}$$

we can simplify the previous estimate, and obtain

$$\rho^N(t)\le c\int_0^t \varphi\big(\rho^{N-1}(s)\big)\mathrm{d}s\,. \tag{3.8}$$

It is well known that this implies

$$\lim_{N\to\infty} \rho^N(t) \to 0\,, \tag{3.9}$$

uniformly on $[0,T]$ for T sufficiently small. Since T depends only on α and the $\mathcal{M}$-norm of q_0, this result can be extended to arbitrarily large times. Thus, the contraction mapping theorem implies the assertion of Step 2.

Step 3. Show that the Lagrangian flow equation (1.9) is satisfied in the limit, and that $\eta \in C^1(\mathbb{R};\mathcal{G})$.

We define the limiting potential vorticity q and the limiting velocity u in the obvious way, and check by direct estimation that

$$q^n \rightharpoonup q \equiv q_0 \circ \eta^{-1} \tag{3.10}$$

weakly in $\mathcal{M}(\mathbb{R}^2)$, and

$$u^n \to u \equiv K^\alpha * q \tag{3.11}$$

in $C(\mathbb{R}^2)$; both limits are uniform over finite intervals of time.

To prove that η, u, and q solve the limit problem (1.9), we consider its integrated version

$$\begin{aligned}
\eta(x,t) - \int_0^t u(\eta(x,s),s)\,\mathrm{d}s
&= \eta(x,t) - \int_0^t u(\eta(x,s),s)\,\mathrm{d}s - \eta^n(x,t) + \int_0^t u^n(\eta^n(x,s),s)\,\mathrm{d}s \\
&\le \bigl|\eta(x,t)-\eta^n(x,t)\bigr| + \int_0^t \bigl|u^n(\eta^n(x,s),s) - u(\eta^n(x,s),s)\bigr|\,\mathrm{d}s \\
&\quad + \int_0^t \bigl|u(\eta^n(x,s),s) - u(\eta(x,s),s)\bigr|\,\mathrm{d}s \\
&\to 0 \text{ uniformly in } x \text{ as } n\to\infty\,.
\end{aligned} \tag{3.12}$$

Thus, the left side must be zero. Since $u(\eta(x,s),s)$ is continuous in x, we can differentiate with respect to t, and find that η satisfies (1.9) and that $\dot\eta$ is in fact continuous. Due to the time-reversibility of the equation, the result extends to negative times as well.

Moreover, one can show—first by formal calculation for smooth function, and then extending by the usual density argument—that the weak solution q defined through (3.10) satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \bigl(\partial_t\phi + u\cdot \operatorname{grad}\phi\bigr)\, q \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.13}$$

for every $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^2)$. This shows that solutions of the vortex method, and hence the equations of second-grade non-Newtonian fluids, preserve the (weak) co-adjoint action.

Step 4. Prove that the solution is unique.

Uniqueness is shown by a direct estimate on the difference of two flow maps. This leads to another log-Gronwall inequality, which can be treated in the same way as the previous ones; we omit all details. □

Remark 8. The homeomorphisms that we consider have the vector space $\mathbb{R}^2$ as the range; we may thus subtract two elements of this class. For homeomorphisms of a compact domain Ω of $\mathbb{R}^2$, one can isometrically embed the set of measure-preserving homeomorphisms of Ω into the vector space $L^2(\Omega, \mathbb{R}^2)$, and take differences in this large space. Similarly, the difference $u^\alpha \circ \eta^\alpha - u \circ \eta$ is not an intrinsic operation, but rather relies on the trivial identification of vector spaces induced by the trivial geometry of $\mathbb{R}^2$. On the other hand, when the configuration space is $\mathcal{D}^s_\mu(M)$, $s > 2$ and M is a compact Riemannian manifold, the map $u^\alpha \circ \eta^\alpha$ is an element of the fiber $T_{\eta^\alpha}\mathcal{D}^s_\mu$ while $u\circ\eta$ is in $T_\eta\mathcal{D}^s_\mu$; thus, in order to compare the two maps, we must parallel transport $u \circ \eta$ into $T_{\eta^\alpha}\mathcal{D}^s_\mu$ along the Riemannian connection.

4. Weak co-adjoint action and reduction

As we described, classical solutions of the two-dimensional averaged Euler equations are geodesics on the Hilbert-class volume-preserving diffeomorphism group $\mathcal{D}^s_\mu$, $s > 2$. We identify the space of classical vorticity solutions $H^{s-1}(M)$ with the reduced space $T_e\mathcal{D}^s_\mu = T\mathcal{D}^s_\mu/\mathcal{D}^s_\mu$ (symmetry reduction by the massive particle relabeling symmetry group $\mathcal{D}^s_\mu$ of hydrodynamics), and note that this space is the union of the $\mathcal{D}^s_\mu$-co-adjoint orbits.

In the case that $M = \mathbb{R}^2$, and for the purpose of studying weak solutions to (1.2) we shall substantially relax the regularity requirements on the configuration space, and use $\mathcal{G}$ in place of $\mathcal{D}^s_\mu$; correspondingly, we shall use the vector space of Radon measure on $\mathbb{R}^2$, which we denote by $\mathcal{M}(\mathbb{R}^2)$, for the reduced space of vorticity functions, in place of the space of H^{s-1} functions.

Recall that the co-Adjoint action of $\mathcal{D}^s_\mu$ on $H^{s-1}(\mathbb{R}^2)$ is given by

$$\mathrm{Ad}^*_\eta(q) = q \circ \eta\,. \tag{4.1}$$

We shall need to define the notion of weak co-adjoint action of $\mathcal{G}$ on $\mathcal{M}(\mathbb{R}^2)$. First, note that the operation $\mathrm{Ad}^*\colon \mathcal{G} \times \mathcal{M}(\mathbb{R}^2) \to \mathcal{M}(\mathbb{R}^2)$ given by $\mathrm{Ad}^*_\eta(q) = q \circ \eta$ is well-defined. Next, define the *weak* co-Adjoint action $\text{w-Ad}^* \colon \mathcal{G} \times \mathcal{M}(\mathbb{R}^2) \to \mathcal{M}(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \text{w-Ad}^*_\eta(q) \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}^2} q \cdot (\phi \circ \eta) \, \mathrm{d}x \, \mathrm{d}t \tag{4.2}$$

for all $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^2)$.

It follows that if η_t is a C^1 curve in $\mathcal{G}$ such that $e = \eta_0$ and $u = (d/dt)|_{t=0}\eta_t$, then we may—computing the time derivative of $\text{w-Ad}^*_{\eta_t}(q)$ at $t = 0$—define the weak analogue of the algebra co-adjoint action by

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \text{w-ad}^*_u(q) \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}^2} q \cdot \big(\partial_t \phi + u \cdot \operatorname{grad} \phi \big) \, \mathrm{d}x \, \mathrm{d}t \tag{4.3}$$

for every $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^2)$. Recall that the classical co-adjoint action is defined by $\mathrm{ad}^*_{u_t}(q_t) = (d/dt)|_{t=0}(\eta_t^* q_t)$ where $(d/dt)|_{t=0}(\eta_t^* q_t) = \partial_t q_t + \mathcal{L}_{u_t} q_t$. In two dimensions, the Lie derivative term $\mathcal{L}_{u_t} q_t$ reduces to $u_t \cdot \operatorname{grad} q_t$.

Theorem 7. *For any $q_0 \in \mathcal{M}(\mathbb{R}^2)$, let $\mathcal{O}_{q_0}$ denote the co-Adjoint orbit $\{q \colon q = q_0 \circ \eta, \eta \in \mathcal{G}\}$. The weak co-adjoint action of $C^0_{\mathrm{div}}(\mathbb{R}^2)$ on $\mathcal{M}(\mathbb{R}^2)$ is well-defined, and solutions of the second-grade fluids equations or of Chorin's vortex blob method with initial data q_0 leave $\mathcal{O}_{q_0}$ invariant.*

Proof. The result immediately follows from the fact that the vanishing of the weak co-adjoint action is equivalent to the weak formulation of (1.2). Theorem 1, giving global well-posedness of weak solutions, then concludes the argument. □

5. Convergence

We can now prove convergence of the flow of the vortex blob method to the flow of the Euler equations. This is done in two steps. First we show that the averaged Euler equation, or vortex method PDE, approximates the Euler equation as $\alpha \to 0$ for bounded vorticity fields. In the second step, we prove that bounded solutions of the averaged Euler equation can be approximated by measure-valued ones. These two results together imply Theorem 2.

Lemma 8. *Let $q_0 \equiv \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then for every $T > 0$ there exists a positive constant $C(T)$ such that*

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^2} \big| \eta^\alpha(x,t) - \eta(x,t) \big| \le C(T) \, \alpha^{\mathrm{e}^{-T}} \, . \tag{5.1}$$

Proof. We estimate the difference of the Euler and Euler-α flow maps:

$$\begin{aligned}
\big|\eta^\alpha(x,t)-\eta(x,t)\big| &\le \int_0^t \big|u^\alpha\circ\eta^\alpha - u\circ\eta\big|\,\mathrm{d}s \\
&\le \int_0^t\int_{\mathbb{R}^2}\big|K^\alpha(\eta^\alpha(x),\eta^\alpha(y)) - K(\eta^\alpha(x),\eta^\alpha(y))\big|\,|\omega_0(y)|\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathbb{R}^2}\big|K(\eta^\alpha(x),\eta^\alpha(y)) - K(\eta(x),\eta^\alpha(y))\big|\,|\omega_0(y)|\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathbb{R}^2}\big|K(\eta(x),\eta^\alpha(y)) - K(\eta(x),\eta(y))\big|\,|\omega_0(y)|\,\mathrm{d}y\,\mathrm{d}s \\
&\equiv \int_0^t (I_1+I_2+I_3)\,\mathrm{d}s
\end{aligned} \tag{5.2}$$

To estimate I_1, we note that on $\mathbb{R}^2$ the difference of the kernels is explicitly given by $|K^\alpha(r)-K(r)| = 1/(2\pi)\,K_1(r/\alpha)/\alpha$, so that

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{\alpha}\,K_1\Big(\frac{|x-y|}{\alpha}\Big)\,|\omega^\alpha(y,s)|\,\mathrm{d}y \\
&\le \int_0^\infty \frac{1}{\alpha}\,K_1\Big(\frac{r}{\alpha}\Big) r\,\mathrm{d}r\,\|\omega^\alpha(s)\|_{L^\infty} \\
&\le c\,\alpha\,\|\omega_0\|_{L^\infty}
\end{aligned} \tag{5.3}$$

The other two integrals can be estimated by using the quasi-Lipschitz conditions, Lemma 3 and Lemma 4, respectively. One finds that

$$I_2 \le c\,\varphi\big(\eta^\alpha(x,s)-\eta(x,s)\big)\,\big(\|\omega_0\|_{L^1}+\|\omega_0\|_{L^\infty}\big)\,, \tag{5.4}$$

and

$$\begin{aligned}
I_3 &= \int_{\mathbb{R}^2}\big|K(\eta(x,s),y) - K(\eta(x,s),\eta^\alpha\circ\eta^{-1}(y,s))\big|\,|\omega(y,s)|\,\mathrm{d}y \\
&\le c\,\sup_{x\in\mathbb{R}^2}\varphi\big(x-\eta^\alpha\circ\eta^{-1}(x,s)\big)\,\big(\|\omega(s)\|_{L^1}+\|\omega(s)\|_{L^\infty}\big) \\
&= c\,\sup_{x\in\mathbb{R}^2}\varphi\big(\eta(x,s)-\eta^\alpha(x,s)\big)\,\big(\|\omega_0\|_{L^1}+\|\omega_0\|_{L^\infty}\big)\,.
\end{aligned} \tag{5.5}$$

By inserting the bounds for I_1 to I_3 back into (5.2) and taking the supremum on both sides, we obtain the log-Gronwall inequality

$$\sup_{x\in\mathbb{R}^2}\big|\eta^\alpha(x,t)-\eta(x,t)\big| \le \int_0^t\Big[\alpha\,K_1 + K_2\sup_{x\in\mathbb{R}^2}\varphi\big(\eta^\alpha(x,s)-\eta(x,s)\big)\Big]\,\mathrm{d}s\,. \tag{5.6}$$

To obtain explicit bounds that are valid on any finite interval of time $[0,T]$, we set

$$\rho(t) = \sup_{x\in\mathbb{R}^2} \bigl|\eta^\alpha(x,t) - \eta(x,t)\bigr| \,, \tag{5.7}$$

and use the tangent approximation of the concave function φ; namely, for any $\varepsilon \in (0,1)$,

$$\varphi(r) \le \varphi(\varepsilon) + \varphi'(\varepsilon)\, r = (-\ln\varepsilon)\, r + \varepsilon \,. \tag{5.8}$$

This makes the right-hand-side of (5.8) linear in r. For notational simplicity, we also rescale α and t such that $K_1 = K_2 = 1$. We substitute (5.8) into (5.6) and obtain the usual Gronwall inequality; it follows that ρ must satisfy the differential inequality

$$\dot\rho \le (-\ln\varepsilon)\,\rho + \varepsilon + \alpha \,, \qquad \rho(0) = 0 \,. \tag{5.9}$$

Setting $\varepsilon = \mathrm{e}^{-1}\alpha^{\exp(-t)}$ and integrating (5.9) with this choice of $\varepsilon(\alpha)$, we find that

$$\rho(t) \le \frac{\mathrm{e}^t - 1}{\mathrm{e}}\, \alpha^{\mathrm{e}^{-t}} + \mathrm{e}^t\, \frac{\alpha^{\mathrm{e}^{-t}} - \alpha}{-\ln\alpha} \,. \tag{5.10}$$

Thus, $\rho = O(\alpha^{\exp(-T)})$ uniformly on $[0,T]$. □

In the following we will consider α as fixed and approximate bounded data by measure valued data. Let η, q, and u denote quantities corresponding to a solution of the Euler-α equation with initial data $q_0 \in L^\infty(\mathbb{R}^2)$, and let η^n, q^n, and u^n denote a sequence of solutions to the Euler-α equation with initial data $q_0^n \in \mathcal{M}(\mathbb{R}^2)$ for every $n \in \mathbb{N}$. Then the following is true.

Lemma 9. *Let $q_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and suppose that q_0 is approximated by a sequence of measures in $\mathcal{M}(\mathbb{R}^2)$ such that $q_0^n \rightharpoonup q_0$ weakly in $\mathcal{M}(\mathbb{R}^2)$, and $\|q_0^n\|_{\mathcal{M}} \to \|q_0\|_{L^1}$. Then, for every $T > 0$,*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^2} \bigl|\eta^n(x,t) - \eta(x,t)\bigr| = 0 \,. \tag{5.11}$$

Proof. As in the proof of Lemma 8, we estimate

$$\begin{aligned}
&\bigl|\eta(x,t) - \eta^n(x,t)\bigr| \\
&\le \int_0^t \biggl| \int_{\mathbb{R}^2} \bigl[K^\alpha(\eta(x,s),\eta(y,s))\, q_0(y) - K^\alpha(\eta^n(x,s),\eta^n(y,s))\, q_0^n(y) \bigr] \,\mathrm{d}y \biggr| \,\mathrm{d}s \\
&\le \int_0^t \biggl| \int_{\mathbb{R}^2} K^\alpha(\eta(x),\eta(y)) \bigl(q_0(y) - q_0^n(y) \bigr) \,\mathrm{d}y \biggr| \,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathbb{R}^2} \left|K^\alpha(\eta(x),\eta(y)) - K^\alpha(\eta(x),\eta^n(y))\right| |q_0^n(y)| \,\mathrm{d}y \,\mathrm{d}s \\
&+\int_0^t \int_{\mathbb{R}^2} \left|K^\alpha(\eta(x),\eta^n(y)) - K^\alpha(\eta^n(x),\eta^n(y))\right| |q_0^n(y)| \,\mathrm{d}y \,\mathrm{d}s \\
\equiv &\int_0^t (J_1 + J_2 + J_3) \,\mathrm{d}s \qquad (5.12)
\end{aligned}
$$

We find, after a change of variables, that

$$
\sup_{x\in\mathbb{R}^2} J_1 = \sup_{x\in\mathbb{R}^2} \left| \int_{\mathbb{R}^2} K^\alpha(x,y)\,(q_0 - q_0^n)(\eta^{-1}(y))\,\mathrm{d}y \right| . \qquad (5.13)
$$

By Lemma 10 below with $\phi(x-y) = K^\alpha(x,y)$ and $q_n(y) = (q_0 - q_0^n)(\eta^{-1}(y))$, this expression converges to zero as $n \to \infty$. Moreover, by Lemma 5,

$$
\begin{aligned}
I_2 &\le \sup_{x\in\mathbb{R}^2} \sup_{y\in\mathbb{R}^2} \left|K^\alpha(\eta(x),\eta(y)) - K^\alpha(\eta(x),\eta^n(y))\right| \|q_0^n\|_{\mathcal{M}} \\
&\le \sup_{y\in\mathbb{R}^2} \frac{c}{\alpha}\,\varphi\!\left(\frac{\eta(y)-\eta^n(y)}{\alpha}\right) \|q_0^n\|_{\mathcal{M}} \qquad (5.14)
\end{aligned}
$$

and

$$
I_3 \le \frac{c}{\alpha}\,\varphi\!\left(\frac{\eta(x)-\eta^n(x)}{\alpha}\right) \|q_0^n\|_{\mathcal{M}} . \qquad (5.15)
$$

By inserting these estimates back into (5.12) and taking the supremum in x on both sides, we obtain an integral inequality that can be solved with the log-Gronwall inequality exactly as in the proof of Lemma 8. The result then follows. □

Lemma 10. *Let $\{q_n\}$ be a sequence of measures in $\mathcal{M}(\mathbb{R}^2)$ converging weakly to zero with uniformly bounded total variation and uniform decay at infinity. Further assume that ϕ is a continuous test function with $\phi \to 0$ as $|x| \to \infty$. Then*

$$
\lim_{n\to\infty} \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x-y)\,q_n(y)\,\mathrm{d}y = 0 . \qquad (5.16)
$$

Proof. Set $M = \sup_n \|q_n\|_{\mathcal{M}}$ and $M' = \sup_x |\phi(x)|$. Let $\varepsilon > 0$ be fixed. By assumption on the $\{q_n\}$, there exists an $R > 0$ such that for every $n \in \mathbb{N}$,

$$
\int_{|y|>R} |q_n(y)|\,\mathrm{d}y < \frac{\varepsilon}{6M'} . \qquad (5.17)
$$

Moreover, there exists an $R' > 0$ such that $|\phi(x)| < \epsilon/(2M)$ for $|x| > R'$. Since ϕ is uniformly continuous on compact sets, there exists $\delta > 0$ such that $|\phi(x)-\phi(x')| < \varepsilon/(3M)$ for all $x, x' \in B(0, 2R+R')$ with $|x-x'| < \delta$. Cover $B(0, R+R')$ with finitely many balls of radius δ and denote the centers of these balls by x_i, $i \in I$. Choose N large enough such that for $n \geq N$,

$$\max_{i\in I}\left|\int_{\mathbb{R}^2} \phi(x_i - y)\, q_n(y)\, \mathrm{d}y\right| < \frac{\varepsilon}{3}\,. \tag{5.18}$$

Then for $|x| < R + R'$ there exists an $i \in I$ such that $|x - x_i| < \delta$, and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \phi(x-y)\, q_n(y)\, \mathrm{d}y\right| \\
&\quad\leq \int_{\mathbb{R}^2} \bigl|\phi(x-y) - \phi(x_i - y)\bigr|\, |q_n(y)|\, \mathrm{d}y + \left|\int_{\mathbb{R}^2} \phi(x_i - y)\, q_n(y)\, \mathrm{d}y\right| \\
&\quad\leq \sup_{|y|\leq R} \bigl|\phi(x-y) - \phi(x_i - y)\bigr|\, \|q_n\|_{\mathcal{M}} \\
&\qquad + 2 \sup_{x\in\mathbb{R}^2} |\phi(x)| \int_{|y|>R} |q_n(y)|\, \mathrm{d}y + \frac{\varepsilon}{3} \\
&\quad\leq \frac{\varepsilon}{3M}\, M + 2M'\, \frac{\varepsilon}{6M'} + \frac{\varepsilon}{3} = \varepsilon\,.
\end{aligned} \tag{5.19}$$

On the other hand, if $|x| \geq R + R'$, then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \phi(x-y)\, q_n(y)\, \mathrm{d}y\right| \\
&\qquad\leq \sup_{|x|\geq R'} |\phi(x)| \int_{|y|<R} |q_n(y)|\, \mathrm{d}y + \sup_{x\in\mathbb{R}^2} |\phi(x)| \int_{|y|<R} |q_n(y)|\, \mathrm{d}y \\
&\qquad\leq \frac{\varepsilon}{2M}\, M + M'\, \frac{\varepsilon}{6M'} < \varepsilon\,.
\end{aligned} \tag{5.20}$$

This completes the proof. □

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References

[1] M. Abramowitz and I.A. Stegun, "Handbook of Mathematical Functions," United States National Bureau of Standards Applied Mathematics Series, Vol. 55, 1964.

[2] V.I. Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits*, Ann. Inst. Fourier (Grenoble) **16** (1966), 319–361.

[3] J.T. Beale and A. Majda, *Vortex methods. II: Higher order accuracy in two and three dimensions*, Math. Comput. **39** (1982), 29–52.

[4] J.T. Beale and A. Majda, *High order accurate vortex methods with explicit velocity kernels*, J. Comput. Phys. **58** (1985), 188–208.

[5] D. Benedetto, C. Marchioro, and M. Pulvirenti, *On the Euler flow in* $\mathbb{R}^2$, Arch. Rational Mech. Anal. **123** (1993), 377–386.

[6] A. Chorin, *Numerical study of slightly viscous flow*, J. Fluid Mech. **57** (1973), 785–796.

[7] G.-H. Cottet, *A new approach for the analysis of Vortex Methods in two and three dimensions*, Ann. Inst. Henri Poincaré, Anal. Non Lineaire **5** (1988), 227–285.

[8] D. Cioranescu and V. Girault, *Weak and classical solutions of a family of second grade fluids*, Internat. J. Non-Linear Mech. **32** (1997), 317–335.

[9] D. Cioranescu and E.H. Ouazar, *Existence and uniqueness for fluids of second grade*, In *Nonlinear Partial Differential Equations*, Vol. 109, pp. 178–197. Collège de France Seminar, Pitman (1984).

[10] J.-M. Delort, *Existence de nappes de tourbillon en dimension deux*, J. Amer. Math. Soc. **4** (1991), 553–586.

[11] J.E. Dunn and R.L. Fosdick, *Thermodynamics, stability and boundedness of fluids of complexity* 2 *and fluids of second grade*, Arch. Rational Mech. Anal. **56** (1974), 191–252.

[12] R.J. DiPerna and A. Majda, *Concentrations in regularizations for 2-D incompressible flow*, Comm. Pure Appl. Math. **40** (1987), 301–345.

[13] D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. **92** (1970), 102–163.

[14] O. Hald, *Convergence of vortex methods for Euler's equations. III*, SIAM J. Numer. Anal. **24** (1987), 538–582.

[15] D.D. Holm, S. Kouranbaeva, M. Oliver, and E.S. Titi, in preparation.

[16] D.D. Holm, J.E. Marsden, and T.S. Ratiu, *Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math. **137** (1998), 1–81.

[17] T. Kato, *On classical solutions of the two-dimensional non-stationary Euler equation*, Arch. Rational Mech. Anal. **25** (1967), 188–200.

[18] F.J. McGrath, *Nonstationary plane flow of viscous and ideal fluids*, Arch. Rational Mech. Anal. **27** (1968), 329–348.

[19] C. Marchioro and M. Pulvirenti, *Hydrodynamics in two dimensions and vortex theory*, Commun. Math. Phys. **84** (1982), 483–503.

[20] C. Marchioro and M. Pulvirenti, "Mathematical Theory of Incompressible Nonviscous Fluids," Springer-Verlag, New York, 1994.

[21] J.E. Marsden, T.S. Ratiu, and S. Shkoller, *A nonlinear analysis of the averaged Euler equations and a new diffeomorphism group*, Geom. Funct. Anal., to appear.

[22] J.E. Marsden and A. Weinstein, *Coadjoint orbits, vortices and Clebsch variables for incompressible fluids*, Phys. D **7** (1983), 305–323.

[23] S. Shkoller, *Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics*, J. Funct. Anal. **160** (1998), 337–365.

[24] S. Shkoller, *The geometry and analysis of non-Newtonian fluids and vortex methods*, E-print Math. AP/9908109 (1999).