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Global well-posedness for the Lagrangian averaged Navier–Stokes (LANS-α) equations on bounded domains

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We prove the global well-posedness and regularity of the (isotropic) Lagrangian averaged Navier–Stokes (LANS-α) equations on a three-dimensional bounded domain with a smooth boundary with no-slip boundary conditions for initial data in the set $\{u \in H^s \cap H^1_0 \mid Au = 0 \text{ on } \partial\Omega, \operatorname{div} u = 0\}$, $s \in [3, 5)$, where A is the Stokes operator. As with the Navier–Stokes equations, one has parabolic-type regularity; that is, the solutions instantaneously become space-time smooth when the forcing is smooth (or zero).

The equations are an ensemble average of the Navier–Stokes equations over initial data in an α-radius phase-space ball, and converge to the Navier–Stokes equations as $\alpha \to 0$. We also show that classical solutions of the LANS-α equations converge almost all in H^s for $s \in (2.5, 3)$, to solutions of the inviscid equations ($\nu = 0$), called the Lagrangian averaged Euler (LAE-α) equations, even on domains with boundary, for time-intervals governed by the time of existence of solutions of the LAE-α equations.

Keywords: Navier–Stokes; averaging; large-scale flow; Euler equations; turbulence

1. Introduction

The Lagrangian averaged Navier–Stokes (LANS-α) equations for a fluid moving in a region Ω in $\mathbb{R}^3$ with boundary $\partial\Omega$ are given by

$$u_t + \nabla_u u + \mathcal{U}^\alpha(u) = -(1 - \alpha^2\Delta)^{-1} \operatorname{grad} p - \nu A u, \tag{1.1 a}$$

$$\operatorname{div} u(t, x) = 0, \tag{1.1 b}$$

$$u = 0 \text{ on } \partial\Omega, \tag{1.1 c}$$

$$u(0, x) = u_0(x), \tag{1.1 d}$$

where

$$\mathcal{U}^\alpha(u) = \alpha^2(1 - \alpha^2\Delta)^{-1} \operatorname{Div}[\nabla u \cdot \nabla u^{\mathrm{T}} + \nabla u \cdot \nabla u - \nabla u^{\mathrm{T}} \cdot \nabla u], \tag{1.2}$$

and $A := -P\Delta$ is the Stokes operator, with P the Leray (also known as Helmholtz–Hodge) projector. The inviscid form of these equations (in a different formulation), as well as related equations for geophysical and other flows, first appeared in the context of averaged fluid models in Holm *et al.* (1998*a*,*b*); the derivation used averaging and asymptotic methods in the variational formulation. Viscosity was added to the conservative dynamics in Chen *et al.* (1998, 1999*a*–*c*). An alternative derivation was given in Holm (1999). We give additional comments on the history of this model below.

Marsden & Shkoller (2001) presented a new derivation by averaging over the set of solutions of the Euler equations with initial data in a phase-space ball of radius α, and treating the dissipative term νAu via stochastic variations. We shall review the salient features of this new procedure below, which we feel ameliorates the prior derivations. We employ the term *Lagrangian averaging*, because one uses a turbulence closure that is based on the behaviour of Lagrangian fluctuations, namely a generalization of the turbulent closure hypothesis of Taylor (1938) often referred to as the *frozen turbulence hypothesis*.

The time-dependent velocity field $u(t, x)$ and pressure function $p(t, x)$ which solve the LANS-α equations are *mean* quantities, and reflect the uncertainty in accurately reproducing the initial data when repeating the same fluids experiment many times. Formally, it is clear that as the parameter $\alpha \to 0$, the Navier–Stokes equations are recovered, as should be expected, since this indicates that there is no uncertainty in specifying the initial data, or equivalently, that the identical initial data are used for every repetition of the fluids experiment.

In dimension two, for which we have global well-posedness theorems for both the Euler and Navier–Stokes equations, Oliver & Shkoller (2001*a*) proved that for initial vorticity $\operatorname{curl} u$ in L^∞, solutions of (1.1) converge globally in time to solutions of the Navier–Stokes equations as $\alpha \to 0$ for all fixed $\nu \geqslant 0$. As we shall remark below, convergence as $\alpha \to 0$ also holds for classical solutions in dimension three on short time-intervals.

There are two types of Lagrangian averaged Navier–Stokes equations: the *isotropic* version (1.1) in which the (fluctuation) covariance tensor is assumed to be a constant multiple of the metric tensor, and the *anisotropic* version, in which the covariance tensor becomes a dynamic variable, coupled with the evolution equations for the velocity and pressure. We refer the reader to Marsden & Shkoller (2001) for details regarding the anisotropic theory.

The main purpose of this paper is to prove global (in time) well-posedness and regularity of solutions for the isotropic LANS-α equations on a compact three dimensional region Ω with smooth boundary $\partial\Omega$, and for initial data in the class $\{u \in H^s \cap H^1_0 \mid Au = 0 \text{ on } \partial\Omega, \operatorname{div} u = 0\}$, $s \in [3, 5)$. We also show that for fixed $\alpha > 0$, solutions of (1.1) converge in $L^\infty(0, \mathrm{T})$, $D(A^{s/2})$, $s \in (2.5, 3)$, to solutions of the inviscid ($\nu = 0$) problem, even in the presence of boundaries, on time-intervals governed by the inviscid equations. Since the Navier–Stokes equations are well-posed in dimension two, all of our results on the LANS-α equations also hold trivially in dimension two.

Global existence for the isotropic LANS-α equations for flow with periodic boundary conditions was proved in Foias *et al.* (2001). Using our formulation (1.1), based on proposition 5 of Shkoller (2001*a*), we are able to extend the well-posedness theory and the global existence result to domains with boundary. However, the reader should be cautioned that for flows on domains with boundary, it may be the *anisotropic* equations that will play the most important role in practice, and we shall examine their global (in time) behaviour in a future publication; we view the isotropic case as an important stepping stone toward that goal.

Foias *et al.* (2001) also computed the dimension of the global attractor for the three-dimensional LANS-α equations on a periodic box; an estimate of the dimension of the global attractor as well as global existence and uniqueness of weak solutions on bounded domains is given in Coutand *et al.* (2001). Global existence for the three-dimensional inviscid Lagrangian averaged Euler (LAE-α) equations (that is, the LANS-α equations with with zero viscosity), remains an open problem.

(*a*) *A brief history*

The isotropic LAE-α (the Lagrangian averaged Euler) equations on all of $\mathbb{R}^n$ first appeared in Holm *et al.* (1998*a*,*b*), and those on compact Riemannian manifolds in Shkoller (1998). The variational formulation of these equations retains the quadratic form of the variational structure for the original Euler equations, so that the equations can be viewed as describing a certain geodesic flow, just as in the work of Arnold (1966) and Ebin & Marsden (1970). The original work on the Lagrangian averaged Euler equations was motivated by the developments of a one-dimensional shallow water theory (see Camassa & Holm 1993) combined with developments in symmetry reduction and Euler–Poincaré theory (see Marsden & Scheurle 1993).

Dissipation was later added to the LAE-α equations to produce the LANS-α equations, also known as the Navier–Stokes-α model.† The papers by Chen *et al.* (1998, 1999*a*–*c*) first added the Navier–Stokes dissipation term to the LAE-α equations. They used physical arguments to write the particular form of the diffusion term. They described the effect of α in the LAE-α as *nonlinear dispersion*, because of its similar effect in the one-dimensional Camassa–Holm shallow water equation. Their physically motivated argument produced the correct form of dissipation on domains that do *not have a boundary* such as the torus; it appeared, however, that in order to extend the model to domains with boundary, an additional boundary condition would be required. Such an additional boundary condition seemed unnatural, and so the extension of the model to bounded domains remained an open problem. The extension to bounded domains was made in Shkoller (2001*a*) and Marsden & Shkoller (2001). The diffusion term was obtained by considering the velocity field as a stochastic process, and replacing deterministic time derivatives with backward-in-time mean stochastic derivatives. This process, which follows the ideas of Chorin (1973) and Peskin (1985), seems more natural to us, generalizes the dissipative term which the above authors obtained, and does not require any additional boundary data.

Remarkably, except for the *crucial* form of the viscosity term, the LANS-α equations are mathematically identical to inviscid *second grade fluid* equations introduced by Rivlin & Erickson (1955). We had initially thought that the second-grade fluid equations had the correct form of viscosity, and in our previous paper (Marsden *et al.* 2000), we had termed those equations the Navier–Stokes-α equations; however, this form of viscosity is *not* the natural dissipation that enters via molecular collision. After we obtained the LANS-α equations using the stochastic methodology, we further appreciated the physical insight in the original papers of Chen *et al.* (1998, 1999*a*–*c*).

The geometric analysis of these equations, including local well-posedness of smooth-in-time solutions in Lagrangian variables and on arbitrary n-dimensional

† In earlier papers, some authors referred to the equations as the *viscous Camassa–Holm (VCH) equations* instead of LANS-α or Navier–Stokes-α.

Riemannian manifolds, was given in Shkoller (1998, 2001*a*) and Marsden *et al.* (2000). Global existence in two dimensions of smooth-in-time solutions was proved in Shkoller (2001*b*). These references also discuss the relationship with the second-grade fluids literature in more detail.

In Oliver & Shkoller (2001*a*), it was shown that the isotropic two-dimensional LAE-α equations are globally well-posed for Radon measure initial vorticity, which includes point-vortex initial data; this fact is not known to be true for the original Euler equations. Correspondingly, while the Kirchhoff point-vortex Hamiltonian ODEs do not generate solutions of the original Euler equations, their counterparts, namely *vortex blob* solutions, *do* generate solutions of the LAE-α equations. The weak solutions to the two-dimensional LAE-α equations induce a *weak* coadjoint action on the vector space of vorticity functions, modelled as the space of Radon measures. The existence of such a weak coadjoint action makes rigorous the formal constructions of Marsden & Weinstein (1983) on the geometry of point-vortex and vortex blob dynamics.

As we described above, the LANS-α equations are a system of PDEs for the mean velocity field, but unlike the Reynolds averaged Navier–Stokes (RANS) or large-eddy simulation (LES) models that add artificial dissipation to the Navier–Stokes equations to filter small scales, the LANS-α equations do not add any artificial viscosity; rather, a nonlinear dispersive mechanism filters the small scales. As such, the LANS-α equations serve as a nice model for turbulent flow. In Chen *et al.* (1999*c*) and Mohseni *et al.* (2001) and works cited therein, it was shown that the LANS-α equations give comparable computational savings as LES models for *forced turbulent flows in periodic domains*; moreover, those papers provided numerical simulations which suggested that the energy spectrum and the energy flux behaviour is preserved by LANS-α at scales larger than α.

For the more demanding case of *decaying turbulence*, a similar computational savings is demonstrated in Mohseni *et al.* (2000). The efficacy of these models for the case of wall-bounded flows remains to be demonstrated; it is quite likely that the anisotropic model is needed for such situations.

(*b*) *A discussion of related mathematical models*

We conclude the introduction with some miscellaneous remarks on related models. The LAE-α equations (3.1) are close in form to the template-matching equations (TME) that occur in computer vision (see Mumford (1998), J. T. Ratnather *et al.* (2000, personal communication), Younes (1998), Dupuis *et al.* (1998), Trouvé & Younes (2000) and references therein). In fact, the TME equations are the same as our LAE-α equations when $\alpha = 0$, and the pressure term and the divergence constraint are omitted. Explicitly, they are

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u + u \operatorname{div} u + (\nabla u)^{\mathrm{T}} \cdot u = 0.$$

(This reduces to $\partial_t u + 3uu_x = 0$ in one dimension.) These are the Euler–Poincaré equations associated with the right invariant L^2 metric on the full diffeomorphism group of the fluid container Ω.

We expect that the averaged equations (equations (3.1) retaining α and with the pressure term and the divergence constraint dropped) and even their anisotropic counterparts may also be of interest in computer vision! The isotropic H^1 template-matching equations are thus the same as the L^2 equations except that u is replaced by $(1-\alpha^2\Delta)u$; in one dimension, these H^1 template-matching equations reduce to the shallow water equations,

$$u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx},$$

which are completely integrable and have peaked soliton solutions (see Camassa & Holm 1993).

In one dimension, both the L^2 and the H^1 template matching equations have smooth-in-time solutions in Lagrangian variables: the L^2 equations reduce to the usual equations for characteristics, while the H^1 equations can be expressed in 'characteristic-like' form (see Shkoller 1998). This is certainly not true for the L^2 equations in higher dimensions, but may continue to hold in higher dimensions for the H^1 equations.

Recently, Misiolek has shown that the shallow water equations are not well-posed for initial data in H^s if $s < 3/2$; however, the existence of H^1 *global* weak solutions has recently been established by Xin & Zhang (2000), although uniqueness does not appear to hold in this class even though numerical simulations seem to choose certain peakon solutions.

2. A review of the Euler equations

(*a*) *The geometry of the Euler equations*

As was shown by Arnold (1966), the flow of the Euler equations of an ideal incompressible fluid is a geodesic of the right-invariant L^2 metric on the group of volume preserving diffeomorphisms. This fact implies that solutions of the linearized Euler equations are Jacobi fields, and that the linear stability problem is completely determined by the sign of the sectional curvatures. Arnold (1966) computed the sectional curvatures of the volume preserving diffeomorphism group of $\mathbb{T}^2$ for the 'tradewind current' solution, and showed that they were negative in most directions. Shkoller (2001*a*) made the same computation for the LAE-α equations, wherein it was shown that the sectional curvatures can flip sign, from negative to positive, when α is taken to be sufficiently large. Thus the LAE-α regularization stabilizes ideal fluid motion.

In addition to the linear stability analysis, well-posedness results may also be obtained in Lagrangian coordinates as in Ebin & Marsden (1970). In fact, the Euler equations become an ODE (in the sense of being a smooth vector field with no derivative loss) on the volume-preserving diffeomorphism group in Lagrangian coordinates, so that local well-posedness results follow directly from the Picard iteration technique. In two dimensions, using the fact that vorticity is conserved, global well-posedness holds in Lagrangian variables (see, for example, Kato 2000; Shkoller 2001*b*), just as in Eulerian variables.

The reduction of the equations from material to spatial (Eulerian) representation may be viewed by the now classical technique of Euler–Poincaré reduction which we briefly review below; see Marsden & Ratiu (1999) and Holm *et al.* (1998*b*) for an exposition and further references. The Euler–Poincaré point of view is a helpful guide to understanding many fluid theories other than the LAE-α equations. We now explain some of these points in more detail.

Let Ω be an open subset of $\mathbb{R}^n$ with C^∞ boundary (possibly empty). The Euler equations for the velocity field u of an ideal, incompressible, homogeneous fluid with density $\rho = 1$ are given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \tag{2.1}$$

with the constraint $\operatorname{div} u = 0$ and the boundary condition that u is tangent to the boundary $\partial\Omega$. The pressure p is determined by the incompressibility constraint. In Cartesian coordinates, these equations are given as follows (using the summation convention for repeated indices):

$$\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} = -\frac{\partial p}{\partial x^i}.$$

We let the flow of the time-dependent vector field $u(t, x)$ be denoted by $\eta(t, x)$ so that

$$\partial_t \eta(t, x) = u(t, \eta(t, x)),$$

with $\eta(0, x) = x$ for all x in Ω. For each t, we denote the map $\eta(t, \cdot)$ by η_t so that $\eta_0 = e$, the identity map. Thus, the map $x \mapsto \eta_t(x)$ gives *the particle placement field* for the fluid. Corresponding to the condition $\operatorname{div} u = 0$, each map η_t is a *volume preserving diffeomorphism*, so that $\det D\eta = 1$, where $D\eta(t, x) = \partial\eta^i/\partial x^j(t, x)$ in coordinates, i.e. the matrix of partial derivatives.

We shall be working with vector fields u of Sobolev class H^s for $s > (n/2) + 1$ and, correspondingly with volume-preserving flow maps η_t in $\mathcal{D}^s_\mu$, where $\mathcal{D}^s_\mu$ denotes the group of H^s-volume preserving diffeomorphisms of the fluid container Ω. We refer the interested reader to Ebin & Marsden (1970) and Shkoller (2001*a*) for some basic properties of Hilbert class diffeomorphism groups for domains (manifolds) with boundary.

Arnold's theorem on the Euler equations may be stated as follows. *A time-dependent velocity field u is a solution of the Euler equations if and only if its Lagrangian flow η_t is a geodesic of the right invariant L^2-metric on $\mathcal{D}^s_\mu$.*

This L^2-metric is defined as follows. The tangent space to $\mathcal{D}^s_\mu$ at the identity is identified with the space $\mathfrak{X}^s_{\text{div}}$, the vector space of H^s divergence free vector fields on Ω that are tangent to the boundary $\partial\Omega$. The right invariant L^2-metric is defined to be the weak Riemannian metric on $\mathcal{D}^s_\mu$ whose value at the identity is

$$\langle u, w \rangle_{L^2} = \int_\Omega u(x) \cdot w(x)\, \mathrm{d}x,$$

where as usual, the pointwise inner product is defined by $u(x) \cdot w(x) = u^i(x) w^i(x)$, and the Euclidean norm is simply $|u(x)|^2 = u(x) \cdot u(x)$. Arnold's theorem is a relatively easy consequence of the general Euler–Poincaré theory.

(*b*) *Vorticity form of the Euler equations*

The Euler equations (2.1) can be written using the Lie derivative as

$$\frac{\partial u^\flat}{\partial t} + \pounds_u u^\flat = \mathrm{d}(\tfrac{1}{2}|u|^2 - p) = -\mathrm{d}p', \tag{2.2}$$

where $u^\flat$ is the one-form associated to the vector field u via the metric (or, equivalently, by lowering the index), and $\mathcal{L}_u u^\flat$ denotes the Lie derivative of the one-form $u^\flat$ along u. Taking the exterior derivative of (2.2) gives the familiar *Lie advection equation* for vorticity,

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_u \omega = 0,$$

where $\omega = \mathrm{d}u^\flat$ is the vorticity, thought of as a two-form. In two dimensions, ω is identified with a scalar and is traditionally thought of as the two-dimensional-curl of the velocity field, so that by taking the curl of (2.1) we obtain

$$\partial_t \omega + \operatorname{grad} \omega \cdot u = 0.$$

In three dimensions, ω may be identified (using the volume-form $\mathrm{d}x$) with a vector field which is traditionally obtained by taking the curl of u. In this case, taking the curl of (2.1) gives the familiar vorticity equation,

$$\partial_t \omega + (u \cdot \nabla)\omega = \operatorname{Def} u \cdot \omega,$$

where the vortex-stretching term appears on the right-hand side.

The vorticity equation is the infinitesimal version of the following *advection property*:

$$\omega_t = (\eta_t)_* \omega_0.$$

Of course in two dimensions, this gives the usual advection of vorticity as a scalar function, while in three (or higher) dimensions, the advection is understood in terms of advection of two-forms.

The Euler equations have both an interesting Hamiltonian structure in terms of Poisson brackets (a Lie–Poisson bracket) and a variational structure. In this paper we shall be working primarily with the variational structure; the Hamiltonian structure, along with references to the literature may be found in Marsden & Weinstein (1983), Arnold & Khesin (1998) and Marsden & Ratiu (1999).

(*c*) *The action principle*

The Lagrangian is given by the total kinetic energy of the fluid; in spatial representation, this Lagrangian is

$$L(u) = \frac{1}{2} \int_\Omega |u(x)|^2 \, \mathrm{d}x. \tag{2.3}$$

The corresponding (unreduced) Lagrangian on $T\mathcal{D}^s_\mu$ is given by

$$\mathcal{L}(\eta, \partial_t \eta) = \frac{1}{2} \int_\Omega |\partial_t \eta(x)|^2 \, \mathrm{d}x. \tag{2.4}$$

Hamilton's principle on $\mathcal{D}^s_\mu$ applied to the Lagrangian $\mathcal{L}$ gives geodesics on this group. Euler–Poincaré reduction techniques (see Marsden & Ratiu 1999) show that this variational principle reduces to the following principle in terms of Eulerian velocities:

$$\delta \int_0^T L(u) \, \mathrm{d}t = 0,$$

which should hold for all variations δu of the form

$$\delta u = \partial_t w + (u \cdot \nabla)w - (w \cdot \nabla)u, \tag{2.5}$$

where w is a time-dependent vector field (representing the infinitesimal particle displacement) vanishing at the temporal endpoints.† One readily checks that this reduced principle yields the standard Euler equations. This simple computation is the heart of Arnold's theorem.

(*d*) *Analytical issues*

While the Eulerian (spatial) representation has been emphasized in most analytical studies of the Euler equations, fluid motion viewed on the Lagrangian (material) side has some distinct advantages. For example, it is shown in Ebin & Marsden (1970) that the flow, solving the Euler equations, on the volume-preserving diffeomorphism group $\mathcal{D}^s_\mu$, $s > (n/2) + 1$, is smooth in time (the results are valid in the Hölder classes $C^{k,\alpha}$ for $k \geqslant 1$ as well). This result holds globally in two dimensions for initial data $u_0 \in \{v \in H^s \mid \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \partial\Omega\}$, $s > (n/2) + 1$ (see Shkoller 2001*b*). A number of interesting consequences of this result were derived, including theorems on the convergence of solutions of the Navier–Stokes equations to solutions of the Euler equations as the viscosity goes to zero when Ω has no boundary. In addition, Hald (1987), Marchioro & Pulvirenti (1994) and others analysed the Lagrangian flow map to prove convergence of the vortex blob algorithm. The fact that vortex blobs give an exact solution to the LAE-α equations warrants another look at some of these issues. In any case, it is clear that the Lagrangian framework is a natural setting to study the behaviour of solutions.

3. The isotropic Lagrangian averaged Euler (LAE-α) equations

(*a*) *The equations*

Let α be a positive constant. In Euclidean space and in Euclidean coordinates, the isotropic LAE-α equations are often written as

$$\partial_t(1-\alpha^2\Delta)u + (u \cdot \nabla)(1-\alpha^2\Delta)u - \alpha^2(\nabla u)^{\mathrm{T}} \cdot \Delta u = -\operatorname{grad} p,$$

or in coordinates as

$$\partial_t(u^i - \alpha^2 u^i_{,kk}) + u^j \frac{\partial}{\partial x^j}(u^i - \alpha^2 u^i_{,kk}) - \alpha^2 \left[\frac{\partial u^j}{\partial x^i}\right] u_{j,kk} = -\frac{\partial p}{\partial x^i},$$

where Δ denotes the componentwise Laplacian, and there is a summation over repeated indices (in Euclidean coordinates, as is common, we make no distinction between indices up or down). As before, we also impose the incompressibility constraint $\operatorname{div} u = 0$, which determines the pressure. We shall additionally impose the no-slip, $u = 0$, boundary conditions; see Shkoller (2001*a*) for the free-slip and mixed-type boundary conditions.

Using the fact that $(\nabla u)^{\mathrm{T}} \cdot u = \operatorname{grad}(\frac{1}{2}|u|^2)$ and modifying the pressure accordingly, we can rewrite the LAE-α equations as follows:

$$\partial_t(1-\alpha^2\Delta)u + (u \cdot \nabla)(1-\alpha^2\Delta)u + (\nabla u)^{\mathrm{T}} \cdot (1-\alpha^2\Delta)u = -\operatorname{grad} p. \tag{3.1}$$

† The constraints on the allowed variations of the fluid velocity field are commonly known as 'Lin constraints'. This itself has an interesting history, going back to Ehrenfest, Boltzmann, Clebsch, Newcomb and Bretherton, but there was little if any contact with the heritage of Lie and Poincaré on the subject.

(*b*) *The geometry of the LAE-α equations*

The Euler–Poincaré theory also shows that the solutions of the isotropic LAE-α equations can be regarded, in a similar way to the Euler equations, as geodesics on certain subgroups of the volume preserving diffeomorphism group, but with respect to an H^1-equivalent metric rather than an L^2 metric. The fact that one has equations for geodesics on the group corresponds simply to the fact that the Lagrangian is quadratic in the velocities; the fact that unique smooth geodesics of such a weak metric exist (recall that the strong topology is H^s with $s > (n/2)+1$) is a consequence of a delicate analysis as performed in Shkoller (2001*a*).

The *anisotropic* LAE-α equations can also be interpreted as geodesic equations of an H^1-equivalent metric, but this metric depends on the covariance tensor which is itself advected in time by the mean flow. These equations also possess unique smooth geodesics as is proved in Marsden & Shkoller (2001).

(*c*) *Rate of deformation tensor*

This tensor will play a basic role in our theory, and is defined by

$$\mathrm{Def}\, u = \tfrac{1}{2}[\nabla u + (\nabla u)^{\mathrm{T}}], \quad (\mathrm{Def}\, u)^i_j = \tfrac{1}{2}(u^i_{,j} + u^j_{,i}).$$

It is often convenient to lower the index; we set $D \equiv \mathrm{Def}\, u^\flat = \frac{1}{2}[\nabla u^\flat + (\nabla u^\flat)^{\mathrm{T}}]$, or in coordinates,

$$D_{ij} = (\mathrm{Def}\, u^\flat)_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

Note that $D \equiv \mathrm{Def}\, u^\flat = \pounds_u g$, the *Killing tensor.*

(*d*) *LAE-α energy law*

There is an energy integral for the isotropic LAE-α equations, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega [|u(x)|^2 + 2\alpha^2|\,\mathrm{Def}\, u(x)|^2]\,\mathrm{d}x = 0. \tag{3.2}$$

With no-slip, $u = 0$, boundary conditions, (3.2) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega [|u(x)|^2 + \alpha^2|\nabla u(x)|^2]\,\mathrm{d}x = 0. \tag{3.3}$$

It is essential, however, to use the energy (3.2), if either the free-slip or mixed-type boundary conditions are used; using (3.3) instead leads to a Stokes problem whose Green's function is not a Fredholm operator.

Another equivalent form of the energy law is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x)\cdot(1-\alpha^2\Delta)u(x)\,\mathrm{d}x = 0. \tag{3.4}$$

(*e*) *Smoothness properties*

Analytical results concerning the regularity of solutions to the inviscid LAE-α problem in Lagrangian coordinates were given in Shkoller (1998) on compact boundary Riemannian manifolds without boundary, and in Marsden *et al.* (2000) on compact Euclidean domains. The problem of how to formulate this system on compact Riemannian manifolds *with boundary* was solved in Shkoller (2001*a*).

(*f*) *Lie derivative and vorticity forms*

The LAE-α may also be expressed using the Lie derivative as

$$\partial_t(1-\alpha^2\Delta)u^\flat + \mathcal{L}_u(1-\alpha^2\Delta)u^\flat = -\mathrm{d}p. \tag{3.5}$$

By applying the exterior derivative (the curl operator in $\mathbb{R}^3$) to (3.5) and letting $\omega = \mathrm{d}u^\flat$, we obtain the vorticity form of the LAE-α equations

$$\partial_t(1-\alpha^2\Delta)\omega + \mathcal{L}_u(1-\alpha^2\Delta)\omega = 0,$$

or equivalently if the vector $\omega = \operatorname{curl} u$, then we see that ω solves

$$\partial_t(1-\alpha^2\Delta)\omega + (u\cdot\nabla)(1-\alpha\Delta)\omega = \nabla u\cdot(1-\alpha^2\Delta)\omega,$$

where the right-hand side again denotes the vortex stretching term. While we have *a priori* control of $\omega(t,\cdot)$ in L^2 for almost all t, we are still far from satisfying the Beale–Kato–Majda condition for ensuring that blow-up cannot occur in three dimensions; however, it appears that the Constantin *et al.* (1996) depletion of the nonlinearity via vorticity alignment occurs (see Oliver & Shkoller 2001*b*).

It also follows from (3.5), that we have the following conservation of *averaged helicity* given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (1-\alpha^2\Delta)u(t,x)\times \operatorname{curl}(1-\alpha^2\Delta)u(t,x)\,\mathrm{d}x = 0.$$

Helicity, which is a Casimir invariant, is interesting in a number of fluid dynamical situations; see Marsden & Weinstein (1983), Moffatt (2000) and references therein.

4. The Lagrangian averaging methodology

We shall now present the main ideas of the derivation of the LAE-α and LANS-α equations from Marsden & Shkoller (2001). We note that there are some interesting links with *optimal prediction theory* (see Chorin *et al.* (1999) and references therein).

Let X denote the vector space of initial velocity fields for which the Euler equations are (at least locally) well-posed, and let S denote the unit sphere in X.

For $\bar{u}_0 \in X$, let $\bar{u}(t,x)$ denote the solution of the Euler equations with $\bar{u}(0,\cdot)=\bar{u}_0$. Similarly, let $u^\epsilon(t,x)$ denote the solution of the Euler equations with initial data u_0^ϵ, where

$$u_0^\epsilon = \bar{u}_0 + \epsilon w, \quad w\in S, \quad \epsilon\in[0,\alpha]$$

for some $\alpha > 0$ and small. Of course, $u^\epsilon(t,x)$ depends on w as well, but we suppress that dependence for notational simplicity.

We let ν denote a chosen measure on the unit sphere S in X, and define the *average* of vector-valued functions $f(\epsilon,w)$ on $[0,\alpha]\times S$ by

$$\langle f\rangle := \frac{\bar{t}^2}{\alpha}\int_0^\alpha\int_S f(\epsilon,w)\nu\,\mathrm{d}\epsilon,$$

where $\bar{t}$ is a characteristic unit of time. This will be our chosen *ensemble averaging operation.*

By uniqueness of solutions, it follows that $u^0(t,x)=\bar{u}(t,x)$. Let $\bar{\eta}$ be the Lagrangian flow of $\bar{u}$, which solves $\partial_t\bar{\eta}(t,x) = \bar{u}(t,\bar{\eta}(t,x))$ with $\bar{\eta}(0,x)=x$. Similarly, let

η^ϵ denote the Lagrangian flow of u^ϵ. We define the *Lagrangian fluctuation volume-preserving diffeomorphism* ξ^ϵ by

$$\xi^\epsilon(t,x) := \eta^\epsilon(t,\bar{\eta}^{-1}(t,x)). \tag{4.1}$$

Clearly, $\xi^0(t,x) = x$, since $\eta^{\epsilon=0}(t,x) = \bar{\eta}(t,x)$ for all $t \geqslant 0$. Let

$$u'(t,x) = \frac{\mathrm{d}}{\mathrm{d}\alpha}|_{\alpha=0} u^\alpha(t,x)$$

denote the associated *Eulerian velocity fluctuation* about $\bar{u}$. The corresponding *Lagrangian fluctuation* (in spatial representation) is given by

$$\xi'(t,x) = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \xi^\alpha(t,x).$$

Similarly, let

$$u''(t,x) = \left.\frac{\mathrm{d}^2}{\mathrm{d}^2\alpha}\right|_{\alpha=0} u^\alpha(t,x),$$

and

$$\xi''(t,x) = \left.\frac{\mathrm{d}^2}{\mathrm{d}^2\alpha}\right|_{\alpha=0} \xi^\alpha(t,x).$$

Our goal is to average over all possible solutions of the Euler equations with initial data u_0^ϵ in an X-ball of radius α about $\bar{u}$; since each solution $u^\epsilon(t,x)$ is obtained from the first variation of the action as we described above, it is appropriate to define the *averaged action function*

$$\bar{S} = \left\langle \frac{1}{2}\int_0^T \int_\Omega |\partial_t u^\epsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \right\rangle. \tag{4.2}$$

Expanding u^ϵ about $\epsilon = 0$, we get

$$u^\epsilon(t,x) = \bar{u}(t,x) + \epsilon u'(t,x) + \tfrac{1}{2}\epsilon^2 u''(t,x) + O(\epsilon^3). \tag{4.3}$$

Since $\bar{u}$ does not depend on either ϵ or w, we see that $\langle \bar{u} \rangle = \bar{u}$; correspondingly, we call $\bar{u}$ the *mean*.

(*a*) *Relationship between Eulerian and Lagrangian fluctuations*

By differentiating (4.1), one obtains the relations,

$$u' = \partial_t \xi' + (\bar{u}\cdot\nabla)\xi' - (\xi'\cdot\nabla)\bar{u}, \tag{4.4 a}$$

$$u'' = \partial_t \xi'' + (\bar{u}\cdot\nabla)\xi'' - 2(\xi'\cdot\nabla)u' - \nabla\nabla\bar{u}(\xi',\xi'), \tag{4.4 b}$$

where, in coordinates,

$$\nabla\nabla\bar{u}(\xi',\xi') = \bar{u}^i_{,jk}\xi'^j\xi'^k.$$

(b) Generalized Taylor or frozen turbulence hypothesis

We shall use a generalization of the classical frozen turbulence hypothesis of Taylor introduced in Taylor (1938). In its classical form, the streamwise scalar component of the fluctuation is considered frozen over the time-scale of the temporal derivative, giving

$$\frac{\partial}{\partial t} = U \frac{\partial}{\partial x},$$

where U is the local mean velocity along the streamwise direction, which is denoted by x.

Our generalized Taylor hypothesis is a collection of assumptions, at each order of α, on the behaviour of the Lagrangian fluctuations. For the present theory, we shall produce a closure to $O(\epsilon^2)$. We make the following assumptions.

$O(\alpha)$ generalized Taylor hypothesis:

$$\partial_t \xi' + (\bar{u} \cdot \nabla)\xi' - (\xi' \cdot \nabla)\bar{u} = 0, \tag{4.5}$$

which states that at $O(\epsilon)$, the first-order Lagrangian fluctuation ξ' is *frozen into the mean flow* as a vector field.†

$O(\alpha^2)$ generalized Taylor hypothesis:

$$\partial_t \xi'' + (\bar{u} \cdot \nabla)\xi'' = 0, \tag{4.6}$$

which states that the second-order fluctuations ξ'' are *isometrically or parallel transported by the mean flow* so that at the $O(\alpha^2)$ level, no stretching of ξ'' may occur.‡

Substituting the relations (4.4 *a*) and (4.4 *b*) into the expansion (4.3) and using the generalized Taylor hypothesis (4.5) and (4.6), we find that

$$u^\epsilon = \bar{u} - \tfrac{1}{2}\epsilon^2 \nabla\nabla\bar{u}(\xi', \xi') + O(\epsilon^3). \tag{4.7}$$

Substitution of (4.7) into the averaged action function (4.2) gives

$$\bar{S} = \frac{1}{2}\int_0^T \int_\Omega [|\bar{u}|^2 + \alpha^2 \langle \nabla\nabla\bar{u} : F, \bar{u}\rangle + O(\alpha^3)]\, \mathrm{d}x\, \mathrm{d}t,$$

where the *Lagrangian covariance tensor* F is defined by

$$F = \langle \xi' \otimes \xi' \rangle.$$

For the purposes of this paper, we shall derive the *isotropic* form of the equations. For the isotropic scenario, we assume that (or, if one prefers, impose the constraint that),

$$F = c\,\mathrm{Id},$$

a constant multiple of the identity. By integration-by-parts and truncation of the averaged action function to $O(\alpha^2)$, we find that

$$\bar{S}^\alpha = \frac{1}{2}\int_0^T \int_\Omega [|\bar{u}^\alpha|^2 + 2\alpha^2 |\,\mathrm{Def}\,\bar{u}^\alpha|^2]\, \mathrm{d}x\, \mathrm{d}t,$$

† This equation may also be written as $\xi' + \pounds_{\bar{u}}\xi'$, where $\pounds$ denotes the Lie derivative operation on vector fields.

‡ One may relax these hypotheses slightly by only requiring their satisfaction up to terms higher order in α.

where we now use the notation $\bar{u}^\alpha$ to indicate that the variable depends on the parameter α. We refer the reader to Marsden & Shkoller (2001) for the anisotropic case.

Hamilton's principle and the Euler–Poincaré theory tells us that we should consider stationary points of the action $\bar{S}^\alpha$ for variations of the form

$$\delta \bar{u}^\alpha = \partial_t(\delta \bar{\eta}^\alpha \circ \bar{\eta^\alpha}^{-1}) + [u, \delta \bar{\eta}^\alpha \circ \bar{\eta^\alpha}^{-1}],$$

where $\bar{\eta}^\alpha$ is the flow of $\bar{u}^\alpha$ and $[v, w]$ is the commutator of vector fields v, w given by $[v, w] = (v \cdot \nabla)w - (w \cdot \nabla)v$. *This action principle yields the isotropic LAE-α equations*, which are the equations (1.1) with $\nu = 0$. The dissipative term νAu comes from a stochastic interpretation of the Lagrangian flow map. As can be seen from the papers of Chorin (1973) and Peskin (1985), by allowing the Lagrangian trajectory to undergo a random walk, the diffusion term naturally arises. From the point-of-view of stochastic ODEs, the deterministic time derivatives are replaced with backward-in-time stochastic mean derivatives. By the Itö formula, the diffusion term naturally arises. Thus, our formulation of the LANS-α equations given by (1.1) is the natural form of the equations in the presence of viscosity. As can be seen, this equation can be solved with the no-slip, $u = 0$, boundary condition, while the condition that $Au =$ on $\partial\Omega$ is automatically satisfied by solutions of the PDE.

For the remainder of the paper, we shall denote $\bar{u}^\alpha$ simply by u.

(*c*) *The isotropic Lagrangian averaged Navier–Stokes (LANS-α) equations*

In their work, Foias *et al.* (2001) considered the LANS-α equations in a three-dimensional periodic box. They reasoned that since it is the momentum $v = (1 - \alpha^2\Delta)u$ that is being transported in the LAE-α equations, given by (3.1), then it is this momentum that needs to be diffused; hence, they considered the following version of the LANS-α equations:

$$\frac{\partial v}{\partial t} + (u \cdot \nabla)v + (\nabla u)^{\mathrm{T}} \cdot v = -\nabla p + \nu \Delta v. \tag{4.8}$$

This form of the PDE agrees with our formulation when no boundaries are present, but requires a generalization of the term $\nu\Delta v$ to $-\nu(1 - \alpha^2\Delta)Au$, and the knowledge of the additional boundary condition to invert this fourth-order operator. The form (4.8) does not reveal what this additional boundary condition must be.

The key to finding the additional boundary condition is to write the LANS-α equations in the from (1.1 *a*) and (1.2). Specifically, notice from (1.1 *a*) that Au must vanish when restricted to the boundary; clearly, $\partial_t u + \nabla_u u$ vanishes on the boundary, as do the remaining terms, since $(1 - \alpha\Delta)^{-1}$ has range the domain $D(1 - \alpha^2\Delta)$. It follows that (1.1) is equivalent to (4.8) on domains with boundary if the additional boundary condition

$$Au = 0 \text{ on } \partial\Omega$$

is imposed.

(*d*) *The 'Reynolds stress'*

When Ω has no boundary, (1.1 *a*) takes the form

$$\partial_t u - \nu\Delta u + \nabla_u u + \operatorname{Div} \tau^\alpha(u) = -\operatorname{grad} p, \tag{4.9}$$

where we may identify

$$\tau^\alpha(u) := \alpha^2(1-\alpha^2\Delta)^{-1}[\nabla u\cdot\nabla u^{\mathrm{T}}+\nabla u\cdot\nabla u-\nabla u^{\mathrm{T}}\cdot\nabla u] \tag{4.10}$$

with the usual Reynolds stress. This form of the equations is useful for comparing the isotropic LANS-α equations with the LES or the RANS models.

We shall introduce another equivalent form of the LANS-α equations below.

5. Global well-posedness

(*a*) *Notation and some interpolation inequalities*

For $s\geqslant 1$, let $\mathcal{V}^s = H^1_0\cap H^s$, $\mathcal{V}^s_\mu = \{u\in\mathcal{V}^s \mid \operatorname{div} u = 0\}$, and for $s\geqslant 3$, let $\dot{\mathcal{V}}^s_\mu = \{u\in\mathcal{V}^s_\mu \mid Au = 0 \text{ on } \partial\Omega\}$. Let $\alpha=(\alpha_1,\dots,\alpha_n)$ denote a multi-index, and

$$|\alpha| = \alpha_1+\cdots+\alpha_n, \qquad D^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}.$$

We have the product rule

$$D^\alpha(fg) = \sum_{\substack{|\beta|\leqslant|\alpha|\\ \alpha-\beta>0}} c_{\alpha,\beta}(D^\beta f)(D^{\alpha-\beta}g).$$

For any integer $s\geqslant 0$, we set

$$D^s u = \{D^\alpha u : |\alpha| = s\}, \qquad \|D^s u\|_{L^p} = \sum_{|\alpha|=s}\|D^\alpha u\|_{L^p}.$$

We let $C>0$ be a generic constant throughout the paper. We will make use of the following standard inequalities in dimension three:

$$|v|_{L^\infty} \leqslant C|D^2 v|^{1/2}_{L^2}|v|^{1/2}_{L^2}, \tag{5.1}$$

$$|v|_{L^4} \leqslant 4|Dv|^{3/4}_{L^2}|v|^{1/4}_{L^2}, \tag{5.2}$$

$$|D^i v|_{L^2} \leqslant C|v|^{1-i/m}_{L^2}|D^m v|^{i/m}_{L^2}. \tag{5.3}$$

The inequality (5.1) is commonly referred to as Agmon's inequality (see Agmon 1965; Agmon *et al.* 1959, 1964; Nirenberg 1959), while (5.2) and (5.3) are often referred to as Ladyzhenskaya inequalities (see Ladyzhenskaya 1963). We shall need the following lemma.

Lemma 5.1. *For* $s\geqslant 3$, $\mathcal{U}^\alpha:\mathcal{V}^s\to\mathcal{V}^s$ *and* $\mathcal{U}^\alpha:\mathcal{V}^2\to H^{1+\sigma}$ *for* $\sigma\in(0,\frac{1}{3})$.

Proof. The first assertion follows from the fact that H^s, $s>1.5$, is a multiplicative algebra. The second follows from the fact that

$$H^r\cdot H^r\subset H^\sigma$$

for

$$r=\tfrac{3}{4}+\tfrac{1}{2}\sigma+\epsilon\theta \quad\text{and}\quad \sigma=\theta(\tfrac{3}{2}+\epsilon).$$

(See, for example, lemma 5.3 in Taylor (1996, ch. 17).) Now take $\theta=1/6$ and $\epsilon=1/2$ so that $\sigma=1/3$. ■

(*b*) *Three forms of the LANS-α equations*

Three equivalent forms of the LANS-α equations will be useful to us.

(i) *LANS-1*

$$\partial_t u + \nabla_u u + \mathcal{U}^\alpha(u) = -\nu A u - (1-\alpha^2\Delta)^{-1}\,\mathrm{grad}\, p \tag{5.4}$$

where the stress term $\mathcal{U}^\alpha$ is defined by (1.2), and where, as usual, $\mathrm{div}\, u(x,t) = 0$ for all $t \geqslant 0$ and $x \in \Omega$ and with the boundary conditions $u = 0$ on $\partial\Omega$ and initial conditions $u(x,0) = u_0(x)$.

This form of the equations shows that $Au = 0$ on the boundary $\partial\Omega$ so that $Au \in \mathrm{Domain}(1-\alpha^2\Delta)$ whenever $u \in H^4$. Notice that because the domain of definition of the operator $(1-\alpha^2\Delta)$ consists of vector fields that vanish on the boundary, the terms $(1-\alpha^2\Delta)^{-1}\,\mathrm{grad}\, p$ and $\mathcal{U}^\alpha(u)$ are zero on $\partial\Omega$.

(ii) *LANS-2*

This form is equivalent to LANS-1 in view of our remark that LANS-1 implies that $Au = 0$ on $\partial\Omega$:

$$\partial_t(1-\alpha^2\Delta)u + \nabla_u[(1-\alpha^2\Delta)u] - \alpha^2\nabla u^{\mathrm{T}}\cdot\Delta u = -\nu(1-\alpha^2\Delta)Au - \mathrm{grad}\, p, \tag{5.5}$$

where $\mathrm{div}\, u = 0$ and $u = Au = 0$ on $\partial\Omega$.

(iii) *LANS-3*

This form is the analogue of the Navier–Stokes equations written in terms of the Helmholtz–Hodge projection:

$$\partial_t u + \nu A u + \mathcal{P}^\alpha[\nabla_u u + \mathcal{U}^\alpha(u)] = 0, \tag{5.6}$$

where for $s \geqslant 1$,

$$\mathcal{P}^\alpha : \mathcal{V}^s \to \mathcal{V}^s_\mu$$

is the *Stokes projector* defined in proposition 2 of Shkoller (2001*a*). as

$$\mathcal{P}^\alpha(w) = w - (1-\alpha^2\Delta)^{-1}\,\mathrm{grad}\, p,$$

where p is a solution of the *Stokes problem*: given $w \in \mathcal{V}^s$, there is a unique vector field v and a function p (unique up to additive constants) such that

$$(1-\alpha^2\Delta)v + \mathrm{grad}\, p = (1-\alpha^2\Delta)w$$

with $\mathrm{div}\, v = 0$ and $v = 0$ on $\partial\Omega$.

Theorem 5.2. *For $u_0 \in \dot{\mathcal{V}}^s_\mu$ and $s \in [3,5)$, there exists a unique solution u to (1.1) in $C([0,\infty), \dot{\mathcal{V}}^s_\mu) \cap C^\infty((0,\infty)\times\Omega)$.*

Proof. We first establish local well-posedness using the contraction mapping theorem; this is standard, but for completeness, we give the argument. We may rewrite LANS-3 as

$$u(t,\cdot) = \mathrm{e}^{-t\nu A}u_0 - \int_0^t \mathrm{e}^{(s-t)\nu A}\mathcal{P}^\alpha[\mathrm{div}(u(s)\otimes u(s)) + \mathcal{U}^\alpha(u(s))]\,\mathrm{d}s =: \Psi u(t).$$

Take s in the interval $[3,5)$. We will find a fixed point of Ψ on $C([0,T],\dot{\mathcal{V}}^s_\mu)$ for some $T>0$. We begin by showing that

$$\Phi : \dot{\mathcal{V}}^s_\mu \to \mathcal{V}^{s-1} \text{ is locally Lipschitz,} \tag{5.7}$$

where $\Phi(u) = \mathcal{P}^\alpha(\operatorname{div}(u\otimes u) + \mathcal{U}^\alpha(u))$. By lemma 5.1, $\mathcal{U}^\alpha : \dot{\mathcal{V}}^s_\mu \to \mathcal{V}^s$, and since H^s, $s \geqslant 3$, is a multiplicative algebra, $u \mapsto u\times u : \dot{\mathcal{V}}^s_\mu \to \mathcal{V}^s$, so $\operatorname{div}(u\otimes u) \in \mathcal{V}^{s-1}$. By ellipticity of the Stokes problem, $\mathcal{P}^\alpha : \mathcal{V}^{s-1} \to \dot{\mathcal{V}}^{s-1}_\mu$ continuously so we have established (5.7). Now,

$$\|\mathrm{e}^{-tA}\|_{\mathcal{L}(\mathcal{V}^{s-1},\dot{\mathcal{V}}^s_\mu)} \leqslant Ct^{-1/2} \text{ for } t\in(0,1]. \tag{5.8}$$

For $\delta>0$ fixed, set

$$Z = \{u \in C([0,T],\dot{\mathcal{V}}^s_\mu) \mid u(0,\cdot) = u_0,\ \|u(t,\cdot)-u_0\|_{H^s} \leqslant \delta\}.$$

We choose T small enough so that $\Psi : Z\to Z$ is a contraction. Since $\mathrm{e}^{-tA} : \dot{\mathcal{V}}^s_\mu \to \dot{\mathcal{V}}^s_\mu$ is a strongly continuous semigroup for $t\geqslant 0$, we can choose T_1 such that

$$\|\mathrm{e}^{tA}u_0 - u_0\|_{H^s} \leqslant \delta/2 \text{ for } t\in[0,T_1].$$

By (5.7), $\|\Phi(u(s))\|_{H^{s-1}} \leqslant K_1$ for $s\in[0,T_1]$ so by (5.8),

$$\left\| \int_0^t \mathrm{e}^{(s-t)A}\Phi(u(s))\,\mathrm{d}s \right\|_{H^s} \leqslant Ct^{1/2},$$

so that for $T_2 \leqslant T_1$ small enough, this will be bounded by $\delta/2$ for $t\in[0,T_2]$, and $\Psi : Z\to Z$ if $T\leqslant T_2$. Finally, by (5.7),

$$\|\Psi(u)(t) - \Psi(v)(t)\|_{H^s} \leqslant Ct^{1/2}\sup\|u(s)-v(s)\|_{H^s}$$

and for small enough t, $Ct^{1/2}<1$, so that $\Psi : Z\to Z$ is a contraction, and hence Ψ has a fixed point.

We shall now use *a priori* energy estimates to extend T to ∞.

(c) An H^1 estimate

We take the L^2 inner product of LANS-2 with u to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|u|^2_{L^2} + \alpha^2|Au|^2_{L^2}) \leqslant -\nu(|A^{1/2}u|^2_{L^2} + \alpha^2|Au|^2_{L^2}) \tag{5.9}$$

from which it follows that

$$u\in L^\infty((0,T],\mathcal{V}^1_\mu)$$

uniformly in $T\geqslant 0$.

(d) An H^2 estimate

We now take the L^2 inner product of the operator A applied to LANS-3 with Au to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Au|^2_{L^2} + \langle A\mathcal{P}^\alpha(\nabla_u u + \mathcal{U}^\alpha(u)), Au\rangle_{L^2} = -\nu|A^{3/2}u|^2_{L^2}. \tag{5.10}$$

Using (5.1) and (5.3), we get

$$\begin{aligned}\langle A\mathcal{P}^\alpha(\nabla_u u + \mathcal{U}^\alpha(u)), Au\rangle &\leqslant C(|u|_{H^1}^{1/2}|u|_{H^2}^2|u|_{H^3}^{1/2} + |u|_{L^2}^{1/2}|u|_{H^2}^{3/2}|u|_{H^3}) \\ &\leqslant C(|u|_{H^1}^{3/2}|A^{3/2}u|_{L^2}^{3/2} + |u|_{H^1}^{1/2}|A^{3/2}u|_{L^2}^{7/4}).\end{aligned}$$

By Young's inequality, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}|Au|_{L^2}^2 \leqslant -C(\nu-\epsilon)|Au|_{L^2}^2 + \frac{C}{\epsilon}(|u|_{H^1}^4 + |u|_{H^1}^6) \tag{5.11}$$

so that taking $0 < \epsilon < \nu$, one obtains

$$u \in L^\infty((0,T], \mathcal{V}_\mu^2).$$

uniformly in $T \geqslant 0$.

(*e*) *An H^3 estimate*

To get an H^3 estimate, we let u_t denote $\partial_t u$, and differentiate the equations LANS-2 with respect to time to get

$$\begin{aligned}\partial_t(1-\alpha^2\Delta)u_t + \nabla_{u_t}(1-\alpha^2\Delta)u + \nabla_u(1-\alpha^2\Delta)u_t - \alpha^2\nabla u_t^{\mathrm{T}}\cdot\Delta u \\ - \alpha^2\nabla u^{\mathrm{T}}\cdot\Delta u_t = -\operatorname{grad} p_t - \nu(1-\alpha^2\Delta)Au_t.\end{aligned}$$

Noting that $u_t \in D(A)$, we take the L^2 inner product with u_t to get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|u_t|_{L^2}^2 + \alpha^2|A^{1/2}u_t|_{L^2}^2) \\ &\qquad \leqslant -\nu(|A^{1/2}u_t|_{L^2}^2 + \alpha^2|Au_t|_{L^2}^2) + |\langle u_t, \nabla_{u_t}u\rangle| + \alpha^2|\langle\Delta u_t, \nabla_u u_t - \nabla_{u_t}u\rangle|.\end{aligned}$$

Now using (5.2), we estimate the next to last term by

$$\begin{aligned}|\langle u_t, \nabla_u u_t\rangle| &\leqslant |u_t|_{L^4}|\nabla u|_{L^4}|u_t|_{L^2} \\ &\leqslant C(|u|_{H^2})|A^{1/2}u_t|_{L_2}^2\end{aligned}$$

and we estimate the last term again using (5.2) and Young's inequality by

$$|\langle\Delta u_t, \nabla_u u_t - \nabla_{u_t}u\rangle| \leqslant \epsilon|Au_t|_{L_2}^2 + \frac{1}{\epsilon}C(|u|_{H^1}, |u|_{H^2})|A^{1/2}u_t|_{L_2}^2.$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^{1/2}u_t|_{L_2}^2 \leqslant \frac{C}{\epsilon}|A^{1/2}u_t|_{L_2}^2.$$

Since u_0 is in H^s for $s \in [3,5)$, it is clear that $u_t(0,\cdot) \in \mathcal{V}_\mu^1$ and hence that

$$u_t \in L^\infty((0,T], \mathcal{V}_\mu^1)$$

for any fixed $T \geqslant 0$. From the above estimates, for almost all t, $u(t,\cdot) \in \mathcal{V}^2$ and $u_t(t,\cdot) \in \mathcal{V}^1$; thus, since H^2 is a multiplicative algebra, we see that for almost all t, $\mathcal{P}^\alpha(\nabla_u u)(t,\cdot)$ is in $\mathcal{V}_\mu^1$. Lemma 5.1 shows that $\mathcal{P}^\alpha(\mathcal{U}^\alpha(u)) \in H^{1+\sigma}$ for $0 < \sigma < 1/3$.

Using LANS-3, this gives

$$\nu Au = -\mathcal{P}^\alpha(\nabla_u u + \mathcal{U}^\alpha(u)) + u_t \in \mathcal{V}_\mu^1$$

so by elliptic regularity of the Stokes operator A, we get

$$u \in L^\infty((0,T), \dot{\mathcal{V}}_\mu^3)$$

for any $T \geqslant 0$, and the H^3 norm of u depends only on the initial data. Using the usual continuation argument, we have a global solution in H^3 for initial data in $\dot{\mathcal{V}}_\mu^3$. A standard boot-strapping argument gives global-wellposedness for initial data in $\dot{\mathcal{V}}_\mu^s$ for $s \in [3,5)$, and shows that the solution becomes instantaneously smooth on $(0,\infty) \times \Omega$. ■

Remark 5.3. Note that the restriction on the initial data $u_0 \in \dot{\mathcal{V}}_\mu^s$ for $s \in [3,5)$ is merely to have continuous dependence on initial data in $\dot{\mathcal{V}}_\mu^s$; for global solutions that are not continuous at $t=0$, we can take initial data in $\dot{\mathcal{V}}_\mu^s$ for all $s \geqslant 3$.

(*f*) *Weak solutions and global attractors*

Coutand *et al.* (2001) prove the global well-posedness of the LANS-α equation for initial data u_0 in H_0^1 together with $\operatorname{div} u_0 = 0$; for such initial data, H^2 absorbing sets are established which yields the existence of maximal compact global attractors in the set $\{u \in H_0^1 \mid \operatorname{div} u = 0\}$. This extends the results of Foias *et al.* (2001) to the case of bounded domains.

(*g*) *Limit of zero viscosity*

Barenblatt & Chorin (1998) state that while Navier–Stokes flows do not, in general, converge to Euler flows on domains with boundary, the averaged Navier–Stokes flow should indeed have this property. The following theorem proves this.

Theorem 5.4. *Let $\alpha > 0$ be fixed and let $s \in (2.5, 3)$. Let $u_0 \in \mathcal{V}_\mu^s$ be given initial data and let u^ν be the corresponding solution of the LANS-α equations. Then there exists a uniform time interval $[0,T]$ such that u^ν converges in $L^\infty[0,T], \mathcal{V}_\mu^s$ to a solution u of the LAE-α equations.*

This is proven in Shkoller (2001*a*). It is interesting to note that for $s \geqslant 3$, boundary layer formation occurs.

(*h*) *Limit of zero α*

Oliver & Shkoller (2001*a*) have proved that velocity solutions of the two-dimensional LAE-α equations on $\mathbb{R}^2$ converge in C^0 to solutions of the Euler equations as $\alpha \to 0$ globally in time for initial vorticity fields in $L^\infty \cap L^1$. On three-dimensional domains without boundary, for $s \geqslant 3$, since $\mathcal{U}^\alpha : \mathcal{V}_\mu^s \to \mathcal{V}^s$, we have convergence as $\alpha \to 0$ of classical solutions in H^s for short time, on intervals which are governed by the existence theory for the Euler equations. Foias *et al.* (2001) have shown that as $\alpha \to 0$, solutions of the LANS-α equations on the three-torus tend to weak solutions of the three-dimensional Navier–Stokes equations. Coutand *et al.* (2001) have shown the same to hold on bounded domains.

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