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Variational methods, multisymplectic geometry and continuum mechanics

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Abstract

This paper presents a variational and multisymplectic formulation of both compressible and incompressible models of continuum mechanics on general Riemannian manifolds. A general formalism is developed for non-relativistic first-order multisymplectic field theories with constraints, such as the incompressibility constraint. The results obtained in this paper set the stage for multisymplectic reduction and for the further development of Veselov-type multisymplectic discretizations and numerical algorithms. The latter will be the subject of a companion paper. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this paper is to give a variational multisymplectic formulation of continuum mechanics from a point of view that will facilitate the development of a corresponding discrete theory, as in the PDE Veselov formulation due to Marsden et al. [20]. This discrete

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theory and its relation to finite element methods will be the subject of a companion paper [21].

In this paper, we restrict our attention to non-relativistic theories, but on general Riemannian manifolds. The relativistic case was considered in [13], where the authors take an alternative approach of inverse fields, effectively exchanging the base and fiber spaces, see also [8–10]. [5]

Two main applications of our theory are considered — fluid dynamics and elasticity — each specified by a particular choice of the Lagrangian density. The resulting Euler–Lagrange equations can be written in a well-known form by introducing the pressure function P and the Piola–Kirchhoff stress tensor $\mathcal{P}$ (Eqs. (2.18) and (2.21), respectively).

We only consider *ideal*, that is non-viscous, fluid dynamics in this paper, both compressible and incompressible cases. In the compressible case, we work out the details for *barotropic* fluids for which the stored energy is a function of the density. These results can be trivially extended to *isentropic* (compressible) fluids, when the stored energy depends also on the entropy. Both the density and the entropy are assumed to be some given functions in material representation, so that our formalism naturally includes *inhomogeneous* ideal fluids. However, in our discussion of symmetries and corresponding conservation laws considered in Section 5, we restrict ourselves, for simplicity only, to fluids that are homogeneous in the reference configuration. We elaborate on this point below.

For the theory of elasticity we restrict our attention to *hyperelastic* materials, that is to materials whose constitutive law is derived from a stored energy function. Similarly, we assume that the material density is some given function which describes a possibly *heterogeneous* hyperelastic material.

A general formalism for treating constrained multisymplectic theories is developed in Section 3. Often, constraints that are treated in the multisymplectic context are dynamically invariant as with the constraint $\operatorname{div} \mathbf{E} = 0$ in electromagnetism (see, e.g., [9]), or $\operatorname{div} \mathbf{E} = \rho$ for electromagnetism interacting with charged matter. Our main example of a constraint in this paper is the incompressibility constraint in fluids, which, when viewed in the standard *Eulerian, or spatial* view of fluid mechanics is often considered to be a *non-local* constraint (because the pressure is determined by an elliptic equation and, correspondingly, the sound speed is infinite), so it is interesting how it is handled in the multisymplectic context, which is, by nature, a local formalism.

In the current work, we restrict our attention to first-order theories, in which both the Lagrangian and the constraints depend only on *first* derivatives of the fields. Moreover, we assume that time derivatives do not enter the constraints, which corresponds, using a chosen space–time splitting, to holonomic constraints on the corresponding infinite-dimensional configuration manifold in *material* representation. We briefly discuss the issues related to

[5] There are a number of reasons, both functional analytic and geometric for motivating a formulation in terms of *direct particle placement fields* rather than on *inverse fields*. For example, in the infinite-dimensional context, this is the setting in which one has the deeper geometric and analytical properties of the Euler equations and related field theories as in [1,7,22,26]. Moreover, the relativistic approach adopted in [13] cannot describe an incompressible fluid or elasticity because the notion of incompressibility is not defined in the relativistic context.

extending this approach to non-holonomic constraints and to space–time covariant field theories in the last section.

Symmetries and corresponding momentum maps and conservation theorems are considered in a separate section (Section 5) since they are very different for different models of a continuous media, e.g. homogeneous fluid dynamics has a huge symmetry, namely the particle relabeling symmetry, while standard elasticity (usually assumed to be inhomogeneous) has much smaller symmetry groups, such as rotations and translations in the Euclidean case. We emphasize that although the rest of the paper describes general heterogeneous continuous media, the results of Section 5.1 only apply to fluid dynamics that is homogeneous in the reference configuration (e.g., the fluid starts out, but need not remain, homogeneous), where the symmetry group is the full group of volume-preserving diffeomorphisms $\mathcal{D}_\mu$. However, these results can be generalized to inhomogeneous fluids, in which case the symmetry group is a *subgroup* $\mathcal{D}_\mu^\rho \subset \mathcal{D}_\mu$ that preserves the level sets of the material density for barotropic fluids, or a *subgroup* $\mathcal{D}_\mu^{\rho,\mathrm{ent}} \subset \mathcal{D}_\mu$ that preserves the level sets of the material density and entropy for isentropic fluids. This puts us in the realm of a multisymplectic version of the Euler–Poincaré theory — one needs to introduce additional *advected quantities* as basic fields to handle this situation (see the discussion on symmetry and reduction in Section 6). We remark also that all continuum mechanics models should satisfy material frame indifference principle, which, as is well known, can be readily accomplished by requiring the stored energy function to be a function of the Cauchy–Green tensor alone (see, e.g. [17,19]).

We finally remark on the notation. The reader is probably aware that typical fluids and elasticity literatures adhere to completely different sets of notations, which both differ substantially from those adopted in multisymplectic theories. In our notations, we follow [9]. The companion paper [21] uses primarily notation from Marsden and Hughes [19] and concentrates on models of continuum mechanics in Euclidean spaces and their variational discretizations.

2. Compressible continuum mechanics

To describe the multisymplectic framework of continuum mechanics, we must first specify the covariant configuration and phase spaces. Once we obtain a better understanding of the geometry of these manifolds we can consider the dynamics determined by a particular covariant Lagrangian.

2.1. *Configuration and phase spaces*

2.1.1. *The jet bundle*

We shall set up a formalism in which a continuous medium is described using sections of a fiber bundle Y over X; here X is the base manifold and Y consists of fibers Y_x at each point $x \in X$. Sections of the bundle $\pi_{XY} : Y \to X$ represent *configurations*, e.g. particle placement fields or deformations.

Let (B, G) be a smooth n-dimensional compact oriented Riemannian manifold with a smooth boundary and let (M, g) be a smooth N-dimensional compact oriented Riemannian manifold. For the non-relativistic case, the base manifold can be chosen to be a space–time manifold represented by the product $X = B \times \mathbb{R}$ of the manifold B together with time; $(x, t) \in X$. Let us set $x^0 = t$, so that $x^\mu = (x^i, x^0) = (x^i, t)$, with $\mu = 0, \ldots, n$, $i = 1, \ldots, n$, denote coordinates on the $(n + 1)$-dimensional manifold X. Construct a trivial bundle Y over X with M being a fiber at each point; i.e., $Y = X \times M \ni (x, t, y)$ with $y \in M$ — the fiber coordinate. This bundle,

$$\pi_{XY} : Y \to X, \qquad (x, t, y) \mapsto (x, t)$$

with π_{XY} being the projection on the first factor, is the covariant configuration manifold for our theory. Let $\mathcal{C} \equiv C^\infty(Y)$ be the set of smooth sections of Y. Then, a section ϕ of $\mathcal{C}$ represents a time dependent configuration.

Let $y^a, i = 1, \ldots, N$ denote fiber coordinates so that a section ϕ has a coordinate representation $\phi(x) = (x^\mu, \phi^a(x)) = (x^\mu, y^a)$. The first jet bundle J^1Y is the affine bundle over Y whose fiber above $y \in Y_x$ consists of those linear maps $\gamma : T_xX \to T_yY$ satisfying $T\pi_{XY} \circ \gamma = \mathrm{Id}_{T_xX}$. Coordinates on J^1Y are denoted $\gamma = (x^\mu, y^a, v^a_\mu)$. For a section ϕ, its tangent map at $x \in X$, denoted $T_x\phi$, is an element of $J^1Y_{\phi(x)}$. Thus, the map $x \mapsto T_x\phi$ is a local section of J^1Y regarded as a bundle over X. This section is denoted $j^1\phi$ and is called the first jet extension of ϕ. In coordinates, $j^1\phi$ is given by $(x^\mu, \phi^a(x), \partial_\mu\phi^a)$, where $\partial_0\phi^a = \partial_t\phi^a$ and $\partial_k\phi^a = \partial_{x^k}\phi^a$.

Notice that we have introduced *two different* Riemannian structures on the configuration bundle. The internal metric on the spatial part B of the base manifold X is denoted by G and the fiber, or field, metric on M is denoted by g. There are two main cases, which we consider in this paper:

1. fluid dynamics on a fixed background with fixed boundaries, when B and M are the same and the fiber metric g coincides with the base metric G; a special case of this is fluid dynamics on a region in Euclidean space;
2. elasticity on a fixed background, when the metric spaces (B, G) and (M, g) are essentially different.

Both approaches result in *background theories*. The case of relativistic fluid and elasticity was considered by Kijowski (see, e.g. [13]).

Define the following function on the first jet bundle:

$$J(x, t, y, v) = \det[v]\sqrt{\frac{\det[g(y)]}{\det[G(x)]}} : J^1Y \to \mathbb{R}. \tag{2.1}$$

We shall see later that its pull-back by a section ϕ has the interpretation of the Jacobian of the linear transformation $D\phi_t$.

A very important remark here is that even though in fluid dynamics metrics g and G coincide, i.e. on each fiber Y_x, g is a copy of G, there is no cancellation because the metric tensors are evaluated at different points. For instance, in (2.1) $g(y)$ does not coincide with $G(x)$ unless $y = x$ or both metrics are constant. Hence, only for fluid dynamics in Euclidean spaces, can one trivially raise and lower indices and drop all metric determinants and derivatives in the expressions in the next sections.

2.1.2. The dual jet bundle

Recall that the dual jet bundle J^1Y^* is an affine bundle over Y whose fiber at $y \in Y_x$ is the set of affine maps from J^1Y to $\Lambda^{n+1}X_x$, where $\Lambda^{n+1}X$ denotes the bundle of $(n+1)$-forms on X. A smooth section of J^1Y^* is an affine bundle map of J^1Y to $\Lambda^{n+1}X$ covering π_{XY}. Fiber coordinates on J^1Y^* are $(\Pi, p_a{}^{\mu})$, which correspond to the affine map given in coordinates by $v^a_{\mu} \mapsto (\Pi + p_a{}^{\mu} v^a_{\mu})\, \mathrm{d}^{n+1}x$.

To define canonical forms on J^1Y^*, another description of the dual bundle is convenient. Let $\Lambda = \Lambda^{n+1}Y$ denote the bundle of $(n+1)$-forms on Y, with fiber over $y \in Y$ denoted by Λ_y and with projection $\pi_{Y\Lambda} : \Lambda \to Y$. Let Z be its "vertically invariant" subbundle whose fiber is given by

$$Z_y = \{z \in \Lambda_y | v \lrcorner w \lrcorner z = 0 \ \text{ for all } \ v, w \in V_yY\},$$

where $V_yY = \{v \in T_yY | T\pi_{XY} \cdot v = 0\}$ is a vertical subbundle. Elements of Z can be written uniquely as

$$z = \Pi\, \mathrm{d}^{n+1}x + p_a{}^{\mu}\, \mathrm{d}y^a \wedge \mathrm{d}^n x_{\mu},$$

where $\mathrm{d}^n x_{\mu} = \partial_{\mu}\, \mathrm{d}^{n+1}x$, so that $(x^{\mu}, y^a, \Pi, p_a{}^{\mu})$ give coordinates on Z.

Equating the coordinates $(x^{\mu}, y^a, \Pi, p_a{}^{\mu})$ of Z and of J^1Y^* defines a vector bundle isomorphism $Z \leftrightarrow J^1Y^*$. This isomorphism can also be defined intrinsically (see [9]).

Define the canonical $(n+1)$-form Θ_{Λ} on Λ by $\Theta_{\Lambda}(z) = (\pi^*_{Y\Lambda} z)$, where $z \in \Lambda$. The canonical $(n+2)$-form is given by $\Omega_{\Lambda} = -\mathrm{d}\Theta_{\Lambda}$. If $i_{\Lambda Z} : Z \to \Lambda$ denotes the inclusion, the corresponding canonical forms on Z are given by $\Theta = i^*_{\Lambda Z}\Theta_{\Lambda}$ and $\Omega = -\mathrm{d}\Theta = i^*_{\Lambda Z}\Omega_{\Lambda}$. In coordinates they have the following representation:

$$\Theta = p_a{}^{\mu}\, \mathrm{d}y^a \wedge \mathrm{d}^n x_{\mu} + \Pi\, \mathrm{d}^{n+1}x, \qquad \Omega = \mathrm{d}y^a \wedge \mathrm{d}p_a{}^{\mu} \wedge \mathrm{d}^n x_{\mu} - \mathrm{d}\Pi \wedge \mathrm{d}^{n+1}x.$$

2.1.3. Ideal fluid

We now recall the classical material and spatial descriptions of ideal (i.e., non-viscous) fluids moving in a fixed region, i.e., with fixed boundary conditions. We set $B = M$ and call it the *reference fluid container*. A fluid flow is given by a family of diffeomorphisms $\eta_t : M \to M$ with $\eta_0 = \mathrm{Id}$, where $\eta_t(M)$ is the fluid configuration at some later time t. Let $x \in M$ denote the original position of a fluid particle, then $y \equiv \eta_t(x) \in M$ is its position at time t; x and y are called *material* and *spatial* points, respectively. The *material velocity* is defined by $V(x, t) = (\partial/\partial t)\eta_t(x)$. The same velocity viewed as a function of (y, t) is called the *spatial velocity*. It is denoted by u; i.e., $u(y, t) = V(x(y), t)$, where $x = \eta_t^{-1}(y)$, so that $u = V \circ \eta_t^{-1} = \dot{\eta} \circ \eta_t^{-1}$.

Thus, in the bundle picture above, the spatial part of the base manifold $B \subset X$ has the interpretation of the reference configuration, while an extra dimension of X corresponds to the time evolution. All later configurations of the fluid are captured by a section ϕ of the bundle

Y, which gets the interpretation of a particle placement field. Pointwise this implies that x in the base point (x, t) represents the material point, while $y \in Y_{(x,t)}$ represents the spatial point and corresponds to a position $y = \phi(x, t) = \eta_t(x)$ of the fluid particle x at time t.

2.1.4. Elasticity

For the theory of elasticity (as well as for fluids with a free boundary), the base and fiber spaces are generally different; (B, G) is traditionally called the *reference configuration*, while (M, g) denotes the ambient space. For classical two- or three-dimensional elasticity, M and B have the same dimension, while for rods and shells models the dimension of the reference configuration B is less than that of the ambient space.

For a fixed time t, sections of the bundle Y, denoted by ϕ_t, play the role of *deformations*; they map reference configuration B into spatial configuration M. Upon restriction to the space of first jets, the fiber coordinates v of $\gamma = (x, y, v) \in J^1Y$ become partial derivatives $\partial\phi^a/\partial x^\mu$; they consist of the time derivative of the deformation $\dot{\phi}^a$ and the *deformation gradient*, $F^a_i = \partial\phi^a/\partial x^i$. The first jet of a section ϕ then has the following local representation $j^1\phi = ((x, t), \phi(x, t), \dot{\phi}(x, t), F(x, t)) : X \to J^1Y$.

Using the map ϕ, one can pull-back and push-forward metrics on the base and fiber manifolds. In particular, a pull-back of the field metric g on M to $B \subset X$ defines the *Green deformation tensor* (also called the right Cauchy–Green tensor) C by $C^\flat = \phi_t^*(g)$, while a push-forward of the base metric G on $B \subset X$ to M defines the inverse of the *Finger deformation tensor b* (also called the left Cauchy–Green tensor): $c = b^{-1} = (\phi_t)_*(G)$. In coordinates,

$$C_{ij}(x, t) = g_{ab}F^a_i F^b_j(x, t), \qquad c_{ab}(y) = G_{ij}(F^{-1})^i_a(F^{-1})^j_b(y), \tag{2.2}$$

where F^{-1} is thought of as a function of y. We remark that C is defined whether or not the deformation is regular, while b and c rely on the regularity of ϕ_t. Another important remark is that operations flat $\flat$ and sharp $\sharp$ are taken with respect to the corresponding metrics on the space, so that, e.g. $(\phi_t^* g)^\sharp \neq \phi_t^*(g^\sharp)$.

Notice that J restricted to the first jets of sections is the *Jacobian* of $D\phi_t$, i.e., the determinant of the linear transformation $D\phi_t$; it is given in coordinates by

$$J(j^1\phi) = \det[F]\sqrt{\frac{\det[g]}{\det[G]}}(j^1\phi) : X \to \mathbb{R}.$$

It is a scalar function of x and t, invariant under coordinate transformations. Notice, also that $J(x, t) > 0$ for regular deformations with $\phi(x, 0) = x$, $F(x, 0) = \mathrm{Id}$ because $J(x, 0) = 1$.

2.2. Lagrangian dynamics

To obtain the Euler–Lagrange equations for a particular model of a continuous medium, one needs to specify a Lagrangian density $\mathcal{L}$. Naturally, it should contain terms corresponding to the kinetic energy and to the potential energy of the medium. Such terms depend on material properties such as mass density ρ as well as on the constitutive relation. The latter is determined by the form of the potential energy of the material. We remark that such an approach excludes from our consideration non-hyperelastic materials whose constitutive laws cannot be obtained from a potential energy function.

2.2.1. Lagrangian density

Let the mass density $\rho : B \to \mathbb{R}$ be given for a particular model of continuum mechanics. The Lagrangian density $\mathcal{L}$: $J^1Y \to \Lambda^{n+1}X$ for a multisymplectic model of continuum mechanics is a smooth bundle map

$$\mathcal{L}(\gamma) = L(\gamma)\,\mathrm{d}^{n+1}x = \mathbb{K} - \mathbb{P} = \tfrac{1}{2}\sqrt{\det[G]}\rho(x) g_{ab} v_0^a v_0^b\,\mathrm{d}^{n+1}x - \sqrt{\det[G]}\rho(x) W(x, G(x), g(y), v_j^a)\,\mathrm{d}^{n+1}x, \tag{2.3}$$

where $\gamma \in J^1Y$ and W is the *stored energy function*. The first term in (2.3) corresponds to the kinetic energy of the matter when restricted to first jet extensions as v_0^a becomes the time derivative $\partial_t \phi^a$ of the section ϕ. The second term reflects the potential energy and depends on the spatial derivatives of the fields (upon restriction to first jet extensions), i.e. on the deformation gradient F.

A choice of the stored energy function specifies a particular model of a continuous medium. While different general functional forms distinguish various broad classes of materials (elastic, fluid, etc.), the specific functional forms determine specific materials. Typically, for elasticity, W depends on the field's partial derivatives through the (Green) deformation tensor C, while for Newtonian fluid dynamics, W is only a function of the Jacobian J (2.1).

2.2.2. Legendre transformations

The Lagrangian density (2.3) determines the Legendre transformation $\mathbb{F}\mathcal{L} : J^1Y \to J^1Y^*$. The conjugate momenta are given by the following expressions:

$$p_a{}^0 = \frac{\partial L}{\partial v_0^a} = \rho g_{ab} v_0^b \sqrt{\det[G]}, \qquad p_a{}^j = \frac{\partial L}{\partial v_j^a} = -\rho \frac{\partial W}{\partial v_j^a}\sqrt{\det[G]}, \tag{2.4}$$

$$\Pi = L - \frac{\partial L}{\partial v_\mu^a} v_\mu^a = \left[-\frac{1}{2} g_{ab} v_0^a v_0^b - W + \frac{\partial W}{\partial v_j^a} v_j^a \right] \rho \sqrt{\det[G]}.$$

Define the *energy density* e by

$$e = p_a{}^0 v_0^a - L \quad \text{or equivalently} \quad e\,\mathrm{d}^{n+1}x = \mathbb{K} + \mathbb{P}, \tag{2.5}$$

then

$$\Pi = -p_a{}^j v_j^a - \sqrt{\det[G]}e.$$

2.2.3. The Cartan form

Using the Legendre transformation (2.4), we can pull-back the canonical $(n+1)$-form from the dual bundle. The resulting form on J^1Y is called the Cartan form and is given by

$$\Theta_{\mathcal{L}} = \rho g_{ab} v_0^b \sqrt{\det[G]}\, \mathrm{d}y^a \wedge \mathrm{d}^n x_0 - \rho \frac{\partial W}{\partial v_j^a} \sqrt{\det[G]}\, \mathrm{d}y^a \wedge \mathrm{d}^n x_j + \left[-\frac{1}{2} g_{ab} v_0^a v_0^b - W + \frac{\partial W}{\partial v_j^a} v_j^a \right] \rho \sqrt{\det[G]}\, \mathrm{d}^{n+1} x. \quad (2.6)$$

We set $\Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}}$.

Theorem 2.1 provides a nicer method for obtaining the Cartan form via the Calculus of Variations and remains entirely on the Lagrangian bundle J^1Y. Moreover, the variational approach is essential for the Veselov-type discretization of our multisymplectic theory. We present it here for the benefit of the reader, but remark that it is not essential for our current exposition and can be omitted on a first reading (see [20] for details).

2.2.4. *Variational approach*

To make the variational derivation of the equations of motion rigorous as well as that of the geometric objects, such as the multisymplectic form and the Noether current, we need to introduce some new notations (see [20]). These are generalizations of the notations used in the rest of the paper. They only apply to the variational derivation described here and later in Section 5.1 and do not influence the formalism and results in the rest of the paper. The reason for such generalizations is very important yet subtle: one should allow for *arbitrary* and not only *vertical* variations of the sections.

Vertical variations are confined to the vertical subbundle $VY \subset TY$, $V_yY = \{\mathcal{V} \in T_yY | T\pi_{XY} \cdot \mathcal{V} = 0\}$; this allows only for *fiber-preserving* variations, i.e., if $\phi(X) \in Y_x$ and $\tilde{\phi}$ is a new section, then $\tilde{\phi} \in Y_x$. In general, one should allow for arbitrary variations in TY, when $\tilde{\phi} \in Y_{\tilde{x}}$ for some $\tilde{x} \neq x$. Introducing a splitting of the tangent bundle into a vertical and a horizontal parts, $T_yY = V_yY \oplus H_yY$ (H_yY is not uniquely defined), one can decompose a general variation into a vertical and horizontal components, respectively.

Explicit calculations show (see [21]) that while both vertical and arbitrary variations result in the same Euler–Lagrange equations, the Cartan form obtained from the vertical variations only is missing one term (corresponding to the $\mathrm{d}^{n+1}x$ form on X); the horizontal variations account precisely for this extra term and make the Cartan form complete.

One can account for general variations either by introducing new "tilted sections", or by introducing some true new sections that compensate for the horizontal variation. The later can be implemented in the following way. Let $U \subset X$ be a smooth manifold with smooth closed boundary. Define the set of smooth maps

$$\mathcal{C} = \{\varphi : U \to Y | \pi_{XY} \circ \varphi : U \to X \ \text{ is an embedding}\}.$$

For each $\varphi \in \mathcal{C}$, set $\varphi_X = \pi_{XY} \circ \varphi$ and $U_X = \pi_{XY} \circ \varphi(U)$, so that $\varphi_X : U \to U_X$ is a diffeomorphism and $\varphi \circ \varphi_X^{-1}$ is a section of Y. The *tangent space* to the manifold $\mathcal{C}$ at a point φ is the set $T_\varphi \mathcal{C}$ defined by

$$\{\mathcal{V} \in C^\infty(X, TY) | \pi_{Y,TY} \circ \mathcal{V} = \varphi \ \text{ and } \ T\pi_{XY} \circ \mathcal{V} = \mathcal{V}_X, \ \text{ a vector field on } \ X\}.$$

Arbitrary (i.e., including both vertical and horizontal) variations of sections of Y can be induced by a family of maps φ defined through the action of some Lie group. Let $\mathcal{G}$ be a Lie group of π_{XY} bundle automorphisms η_Y covering diffeomorphisms η_X. Define the *action* of $\mathcal{G}$ on $\mathcal{C}$ by composition: $\eta_Y \cdot \varphi = \eta_Y \circ \varphi$. Hence, while $\varphi \circ \varphi_X^{-1}$ is a section of $\pi_{U_X,Y}$, $\eta_Y \cdot \varphi$ induces a section $\eta_Y \circ (\varphi \cdot \varphi_X^{-1}) \circ \eta_X^{-1}$ of $\pi_{\eta_X(U_X),Y}$.

A one parameter family of variations can be obtain in the following way. Let $\varepsilon \mapsto \eta_Y^\varepsilon$ be an arbitrary smooth path in $\mathcal{G}$ with $\eta_Y^0 = e$, and let $\mathcal{V} \in T_\varphi \mathcal{C}$ be given by

$$\mathcal{V} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \eta_Y^\varepsilon \cdot \varphi.$$

Define the *action function*

$$S(\varphi) = \int_{U_X} \mathcal{L}(j^1(\varphi \circ \varphi_X^{-1})) : \mathcal{C} \to \mathbb{R},$$

and call φ a *critical point (extremum)* of S if

$$\mathrm{d}S(\varphi) \cdot \mathcal{V} \equiv \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} S(\eta_Y^\varepsilon \cdot \varphi) = 0.$$

The Euler–Lagrange equations and the Cartan form can be obtained by analyzing this condition. We summarize the results in the following theorem from Marsden et al. [20] which illustrates the application of the variational principle to multisymplectic field theory.

Theorem 2.1. *Given a smooth Lagrangian density* $\mathcal{L} : J^1Y \to \Lambda^{n+1}(X)$, *there exist a unique smooth section* $\mathrm{D}_{\mathrm{EL}}\mathcal{L} \in C^\infty(Y'', \Lambda^{n+1}(X) \otimes T^*Y)$ (Y'' *being the space of second jets of sections*) *and a unique differential form* $\Theta_\mathcal{L} \in \Lambda^{n+1}(J^1Y)$ *such that for any* $\mathcal{V} \in T_\phi \mathcal{C}$, *and any open subset* U_X *such that* $\overline{U}_X \cap \partial X = \emptyset$,

$$\mathrm{d}S(\varphi) \cdot \mathcal{V} = \int_{U_X} \mathrm{D}_{\mathrm{EL}}\mathcal{L}(j^2(\varphi \circ \varphi_X^{-1})) \cdot \mathcal{V} + \int_{\partial U_X} j^1(\varphi \circ \varphi_X^{-1})^*[j^1(\mathcal{V}) \lrcorner \Theta_\mathcal{L}]. \tag{2.7}$$

Furthermore,

$$\mathrm{D}_{\mathrm{EL}}\mathcal{L}(j^2(\varphi \circ \varphi_X^{-1})) \cdot \mathcal{V} = j^1(\varphi \circ \varphi_X^{-1})^*[j^1(\mathcal{V}) \lrcorner \Omega_\mathcal{L}] \quad \text{in } U_X. \tag{2.8}$$

In coordinates, the action of the Euler–Lagrange derivative $\mathrm{D}_{\mathrm{EL}}\mathcal{L}$ *on* Y'' *is given by*

$$\begin{aligned}\mathrm{D}_{\mathrm{EL}}\mathcal{L}(j^2(\varphi \circ \varphi_X^{-1})) = \Bigg[& \frac{\partial L}{\partial y^a}(j^1(\varphi \circ \varphi_X^{-1})) - \frac{\partial^2 L}{\partial x^\mu \partial v^a_\mu}(j^1(\varphi \circ \varphi_X^{-1})) \\ & - \frac{\partial^2 L}{\partial y^b \partial v^a_\mu}(j^1(\varphi \circ \varphi_X^{-1})) \cdot (\varphi \circ \varphi_X^{-1})^b_{,\mu} \\ & - \frac{\partial^2 L}{\partial v^b_\nu \partial v^a_\mu}(j^1(\varphi \circ \varphi_X^{-1})) \cdot (\varphi \circ \varphi_X^{-1})^b_{,\mu\nu} \Bigg] \mathrm{d}y^a \wedge \mathrm{d}^{n+1}x,\end{aligned} \tag{2.9}$$

while the form $\Theta_{\mathcal{L}}$ matches the definition of the Cartan form obtained via Legendre transformation and has the coordinate expression

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial v^a_\mu}\, \mathrm{d}y^a \wedge \mathrm{d}^n x_\mu + \left(L - \frac{\partial L}{\partial v^a_\mu} v^a_\mu \right) \mathrm{d}^{n+1}x. \tag{2.10}$$

Corollary 2.1. *The $(n+1)$-form $\Theta_{\mathcal{L}}$ defined by the variational principle satisfies the relationship*

$$\mathcal{L}(\mathfrak{z}) = \mathfrak{z}^* \Theta_{\mathcal{L}}$$

for all holonomic sections $\mathfrak{z} \in C^\infty(\pi_{X,J^1Y})$.

Another important general theorem, which we quote from Marsden et al. [20], is the so-called *multisymplectic form formula*

Theorem 2.2. *If ϕ is a solution of the Euler–Lagrange equation* (2.9), *then*

$$\int_{\partial U_X} (j^1(\varphi \circ \varphi_X^{-1}))^* \left[j^1\mathcal{V} \lrcorner\, j^1\mathcal{W} \lrcorner\, \Omega_{\mathcal{L}} \right] = 0 \tag{2.11}$$

for any $\mathcal{V}, \mathcal{W}$ which solve the first variation equations of the Euler–Lagrange equations, i.e. any tangent vectors to the space of solutions of (2.9).

This result is the multisymplectic analog of the fact that the time t map of a mechanical system consists of canonical transformations, see [20] for the proofs.

Finally, we remark that in order to obtain vertical variations we can require φ_X (and, hence, φ_X^{-1}) to be the identity map on X. Then, $\phi = \varphi \circ \varphi_X^{-1}$ becomes a true section of the bundle Y.

2.2.5. Euler–Lagrange equations

Treating $(J^1Y, \Omega_{\mathcal{L}})$ as a multisymplectic manifold, the Euler–Lagrange equations can be derived from the following condition on a section ϕ of the bundle Y:

$$(j^1\phi)^*(\mathcal{W} \lrcorner\, \Omega_{\mathcal{L}}) = 0$$

for any vector field $\mathcal{W}$ on J^1Y (see [9] for the proof). This translates to the following familiar expression in coordinates:

$$\frac{\partial L}{\partial y^a}(j^1\phi) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^a_\mu}(j^1\phi) \right) = 0, \tag{2.12}$$

which is equivalent to Eq. (2.9).

Substituting the Lagrangian density (2.3) into Eq. (2.12), we obtain the following Euler–Lagrange equation for a continuous medium:

$$\rho g_{ab} \left(\frac{\mathrm{D}_g \dot{\phi}}{\mathrm{D}t} \right)^b - \frac{1}{\sqrt{\det[G]}} \frac{\partial}{\partial x^k} \left(\rho \frac{\partial W}{\partial v^a_k}(j^1\phi) \sqrt{\det[G]} \right) = -\rho \frac{\partial W}{\partial g_{bc}} \frac{\partial g_{bc}}{\partial y^a}(j^1\phi), \tag{2.13}$$

where

$$\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b \equiv \frac{\partial\dot{\phi}^b}{\partial t} + \gamma^a_{bc}\dot{\phi}^b\dot{\phi}^c$$

is the covariant time derivative, which corresponds to *material acceleration*, with

$$\gamma^c_{ab} = \frac{1}{2}g^{cd}\left(\frac{\partial g_{ad}}{\partial y^b} + \frac{\partial g_{bd}}{\partial y^a} - \frac{\partial g_{ab}}{\partial y^d}\right),$$

being the Christoffel symbols associated with the 'field' metric g. We remark that all terms in this equation are functions of x and t and hence have the interpretation of material quantities.

Eq. (2.13) is a PDE to be solved for a section $\phi(x, t)$ for a given type of potential energy W. As the gravity here is treated parametrically, the term on the right-hand side of (2.13) can be thought of as a derivative with respect to a parameter, and we can define a multisymplectic analog of the Cauchy stress tensor σ as follows:

$$\sigma^{ab} = \frac{2\rho}{J}\frac{\partial W}{\partial g_{ab}}(j^1\phi) : X \to \mathbb{R}, \tag{2.14}$$

where $J = \det[F]\sqrt{\det[g]/\det[G]}$ is the Jacobian. Eq. (2.14) is known in the elasticity literature as the Doyle–Ericksen formula (recall that our ρ corresponds to ρ_{Ref}, so that the Jacobian J in the denominator disappears).

Another important remark is that the balance of moment of momentum

$$\sigma^{\mathrm{T}} = \sigma$$

follows from definition (2.14) and the symmetry of the metric tensor g.

Finally, in the case of Euclidean manifolds with constant metrics g and G, Eq. (2.13) simplifies to

$$\rho\frac{\partial^2\phi_a}{\partial t^2} = \frac{\partial}{\partial x^k}\left(\rho\frac{\partial W}{\partial v^a_k}(j^1\phi)\right). \tag{2.15}$$

2.2.6. *Barotropic fluid*

For standard models of barotropic fluids, the potential energy of a fluid depends only on the Jacobian of the fluid's "deformation", so that $W = W(J(g, G, v))$. For a general inhomogeneous barotropic fluid, the material density is a given function $\rho(x)$. In material representation, this formalism also includes the case of isentropic fluids in which there is a possible dependence on entropy. Since, in that case, entropy is advected, this dependency in the material representation is subsumed by the dependency of the stored energy function on the deformation gradient. [6]

[6] In spatial representation, of course one has to introduce the entropy as an independent variable, but this naturally happens via reduction. See [11] for related results from the point of view of the Euler–Poincaré theory with advected quantities.

The Legendre transformation can be thought of as defining the pressure function P. Notice that

$$p_a{}^i = -\rho \frac{\partial W}{\partial v_i^a}\sqrt{\det[G]} = -\rho \frac{\partial W}{\partial J}\frac{\partial J}{\partial v_i^a}\sqrt{\det[G]} = -\rho \frac{\partial W}{\partial J} J (v^{-1})_a{}^i \sqrt{\det[G]},$$

and define the pressure function to be

$$P(\phi, x) = -\rho(x)\frac{\partial W}{\partial J}(j^1\phi(x)) : \mathcal{C} \times X \to \mathbb{R}. \tag{2.16}$$

Then for a given section ϕ, $P(\phi) : X \to \mathbb{R}$ has the interpretation of the *material pressure* which is a function of the material density. In this case, the Cauchy stress tensor defined by (2.14) is proportional to the metric with the coefficient being minus the pressure itself:

$$\sigma^{ab}(x) = \frac{2\rho}{J}\frac{\partial W}{\partial J}\frac{\partial J}{\partial g_{ab}}(j^1\phi) = -\frac{2P}{J} J \frac{1}{2} g^{ab}(j^1\phi) = -P(x) g^{ab}(y(x)).$$

We remark that this can be a defining equation for the pressure from which (2.16) would follow. With this notation the left-hand side of the Euler–Lagrange equations (2.13) becomes

$$\begin{aligned} &\rho g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b - \frac{1}{\sqrt{\det[G]}}\frac{\partial}{\partial x^k}\left(-PJ\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k \sqrt{\det[G]}\right) \\ &= \rho g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b + \frac{\partial P}{\partial x^k} J \left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k + \frac{P\det(\partial\phi/\partial x)}{\sqrt{\det[G]}}\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k \frac{\partial\det[g]}{\partial x^k} \\ &+(\mathrm{I}) + (\mathrm{II}) = \rho g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b + \frac{\partial P}{\partial x^k} J \left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k + \frac{P}{2} J g^{bc}\frac{\partial g_{bc}}{\partial y^a}, \end{aligned} \tag{2.17}$$

where terms (I) and (II) arise from differentiating $\det[v]$ and $(v^{-1})_a{}^k$ and cancel each other. The right-hand side of (2.13) is given by

$$-\rho\frac{\partial W}{\partial g_{bc}}\frac{\partial g_{bc}}{\partial y^a} = -\rho\frac{\partial W}{\partial J}\frac{\partial J}{\partial g_{bc}}\frac{\partial g_{bc}}{\partial y^a} = \frac{P}{2} J g^{bc}\frac{\partial g_{bc}}{\partial y^a}.$$

Notice that the last term in (2.17) and in the equation above coincide, so that the Euler–Lagrange equations for the barotropic fluid have the following form:

$$\rho g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b = -\frac{\partial P}{\partial x^k} J \left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k, \tag{2.18}$$

where the pressure depends on the section ϕ and the density ρ and is defined by (2.16). Both the metric g_{ab} and the Christoffel symbols γ_{ab}^c in the covariant derivative are evaluated at $y = \phi(x, t)$.

One can re-write (2.18) introducing the *spatial density* $\rho_{\mathrm{sp}} = \rho/J$ and defining the *spatial pressure* $p(y)$ by the relation $P(x) = p(y(x)) = p(\phi_t(x))$. This yields

$$\frac{\mathrm{D}_g V}{\mathrm{D}t}(x, t) = -\frac{1}{\rho_{\mathrm{sp}}}\,\mathrm{grad}\, p \circ \phi(x, t),$$

where $V = \dot{\phi}$. Compare this to the equations for incompressible ideal hydrodynamics in Section 4.

2.2.7. Elasticity

The Legendre transformation defines the first Piola–Kirchhoff stress tensor ${\mathcal{P}_a}^i$. It is given, up to the multiple of $-1/\sqrt{\det[G]}$ by the matrix of the partial derivatives of the Lagrangian with respect to the deformation gradient:

$$ {\mathcal{P}_a}^i(\phi, x) = \rho(x)\frac{\partial W}{\partial v_i^a}(j^1\phi(x)), \tag{2.19} $$

and for a given section ϕ, ${\mathcal{P}_a}^i$ is a stress tensor defined on X.

Notice that the first Piola–Kirchhoff stress tensor is proportional to the spatial momenta, ${\mathcal{P}_a}^i = -{p_a}^i/\sqrt{\det[G]}$. The coefficient $\sqrt{\det[G]}$ arises from the difference in the volume forms used in standard and multisymplectic elasticity. In the former, the Lagrangian density is integrated over a space area using the volume form $\mu_G = \sqrt{\det[G]}\,\mathrm{d}^n x$ associated with the metric G, while in the latter, the integration is done over the space–time using $\mathrm{d}^{n+1}x = \mathrm{d}t \wedge \mathrm{d}^n x$. We also remark that though traditionally the first Piola–Kirchhoff stress tensor is normally taken with both indices up, our choice is more natural in the sense that it arise from the Lagrange transformation (2.19) which relates ${\mathcal{P}_a}^i$ with the spatial momenta.

Using definitions (2.14) and (2.19), we can re-write Eq. (2.13) in the following form:

$$ \rho g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b = {\mathcal{P}_a}^i_{|i} + \gamma^b_{ac}({\mathcal{P}_b}^j F^c_j - J g_{bd}\sigma^{dc}), \tag{2.20} $$

where we have introduced a *covariant divergence* according to

$$ {\mathcal{P}_a}^i_{|i} = \mathrm{DIV}\,\mathcal{P} = \frac{\partial {\mathcal{P}_a}^i}{\partial x^i} + {\mathcal{P}_a}^j \Gamma^k_{jk} - {\mathcal{P}_b}^i \gamma^b_{ac} F^c_i. $$

Here Γ^i_{jk} are the Christoffel symbols corresponding to the base metric G on $B \subset X$ (see, e.g., [19] for an exposition on covariant derivatives of two-point tensors).

We emphasize that in (2.20) there is no a priori relationship between the first Piola–Kirchhoff stress tensor and the Cauchy stress tensor, i.e., W has the most general form $W(x, G, g, v)$. Such a relationship can, however, be derived from the fact that for standard models of elasticity the stored energy function W depends on the deformation gradient F (i.e. on v) and on the field metric g only via the Green deformation tensor C given by (2.2), i.e. $W = W(C(v, g))$. Thus, the partial derivatives of W with respect to g and v are related, and the following equation:

$$ {\mathcal{P}_a}^i = J{(\sigma F^{-1})_a}^i $$

follows from definitions (2.14) and (2.19). This relation immediately follows from the form of the stored energy function; it recovers the Piola transformation law, which in conventional elasticity relates the first Piola–Kirchhoff stress tensor and the Cauchy stress tensor. Substituting this relation in (2.20) one easily notices that the last term on the right-hand side cancels, so that the Euler–Lagrange equation for the standard elasticity model can be written in the following covariant form:

$$\rho \frac{\mathrm{D}_g V}{\mathrm{D}t} = \mathrm{DIV}\,\mathcal{P}, \tag{2.21}$$

where $V = \dot{\phi}$. For elasticity in a Euclidean space, this equation simplifies and takes a well-known form

$$\rho \frac{\partial V^a}{\partial t} = \frac{\partial \mathcal{P}^{ai}}{\partial x^i}.$$

3. Constrained multisymplectic field theories

Multisymplectic field theory is a formalism for the construction of Lagrangian field theories. This is to be contrasted with the formalism in which one takes the view of infinite-dimensional manifolds of fields as configuration spaces. The multisymplectic view makes explicit use of the fact that many Lagrangian field theories are local theories, that is, the Lagrangian depends only pointwise on the values of the fields and their derivatives. In formulating a constrained multisymplectic theory, we will therefore only be concerned with the imposition of pointwise constraints $\Phi(\gamma)$, $\gamma \in J^1Y$, depending on point values of the fields and their derivatives. In the current work we also restrict our attention to first-order theories, in which only first derivatives of the fields are considered.

Despite the pointwise nature of the Lagrangian $\mathcal{L}(\gamma)$, $\gamma \in J^1Y$, the variational principle assumes variations of local sections over some region $U \subset X$, i.e., it is the action $S(\phi) = \int_U \mathcal{L}(j^1\phi)$ as a function of sections that is being minimized. In order to use the theory of Lagrange multipliers to impose the constraints, it is therefore necessary to form a function $\Psi(\phi)$ of local sections which is defined through point values of the constraint $\Phi(j^1\phi)$ evaluated at the first jets of sections. It is then possible, however, to use the pointwise nature of the Lagrangian and the constraint function to derive a purely local condition, the Euler–Lagrange equations, for the constrained field variables. We will make these ideas precise in Section 3.2.

For holonomic constraints it is well known that Hamilton's principle constrained to the space of allowable configurations gives the correct equations of motion. Hamilton's principle can be naturally extended by either extremizing over the space of motions satisfying the constraints (so-called vakonomic mechanics), which is appropriate for optimal control, but not for dynamics, or by requiring stationarity of the action with respect to variations which satisfy the constraints (the Lagrange–d'Alembert or virtual work principle). The equations of motion derived in each case are, however, different.

Derivations from balance laws [12], evidence from experiments [16] and comparison to Gauss' principle of least constraint and the Gibbs–Appell equations [15] indicates that it is the Lagrange–d'Alembert principle which gives the correct equations of motion; see [3] for further discussion and references.

While the subject of linear and affine non-holonomic constraints is relatively well understood (see [4]), it is less clear how to proceed for non-linear non-holonomic constraints. Part of the problem lies in the lack of examples for which the correct equations are clear from physical grounds. In the context of constrained field theories, however, there are many cases where non-linear constraints involving spatial derivatives of the fields need to be applied, such as incompressibility in fluid mechanics, and it is clear what the physically correct equations should be. Here, we deliberately avoid the use of the term non-holonomic to avoid confusion with its standard meaning in the ODE context, where it applies only to time derivatives. Other examples of non-linearly constrained field theories include constrained director models of elastic rods and shells.

The fact that the constraints involve only spatial and not time derivatives means that imposing the constraints is equivalent to restricting the infinite-dimensional configuration manifold used to formulate the theory as a traditional Hamiltonian or Lagrangian field theory. In this case, the constraint is simply a holonomic or configuration constraint and it is known that restricting Hamilton's principle to the constraint submanifold gives the correct equations for the system.

3.1. Lagrange multipliers

The Lagrange multiplier theorem naturally makes use of the dual of the space of constraints. In a finite-dimensional setting this is a well-defined object, with all definitions being equivalent. When considering infinite-dimensional constraint spaces, however, the issue of what is being used as the dual becomes less clear and more important.

We shall consider constrained multisymplectic field theories for which the constraint space is the space of smooth sections of a particular vector bundle. In the case of the incompressibility constraint, the vector space is one-dimensional and the constraint bundle is, effectively, the space of real valued functions on the base space X. A dual of the constraint space is then defined with respect to an inner product structure on the vector bundle. This is made explicit in the following statement of the Lagrange multiplier theorem where we assume that fields and Lagrange multipliers are sufficiently regular (see [18]).

Theorem 3.1 (Lagrange multiplier theorem). *Let $\pi_{\mathcal{M},\mathcal{E}} : \mathcal{E} \to \mathcal{M}$ be an inner product bundle over a smooth manifold $\mathcal{M}$, Ψ a smooth section of $\pi_{\mathcal{M},\mathcal{E}}$, and $h : \mathcal{M} \to \mathbb{R}$ a smooth function. Setting $\mathcal{N} = \Psi^{-1}(0)$, the following are equivalent:*

1. *$\varphi \in \mathcal{N}$ is an extremum of $h|_{\mathcal{N}}$,*
2. *there exists an extremum $\bar{\varphi} \in \mathcal{E}$ of $\bar{h} : \mathcal{E} \to \mathbb{R}$ such that $\pi_{\mathcal{M},\mathcal{E}}(\bar{\varphi}) = \varphi$,*

where $\bar{h}(\bar{\varphi}) = h(\pi_{\mathcal{M},\mathcal{E}}(\bar{\varphi})) - \langle \bar{\varphi}, \Psi(\pi_{\mathcal{M},\mathcal{E}}(\bar{\varphi}))\rangle_{\mathcal{E}}$.

If $\mathcal{E}$ is a trivial bundle over $\mathcal{M}$, then in coordinates of the trivialization we have $\bar{\varphi} = (\varphi, \lambda)$, where $\lambda : \mathcal{M} \to \mathcal{E}/\mathcal{M}$ is a Lagrange multiplier function.

In the next section we shall use this theorem to relate the constrained Hamilton's principle with the extremum of the augmented action integral which contains the constraint paired with a Lagrange multiplier. Both of them result in constrained Euler–Lagrange equations. We shall furthermore demonstrate that using the trivialization coordinates, these equations can be equivalently obtained from a Lagrangian defined on an extended configuration bundle. In this picture, the Lagrange multiplier corresponds to a new field, which extends the dimension of the fiber space, and the augmented Lagrangian contains an additional part corresponding to the pairing of this field with the constraint. The Euler–Lagrange equations of motion then follow from *unconstrained* Hamilton's principle in a standard way.

3.2. Multisymplectic field theories

In the setting above, the *configuration bundle* is a fiber bundle $\pi_{X,Y} : Y \to X$ and $\pi_{Y,J^1Y} : J^1Y \to Y$ is the corresponding first jet bundle with x^μ and y^a being a local coordinate system on X and Y, respectively, and v^a_μ the fiber coordinates on J^1Y.

Choose the *configuration manifold* $\mathcal{M}$ to be the space $\mathcal{C}$ of smooth sections ϕ of $\pi_{X,Y}$. Recall that for a Lagrangian density $\mathcal{L} : J^1Y \to \Lambda^{n+1}X$, a section $\phi \in \mathcal{M}$ is said to satisfy Hamilton's principle if ϕ is an extremum of the action function $S(\phi) = \int_X \mathcal{L}(j^1\phi) : \mathcal{M} \to \mathbb{R}$. Choose the h above to be the action function S and use $\bar{S}$ instead of $\bar{h}$.

To apply the Lagrange multiplier theorem we need to define constraints as a section of some bundle $\mathcal{E} \to \mathcal{M}$ (below called the constraint bundle). As mentioned above, we restrict our attention to constraints Φ which depend only on point values of the fields and their derivatives. Using such constraints we can construct induced constraints Ψ according to (3.1). This is made precise below. We point out, however, that our treatment excludes inherently global constraints, such as those on the inverse Laplacian of the field, which cannot be derived from pointwise values.

On the other hand, we also exclude from the consideration a (simple) case when the constrained subbundle of J^1Y can be trivially realized as the first jet of some subbundle of Y.

Define an inner product vector bundle $\pi_{X,\mathcal{V}} : \mathcal{V} \to X$ with the inner product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ whose fibers are isomorphic to $\mathbb{R}^n$. Let $\mathcal{C}^\infty(\mathcal{V})$ be the inner product space of smooth sections of $\pi_{X,\mathcal{V}}$ with the inner product given by

$$\langle a, b \rangle = \int_X \langle a(x), b(x) \rangle_{\mathcal{V}} \, \mathrm{d}^{n+1}x.$$

The constraint function is an $\mathbb{R}^n$-valued function on J^1Y:

$$\Phi : J^1Y \to \mathbb{R}^n.$$

We say that a point $\gamma \in J^1Y$ satisfies the constraint if $\Phi(\gamma) = 0$. By restricting Φ to the space of first jets of sections ϕ of Y, we can define the *induced constraint function* Ψ from Φ by setting

$$\Psi(\phi)(x) = \Phi((j^1\phi)(x)) \tag{3.1}$$

for all $\phi \in \mathcal{M}$ and $x \in X$. By construction, Ψ is a map from the space $\mathcal{M}$ of sections of Y to the space $\mathcal{C}^\infty(\mathcal{V})$ of sections of $\mathcal{V}$, hence it can be thought of as a smooth section

$\Psi : \mathcal{M} \to \mathcal{E}$ of the *constraint bundle* $\mathcal{E}$. This bundle is the trivial inner product bundle given by $\mathcal{M} \times \mathcal{C}^\infty(\mathcal{V})$ over $\mathcal{M}$. Then, a configuration $\phi \in \mathcal{M}$ is said to *satisfy the constraints* if $\Phi((j^1\phi)(x)) = 0$ for all $x \in X$, i.e., the section $\Psi(\phi)$ must be a zero function on X. This implies that the space of configurations which satisfy the constraints is given by $\mathcal{N} = \Psi^{-1}(0)$.

The *constrained Hamilton's principle* now seeks a $\phi \in \mathcal{N}$ which is an extremum of $S|_{\mathcal{N}}$. The Lagrange multiplier theorem given in the previous section can be applied to conclude that this is equivalent to the existence of $\bar{\phi} \in \mathcal{E}$ with $\pi_{\mathcal{M},\mathcal{E}}(\bar{\phi}) = \phi$ which is an extremum of $\bar{S}$. Using the coordinates of the trivialization of $\mathcal{E}$ we can write $\bar{\phi} = (\phi, \lambda)$, where $\phi = \pi_{\mathcal{M},\mathcal{E}}(\bar{\phi})$ is the base point, i.e. section ϕ of Y, and λ is thought of as a section of the bundle $\pi_{X,\mathcal{V}}$, i.e. an $\mathbb{R}^n$-valued function on X. Then $\bar{S} : \mathcal{E} \to \mathbb{R}$ is given by

$$\bar{S}(\bar{\phi}) = S(\phi) - \langle \lambda, \Psi(\phi) \rangle_{\mathcal{E}} = \int_X L((j^1\phi)(x))\, \mathrm{d}^{n+1}x - \int_X \langle \lambda(x), \Phi((j^1\phi)(x)) \rangle_{\mathcal{V}}\, \mathrm{d}^{n+1}x = \int_X [L((j^1\phi)(x)) - \langle \lambda(x), \Phi((j^1\phi)(x)) \rangle_{\mathcal{V}}]\, \mathrm{d}^{n+1}x.$$

In the next section, we demonstrate these constructions for the incompressibility constraint for continuum theories.

The requirement that $\bar{S}$ be stationary with respect to variations in λ at the point $\bar{\phi}$ implies that

$$0 = \frac{\delta \bar{S}}{\delta \lambda}(\bar{\phi}) \cdot \delta\lambda = \int_X [-\langle \delta\lambda(x), \Phi((j^1\phi)(x)) \rangle_{\mathcal{V}}]\, \mathrm{d}^{n+1}x$$

for all variations $\delta\lambda$, and thus that $\Phi((j^1\phi)(x)) = 0$ for all $x \in X$. This therefore recovers the condition that ϕ must satisfy the constraints.

Stationarity of $\bar{S}$ with respect to variations in ϕ can be used to derive the *constrained Euler–Lagrange equations*, which have the form

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^a_\mu}((j^1\phi)(x))\right) - \frac{\partial L}{\partial y^a}((j^1\phi)(x)) + \left\langle \lambda(x), \frac{\partial \Phi}{\partial y^a}((j^1\phi)(x)) \right\rangle - \frac{\partial}{\partial x^\mu}\left\langle \lambda(x), \frac{\partial \Phi}{\partial v^a_\mu}((j^1\phi)(x)) \right\rangle = 0. \tag{3.2}$$

Alternatively, one can handle the constraints by introducing another bundle, denoted by E, which is a product bundle over X with fibers diffeomorphic to $Y_x \times \mathcal{V}_x$. One can think of E as a configuration bundle of the corresponding *unconstrained* system whose Lagrangian contains an additional part corresponding to the pairing of the constraint with the Lagrange multiplier:

$$L_\Phi = L + \langle \lambda, \Phi \rangle_{\mathcal{V}}.$$

The Euler–Lagrange equations of motion then follow from *unconstrained* Hamilton's principle in a standard way and coincide with (3.2). We work out the details for the incompressibility constraint in the next section.

4. Incompressible continuum mechanics

In this section, we shall consider the incompressibility constraint using the multisymplectic description of continuum mechanics. The main issue is a proper interpretation of the constraint using the Lagrange multiplier formalism developed in the previous section.

4.1. Configuration and phase spaces

Here, we briefly summarize the results. See the analogous parts of Section 2 for more details.

4.1.1. Extended covariant configuration bundle

The fibers of $\mathcal{V}$ in this case are one-dimensional and sections $\bar{\phi} = (\phi, \lambda)$ of E contain both the deformation field and the Lagrange multiplier, i.e., E denotes a bundle over X whose fibers are diffeomorphic to the product manifold $M \times \mathbb{R}$ with the projection map

$$\pi_{XE} : E \to X, \qquad (x, t, y, \lambda) \mapsto (x, t).$$

Here, λ is a section of the trivial bundle $X \times \mathbb{R}$ over X, which can be thought of as a function $\lambda(x, t)$ on X. The phase space is then the first jet bundle J^1E with coordinates $\bar{\gamma} = (x^\mu, y^a, \lambda, v^a_\mu, \beta_\mu)$; the first jet extension of a section $\bar{\phi} = (\phi, \lambda)$ has the following coordinate representation $(x^\mu, y^a, \lambda, \partial_\mu \phi^a, \partial_\mu \lambda)$.

4.1.2. The dual jet bundle

We can consider the affine dual bundle J^1E^* as a “vertically invariant” subbundle Z of the bundle $\Lambda = \Lambda^{n+1}E$ of all $(n+1)$-forms on E. Elements of Z can be written uniquely as

$$z = \Pi\, \mathrm{d}^{n+1}x + p_a{}^\mu\, \mathrm{d}y^a \wedge \mathrm{d}^n x_\mu + \pi^\mu\, \mathrm{d}\lambda \wedge \mathrm{d}^n x_\mu,$$

where $\mathrm{d}^n x_\mu = \partial_\mu \,\lrcorner\, \mathrm{d}^{n+1}x$, so that $(x^\mu, y^a, \lambda, \Pi, p_a{}^\mu, \pi^\mu)$ give coordinates on Z.

The canonical $(n+1)$-form is constructed in a standard manner and in the above coordinates has the following representation:

$$\Theta = p_a{}^\mu\, \mathrm{d}y^a \wedge \mathrm{d}^n x_\mu + \pi^\mu\, \mathrm{d}\lambda \wedge \mathrm{d}^n x_\mu + \Pi\, \mathrm{d}^{n+1}x.$$

We set $\Omega = -\mathrm{d}\Theta$.

The *primary constraint manifold* $\mathfrak{C}$ is a subbundle of the dual jet bundle and corresponds to the incompressibility constraint. The pull-back of the inclusion map $i_{\mathfrak{C}} : \mathfrak{C} \to J^1E^*$ defines a degenerate $(n+2)$-form $\Omega_{\mathfrak{C}}$ on $\mathfrak{C}$. We shall discuss the explicit form of the constraint in the next section.

4.1.3. Incompressibility constraint

Recall that such a constraint in, e.g. compressible fluid dynamics, is a reflection of the divergence-free property of the Eulerian fluid velocity and, hence, has a pointwise character.

The divergence-free character of the velocity field arises from the requirement that the particle placement map be volume-preserving at each instant of time. Then, according to the general theory of constrained multisymplectic fields outlined above, it can be obtained from a pointwise constraint Φ defined on the first jet bundle J^1Y.

For $\gamma \equiv (x^\mu, y^a, v^a_\mu) \in J^1Y$ we impose the constraint $\Phi(\gamma) = 0$ on the Jacobian of the deformation, where

$$\Phi : J^1Y \to \mathbb{R}, \qquad \gamma \mapsto J(\gamma) - 1, \quad J(\gamma) = \det[v]\sqrt{\frac{\det[g(y)]}{\det[G(x)]}}, \tag{4.1}$$

where we have used the definition of J given in (2.1). Restricting Φ to the first jet of a section ϕ results in a constraint on the matrix of spatial partial derivatives $\partial_j \phi^a$.

For the Lagrange multiplier itself, we choose the following ansatz

$$\lambda(x) = \sqrt{\det[G]}P(x) : X \to \mathbb{R}, \tag{4.2}$$

where P will be shown later to have the interpretation of the *material* pressure. Eq. (4.2) guarantees that λ transforms like a density under the transformations of the base manifold X, so that the pairing of λ and Φ, defined by integrating over X, has the correct transformation law.

4.2. Lagrangian dynamics

As we have already mentioned, the main distinguishing feature of incompressible models of continuum mechanics is the presence of the constraint (4.1). We shall now explain how this modification to the Lagrangian alters the Legendre transform as well as the Euler–Lagrange equations.

4.2.1. The Lagrangian density

The Lagrangian density $\mathcal{L} : J^1E \to \Lambda^{n+1}X$ for the multisymplectic model of incompressible continuum mechanics is a smooth bundle map defined by

$$\mathcal{L}_\Phi(\bar{\gamma}) = (L(\gamma) + \lambda \cdot \Phi(\gamma))\, \mathrm{d}^{n+1}x = \mathbb{K} - \mathbb{P} + \lambda \cdot \Phi\, \mathrm{d}^{n+1}x, \tag{4.3}$$

where L (i.e. $\mathbb{K}$ and $\mathbb{P}$) is given by (2.3) and depends on the choice of the stored energy function W.

4.2.2. The Legendre transformation

For the above choice of the Lagrangian, the Legendre transform thought of as a fiber-preserving bundle map $\mathbb{F}\mathcal{L}_\Phi : J^1E \to J^1E^*$ over E is degenerate due to the constrained character of the dynamics. Indeed, the Lagrange multiplier λ is a cyclic variable as the Lagrangian (4.3) does not depend on its derivatives, β_μ. Hence, the conjugate momentum to λ is identically zero: $\pi^\mu \equiv \partial L_\Phi/\partial\beta_\mu = 0$. The set $\{\pi^\mu = 0\}$ defines the primary constraint set as a subset of the dual bundle J^1E^* to which we restrict the Legendre transformation to make it non-degenerate. The rest of the momenta are given by the following expressions:

$$p_a{}^0 = \frac{\partial L_\Phi}{\partial v_0^a} = \rho g_{ab} v_0^b \sqrt{\det[G]}, \quad p_a{}^j = \frac{\partial L_\Phi}{\partial v_j^a} = \left(PJ(v^{-1})_a{}^j - \rho \frac{\partial W}{\partial v_j^a} \right) \sqrt{\det[G]},$$

$$\Pi = \left[-\rho \frac{1}{2} g_{ab} v_0^a v_0^b + \rho \frac{\partial W}{\partial v_j^a} v_j^a - \rho W - P(J(n-1)+1) \right] \sqrt{\det[G]}. \tag{4.4}$$

4.2.3. Euler–Lagrange equations

Using the trivialization (ϕ, λ), we now consider the Euler–Lagrange equations for a section $\bar{\phi}$ of E, both with respect to the deformation ϕ and with respect to the Lagrange multiplier λ. The former can be written in coordinates as follows:

$$\frac{\partial L_\Phi}{\partial y^a}(j^1\bar{\phi}) - \frac{\partial}{\partial x^\mu} \left(\frac{\partial L_\Phi}{\partial v_\mu^a}(j^1\bar{\phi}) \right) = 0. \tag{4.5}$$

The Euler–Lagrange equation for λ trivially recovers the constraint $\Phi = 0$ itself restricted to the first jet:

$$\frac{\partial L_\Phi}{\partial \lambda}(j^1\bar{\phi}) - \frac{\partial}{\partial x^\mu} \left(\frac{\partial L_\Phi}{\partial \beta_\mu}(j^1\bar{\phi}) \right) = \Phi(j^1\phi)\, \mathrm{d}^{n+1}x = (J(j^1\phi) - 1)\, \mathrm{d}^{n+1}x = 0. \tag{4.6}$$

These two equations are to be solved for the Lagrange multiplier λ (equivalently, for the pressure P) and for the section ϕ.

Substituting Lagrangian (4.3) into (4.5), we obtain the Euler–Lagrange equation (2.13) modified by the pressure term:

$$\rho g_{ab} \left(\frac{\mathrm{D}_g \dot{\phi}}{\mathrm{D}t} \right)^b - \frac{1}{\sqrt{\det[G]}} \frac{\partial}{\partial x^k} \left(\rho \frac{\partial W}{\partial v_k^a}(j^1\phi) \sqrt{\det[G]} \right)$$
$$= -\rho \frac{\partial W}{\partial g_{bc}} \frac{\partial g_{bc}}{\partial y^a}(j^1\phi) - \frac{\partial P}{\partial x^k}(v^{-1})_a{}^k J(j^1\phi). \tag{4.7}$$

Notice that in the case of parameterized *non-constant* metrics, the extra pressure term in (4.4) gives rise to the term

$$\frac{\partial}{\partial y^b}((PJ(v^{-1})_a{}^j \sqrt{\det[G]})(j^1\phi)) \frac{\partial y^b}{\partial x^j},$$

which follows from the chain rule applied to $\partial_{x^j} g(y(x))$. This term exactly cancels another term coming from differentiating the constraint with respect to y again due to the composition $g = g(y)$:

$$\lambda \frac{\partial \Phi}{\partial y^a} = \frac{P}{2} J g^{bc} \frac{\partial g_{bc}}{\partial y^a} \sqrt{\det[G]},$$

and other cancellations occur as in Eq. (2.17).

In the case of Euclidean manifolds with constant metrics g and G, the Euler–Lagrange equations simplify to

$$\rho\frac{\partial^2\phi_a}{\partial t^2}=\frac{\partial}{\partial x^k}\left(\rho\frac{\partial W}{\partial v_k^a}(j^1\phi)\right)-\frac{\partial P}{\partial x^k}(v^{-1})_a{}^k J(j^1\phi) \tag{4.8}$$

together with the constraint (4.6).

4.3. Incompressible ideal hydrodynamics

For fluid dynamics, the stored energy term in the Lagrangian is a constant function precisely because of the incompressibility constraint. Indeed, as we have mentioned above, W in ideal fluid models is a function of the Jacobian J, but the latter is constrained to be 1. For simplicity, consider an ideal homogeneous incompressible fluid, so that the material density ρ is constant, and we can set $\rho = 1$ (for inhomogeneous fluids the dependence of material density on the point x is implicit in the pressure function P).

The Lagrangian is given by (4.3) with $\mathbb{P} = \text{const}$. Hence, two terms in Eq. (4.7) which correspond to the derivatives of W vanish, so that the dynamics of an incompressible ideal fluid is described by

$$g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b=-\frac{\partial P}{\partial x^k}J\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k, \tag{4.9}$$

together with the constraint

$$J(j^1\phi)=\left(\frac{\sqrt{\det[G\circ\phi]}}{\sqrt{\det[G]}}\det\left(\frac{\partial\phi}{\partial x}\right)\right)(x,t)=1, \tag{4.10}$$

where we have used the fact that $g = G$.

Compare (4.9) with (2.18) and notice that the incompressibility constraint $J(j^1\phi) = 1$ implies that the *spatial density* $\rho_{\text{sp}} = \rho/J$ is constant, e.g., 1. Introducing the *spatial pressure* $p = P \circ \phi_t^{-1}$, the above equation can be written as

$$\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}(x,t)=-\text{grad}\, p\circ\phi(x,t), \tag{4.11}$$

where we have set $\rho_{\text{sp}} = 1$. We remark again that the covariant derivative is evaluated at $y = \phi(x, t)$.

4.3.1. A new look at the pressure

Here, we shall demonstrate that the same equations of motion are obtained if the potential energy in the Lagrangian (4.3) is not set to a constant, but rather is treated as a function of the Jacobian, $W = W(J)$. This will also clarify the relation between the two definitions of pressure that we have thus far examined.

Recall the definition of the pressure function for barotropic fluids given by (2.16) as a partial derivative of the stored energy function W with respect to the Jacobian J. Compare this to the definition (4.2) of the pressure as a Lagrange multiplier corresponding to the incompressibility constraint (4.1) (modulo a $\sqrt{\det[G]}$ term). In this section, we shall denote these objects by P_W and P_λ, respectively:

$$P_W = -\rho \frac{\partial W}{\partial J}, \qquad P_\lambda = \frac{1}{\sqrt{\det[G]}}\lambda.$$

The resulting Euler–Lagrange equations can be obtained by combining (2.18) with (4.9) and are given by

$$g_{ab}\left(\frac{\mathrm{D}_g \dot{\phi}^b}{\mathrm{D}t}\right)^b = -\frac{\partial (P_W + P_\lambda)}{\partial x^k} J \left(\left(\frac{\partial \phi}{\partial x}\right)^{-1}\right)_a^k,$$

together with the constraint (4.10). We can define a new pressure function

$$P = P_W + P_\lambda. \tag{4.12}$$

Notice that when the constraint $J = 1$ is enforced by the Euler–Lagrange equation (4.6), $P_W(J) = \text{const.}$, so that $P = P_\lambda + \text{const.}$ This is equivalent to a re-definition of the Lagrange multiplier λ. At the same time, the above Euler–Lagrange equation coincides with (4.9) because $\partial_k P = \partial_k P_\lambda$.

4.3.2. Relation to standard ideal hydrodynamics

Recall the Lie–Poisson description of fluid dynamics as a right invariant system on the group $\mathcal{D}_\mu(M)$ of volume-preserving diffeomorphisms of a Riemannian manifold (M, G). Here, we follow [2,23], using our notations. The Lie algebra of $\mathcal{D}_\mu(M)$ is the algebra of divergence-free vector fields on M tangential to the boundary with minus the Jacobi–Lie bracket. The L^2 inner product on this algebra is given by

$$\langle u, v\rangle_{L^2} = \int_M \langle u(x), v(x)\rangle_G \mu,$$

where μ is the Riemannian volume form on M.

We extend this metric by right invariance to the entire group. The resulting Riemannian manifold with right invariant L^2 metric, denoted by $(\mathcal{D}_\mu(M), L^2)$, is the configuration space for the Lie–Poisson or Euler–Poincaré model of ideal hydrodynamics. Its tangent bundle is the phase space, so that $(\eta_t, \dot{\eta}_t)$ are the basic "coordinates"; here $\eta_t \in \mathcal{D}_\mu(M)$ is a diffeomorphism that transforms the reference fluid configuration to its configuration at time t. Then, using the kinetic energy of fluid particles as a Lagrangian, one obtains the following covariant equations of motion:

$$\frac{\mathrm{D}\dot{\eta}}{\mathrm{D}t}(x) = -\operatorname{grad} p \circ \eta(x), \tag{4.13}$$

where

$$\frac{\mathrm{D}\dot{\eta}}{\mathrm{D}t} = \ddot{\eta} + \Gamma_\eta(\dot{\eta}, \dot{\eta}),$$

denotes covariant material time derivative with respect to the metric (Γ_η denotes the connection associated to the metric) and p is the *spatial pressure*. Notice that covariant derivative is evaluated at $\eta(x)$.

Now define $\eta_t(x) = \eta(t, x)$ to be the flow of the time dependent vector field $u(t, x)$, so that $\partial_t \eta(t, x) = u(t, \eta(t, x))$. Then composing (4.13) on the right with η^{-1} gives the classical Eulerian description of incompressible ideal fluids:

$$\partial_t u(t, x) + (u \cdot \nabla)u = -\text{grad}\, p, \qquad \text{div}\, u = 0.$$

Taking the divergence of both sides of this expression yields the equation for the pressure

$$\Delta p = -\text{div}((u \cdot \nabla)u). \tag{4.14}$$

One readily notices that Eqs. (4.11) and (4.13) coincide provided $\eta_t(x) = \phi(x, t)$. Upon this identification, the Euler–Lagrange equations for the multisymplectic model of incompressible ideal hydrodynamics recover the well-known evolution of fluid diffeomorphisms (4.13). Similarly, taking the divergence of both sides of (4.11) results in the Poisson equation on the pressure (4.14).

4.4. Incompressible elasticity

In a manner similar to the previous section, we modify the elasticity Lagrangian by the constraint and extend the phase space to include the Lagrange multiplier. Recall that the stored energy is a function of the Green deformation tensor $W = W(C)$ and use the definition of the first Piola–Kirchhoff stress tensor $\mathcal{P}_a{}^i$ (2.19) to write down the equations of motion:

$$\rho g_{ab} \left(\frac{\mathrm{D}_g \dot{\phi}}{\mathrm{D}t} \right)^b = \mathcal{P}_a{}^i{}_{|i} - \frac{\partial P}{\partial x^k} J \left(\left(\frac{\partial \phi}{\partial x} \right)^{-1} \right)^k_a,$$

together with the constraint (4.6). The above equation can be written in a fully covariant form

$$\rho \frac{\mathrm{D}_g V}{\mathrm{D}t} = \text{DIV}\, \mathcal{P} - \text{grad}\, p \circ \phi,$$

where $V = \dot{\phi}$ is the velocity vector field, $\mathcal{P}$ the first Piola–Kirchhoff stress tensor, and p the spatial pressure.

5. Symmetries, momentum maps and Noether's theorem

We already mentioned in Section 1 that homogeneous fluid dynamics has a huge symmetry, namely the particle relabeling symmetry, while standard elasticity (usually assumed to be inhomogeneous) has much smaller symmetry groups, such as rotations and translations in the Euclidean case. While inhomogeneous fluids (especially the compressible ones) are of great interest, the results worked out in Section 5.1 only apply to homogeneous fluid dynamics, when the symmetry group is the full group of volume-preserving diffeomorphisms $\mathcal{D}_\mu$. However, these results can be generalized to inhomogeneous fluids, in which case the symmetry group is a *subgroup* $\mathcal{D}_\mu^\rho \subset \mathcal{D}_\mu$ that preserves the level sets of the material density for barotropic fluids, or a *subgroup* $\mathcal{D}_\mu^{\rho,\mathrm{ent}} \subset \mathcal{D}_\mu$ that preserves the level sets of the material density and entropy for isentropic fluids.

A general model of continuum mechanics will have the metric g isometry as its symmetry. In particular, the group of rotations and translations is a symmetry for models of fluid dynamics and elasticity in Euclidean spaces. The later is treated in [21], where the overall emphasis is on continuum mechanics in Euclidean spaces.

The only symmetry which is universal for *non-relativistic* continuum mechanics is the time translation invariance. This is due to the fact that the base manifold is a tensor product of the spatial part and the time direction, rather than a space–time, so that all material quantities, such as density ρ, metric G, etc. depend only on $x \in B \subset X$. In this section, we shall treat these symmetries separately. We start with the particle relabeling symmetry, introducing the necessary notations.

5.1. Relabeling symmetry of ideal homogeneous hydrodynamics

In this section, we shall consider both the barotropic model and the incompressible model of ideal homogeneous fluids with fixed boundaries at the same time. Their corresponding Lagrangians differ only by the constraint term and both are equivariant with respect to the action of the group of volume-preserving diffeomorphisms.

5.1.1. The group action

The action of the diffeomorphism group $\mathcal{D}_\mu(B)$ on the (spatial part of the) base manifold $B \subset X$ captures precisely the meaning of particle relabeling. For any $\eta \in \mathcal{D}_\mu(B)$, denote this action by $\eta_X : (x, t) \mapsto (\eta(x), t)$. The lifts of this action to the bundles Y and E are given by $\eta_Y : (x, t, y) \mapsto (\eta(x), t, y)$ and $\eta_E : (x, t, y, \lambda) \mapsto (\eta(x), t, y, \lambda)$, respectively. Both lifts are fiber-preserving and act on the fibers themselves by the identity transformation. The coordinate expressions have the following form:

$$\eta_X^0 = \mathrm{Id} \cdot t, \qquad \eta_X^i = \eta^i(x), \qquad \eta_Y^a = \delta_b^a y^b, \qquad \eta_E^a = (\delta_b^a y^b, \mathrm{Id} \cdot \lambda). \tag{5.1}$$

5.1.2. Jet prolongations

The jet prolongations are natural lifts of automorphisms of Y to automorphisms of its first jet J^1Y and can be viewed as covariant analogs of the tangent maps (see [9]).

Let γ be an element of J^1Y and $\bar{\gamma}$ be a corresponding element of the extended phase space J^1E, in coordinates $\gamma = (x^\mu, y^a, v_\mu^a)$ and $\bar{\gamma} = (x^\mu, y^a, \lambda, v_\mu^a, \beta_\mu)$. The prolongation of η_Y is defined by

$$\eta_{J^1Y}(\gamma) = T\eta_Y \circ \gamma \circ T\eta_X^{-1}, \qquad \eta_{J^1E}(\bar{\gamma}) = T\eta_E \circ \bar{\gamma} \circ T\eta_X^{-1}. \tag{5.2}$$

We shall henceforth consider η_{J^1E}, since it includes η_{J^1Y} as a special case. In coordinates, we have

$$\eta_{J^1E}(\bar{\gamma}) = \left(\eta^k(x), t;\, y^b, \lambda;\, v_0^a, v_m^a \left(\left(\frac{\partial \eta}{\partial x} \right)^{-1} \right)_j^m ;\, \beta_0, \beta_m \left(\left(\frac{\partial \eta}{\partial x} \right)^{-1} \right)_j^m \right).$$

If ξ is a vector field on E whose flow is η_ϵ, then its prolongation $j^1\xi$ is the vector field on J^1E whose flow is $j^1(\eta_\epsilon)$, i.e. $j^1\xi \circ j^1(\eta_\epsilon) = (\mathrm{d}/\mathrm{d}\epsilon) j^1(\eta_\epsilon)$. In particular, the vector field ξ corresponding to η_E given by (5.1) has coordinates $(\xi^i, 0, 0, 0)$ and is divergence-free; its prolongation $j^1\xi$, which corresponds to the prolongation η_{J^1E} of η_E, has the following coordinate expression:

$$j^1\xi = \left(\xi^i, 0; 0, 0; 0, -v^a_m \frac{\partial \xi^m}{\partial x^j}; 0, -\beta_m \frac{\partial \xi^m}{\partial x^j}\right). \tag{5.3}$$

5.1.3. Noether's theorem

Suppose the Lie group $\mathcal{G}$ acts on $\mathcal{C}$ and leaves the action S invariant. This is equivalent to the Lagrangian density $\mathcal{L}$ being *equivariant* with respect to $\mathcal{G}$, i.e., for all $\eta \in \mathcal{G}$ and $\gamma \in J^1Y$,

$$\mathcal{L}(\eta_{J^1Y}(\gamma)) = (\eta_X^{-1})^*\mathcal{L}(\gamma),$$

where $(\eta_X^{-1})^*\mathcal{L}(\gamma) = (\eta_X)_*\mathcal{L}(\gamma)$ is a push-forward; this equality means equality of $(n+1)$-forms at $\eta(x)$. Denote the *covariant momentum map* on J^1Y by $J_\mathcal{L} \in L(\mathfrak{g}, \Lambda^n(J^1Y))$. It is defined by the following expression:

$$j^1(\xi) \lrcorner \Omega_\mathcal{L} = \mathrm{d}J_\mathcal{L}(\xi), \tag{5.4}$$

and can be thought of as a Lie algebra valued n-form on J^1Y.

Recall that ϕ is a solution of the Euler–Lagrange equations if and only if

$$(j^1\phi)^*(\mathcal{W} \lrcorner \Omega_\mathcal{L}) = 0$$

for any vector field $\mathcal{W}$ on J^1Y. In particular, setting $\mathcal{W} = j^1(\xi)$ and applying $(j^1\phi)^*$ to (5.4), we obtain the following basic Noether conservation law.

Theorem 5.1. *Assume that group $\mathcal{G}$ acts on Y by π_{XY}-bundle automorphisms and that the Lagrangian density $\mathcal{L}$ is equivariant with respect to this action for any $\gamma \in J^1Y$. Then, for each $\xi \in \mathfrak{g}$*

$$\mathbf{d}((j^1\phi)^*J_\mathcal{L}(\xi)) = 0 \tag{5.5}$$

for any section ϕ of π_{XY} satisfying the Euler–Lagrange equations. The quantity $(j^1\phi)^*J_\mathcal{L}(\xi)$ is called the Noether current.

See [9] for a proof.

5.1.4. The variational route to Noether's theorem

The variational route to the covariant Noether's theorem on J^1Y was first presented in [20, pp. 374–375]. We shall briefly describe this formulation now.

Recall the notations of the maps $\varphi : U \to Y$ and the corresponding induced local sections $\varphi \circ \varphi_X^{-1}$ of Y from Section 2.2. Here again it is important to allow for both vertical and horizontal variations of the sections. Vertical variations alone capture only fiber-preserving symmetries (i.e., spatial symmetries), while taking arbitrary variations allows for both material and spatial symmetries to be included.

The invariance of the action $S = \int_{U_X} \mathcal{L}$ under the Lie group action is formally represented by the following expression:

$$S(\eta_Y \cdot \varphi) = S(\varphi) \quad \text{for all} \quad \eta_Y \in \mathcal{G}. \tag{5.6}$$

Eq. (5.6) implies that for each $\eta_Y \in \mathcal{G}$, $\eta_Y \cdot \varphi$ is a solution of the Euler–Lagrange equations, whenever φ is a solution. We restrict the action of $\mathcal{G}$ to the space of solutions, and let $\xi_{\mathcal{C}}$ be the corresponding infinitesimal generator on $\mathcal{C}$ restricted to the space of solutions; then

$$0=(\xi_{\mathcal{C}} \lrcorner \, \mathrm{d}\mathcal{S})(\varphi) = \int_{\partial U_X} j^1(\varphi \circ \varphi_X^{-1})^*[j^1(\xi) \lrcorner \, \Theta_{\mathcal{L}}] = \int_{U_X} j^1(\varphi \circ \varphi_X^{-1})^*[j^1(\xi) \lrcorner \, \Omega_{\mathcal{L}}],$$

since the Lie derivative $\mathfrak{L}_{j^1(\xi)}\Theta_{\mathcal{L}} = 0$ by (5.6) and Corollary 2.1.

Using (5.4), we find that $\int_{U_X} \mathrm{d}[j^1(\varphi \circ \varphi_X^{-1})^* J_{\mathcal{L}}(\xi)] = 0$, and since this holds for arbitrary regions U_X, the integrand must also vanish. Recall that $\phi = \varphi \circ \varphi_X^{-1}$ is a true section of the bundle Y, so that this is precisely a restatement of Theorem 5.1.

5.1.5. Covariant canonical transformations

The computations of the momentum map from definition (5.4) can be simplified significantly in some special cases which we discuss here. A π_{XJ^1Y}-bundle map $\eta_{J^1Y} : J^1Y \to J^1Y$ covering the diffeomorphism $\eta_X : X \to X$ is called a *covariant canonical transformation* if $\eta_{J^1Y}^* \Omega_{\mathcal{L}} = \Omega_{\mathcal{L}}$. It is called a *special covariant canonical transformation* if $\eta_{J^1Y}^* \Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$. Recall that forms $\Omega_{\mathcal{L}}$ and $\Theta_{\mathcal{L}}$ can be obtained either by variational arguments or by pulling back canonical forms Ω and Θ from the dual bundle using the Legendre transformation $\mathbb{F}\mathcal{L}$.

From Gotay et al. [9], any η_{J^1Y} which is obtained by lifting some action η_Y on Y to J^1Y, is automatically a special canonical transformation. In this case the momentum mapping is given by

$$J_{\mathcal{L}}(\xi) = j^1\xi \lrcorner \, \Theta_{\mathcal{L}}. \tag{5.7}$$

We remark that the validity of this expression does not rely on the way in which the Cartan form was derived, i.e., for simplicity of the computations in concrete examples, one can forgo the issues of vertical vs. arbitrary variations in the variational derivation and obtain the Cartan form directly from the dual bundle by means of Legendre transformations. Then, evaluating this form on the prolongation of a vector of an infinitesimal generator gives the momentum n-form.

5.1.6. Equivariance of the Lagrangian

To apply Theorem 5.1 to our case we need to establish equivariance of the fluid Lagrangians.

Proposition 5.1. *The Lagrangian of an ideal homogeneous barotropic fluid* (2.3) *and the Lagrangian of an ideal homogeneous incompressible fluid* (4.3) *are equivariant with respect to the $\mathcal{D}_\mu(B)$ action* (5.1):

$$\mathcal{L}(\eta_{J^1E}(\bar{\gamma})) = (\eta_X^{-1})^* \mathcal{L}(\bar{\gamma})$$

for all $\bar{\gamma} \in J^1E$.

Proof. First observe that the material density of an ideal homogeneous (compressible or incompressible) fluid is constant. Notice also that Lagrangians (2.3) and (4.3) differ only in the potential energy terms. Both these terms are functions of the Jacobian, which is equivariant with respect to the action of volume-preserving diffeomorphisms given by (5.3). Indeed,

$$\begin{aligned} f(J(\eta_{J^1E}(\bar{\gamma})))\,\mathrm{d}^{n+1}x &= f\left(\frac{\sqrt{\det[g]}}{\sqrt{\det[G]}}\det(v)\det\left(\frac{\partial\eta}{\partial x}\right)^{-1}\right)\mathrm{d}^{n+1}x \\ &= (\eta_X^{-1})^*(f(J(\bar{\gamma}))\,\mathrm{d}^{n+1}x), \end{aligned}$$

due to the fact that $\det \partial_i \eta^j = 1$ for a volume-preserving diffeomorphism η; here f can be any function, e.g. the stored energy W or the constraint Φ.

For the same reason, and the fact that (5.3) acts trivially on v_0^a, the kinetic part of both Lagrangians is also equivariant. □

Proposition 5.1 enables us to use (5.7) for explicit computations of the momentum maps for the relabeling symmetry of homogeneous hydrodynamics. We shall consider barotropic and incompressible ideal fluids separately because their Lagrangians and, hence, their momentum mappings are different.

5.1.7. Barotropic fluid

Using (5.7), we can compute the Noether current corresponding to the relabeling symmetry of the Lagrangian (2.3) to be

$$\begin{aligned} j^1(\phi)^* J_{\mathcal{L}}(\xi) &= (\tfrac{1}{2}\rho g_{ab}\dot{\phi}^a\dot{\phi}^b - \rho W - PJ)\sqrt{\det[G]}\xi^k\,\mathrm{d}^n x_k \\ &\quad -(g_{ab}\dot{\phi}^b\phi^a_{,k})\rho\sqrt{\det[G]}\xi^k\,\mathrm{d}^n x_0, \end{aligned} \tag{5.8}$$

where $j^1\xi$ is the prolongation of the vector field ξ and is given by (5.3).

The differential of this quantity restricted to the solutions of the Euler–Lagrange equation is identically zero according to Theorem 5.1. Conversely, requiring the differential of (5.8) to be zero for arbitrary sections ϕ recovers the Euler–Lagrange equation. Indeed, computing the exterior derivative and taking into account that the vector field ξ is divergence-free, we obtain

$$g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b = -\frac{\partial P}{\partial x^k}J\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k,$$

which coincides with the Euler–Lagrange equation (2.18).

5.1.8. Incompressible ideal fluid

Similar computations using Lagrangian (4.3) with the potential energy set to a constant give the following expression for the Noether current corresponding to the relabeling symmetry:

$$\begin{aligned} j^1(\phi)^* J_{\mathcal{L}}(\xi) &= (\tfrac{1}{2}\rho g_{ab}\dot{\phi}^a\dot{\phi}^b - P)\sqrt{\det[G]}\xi^k\,\mathrm{d}^n x_k \\ &\quad -(g_{ab}\dot{\phi}^b\phi^a_{,k})\rho\sqrt{\det[G]}\xi^k\,\mathrm{d}^n x_0. \end{aligned} \tag{5.9}$$

The assumptions of Theorem 5.1 are satisfied; hence the exterior differential of this Noether current $\mathrm{d}(j^1(\phi)^* J_{\mathcal{L}}(j^1\xi))$ is equal to zero for all section ϕ which are solutions of the Euler–Lagrange equations.

Now consider the inverse statement. That is, let us analyze whether the Noether conservation law implies the Euler–Lagrange equations for incompressible ideal fluids. Computing the exterior differential of (5.9) for an arbitrary section $\bar{\phi} = (\phi, \lambda)$, we obtain

$$g_{ab}\left(\frac{\mathrm{D}_g\dot{\phi}}{\mathrm{D}t}\right)^b = -\frac{\partial P}{\partial x^k}\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)^k_a.$$

Here, we have used the fact that ξ is a divergence-free vector field on X. This is precisely the Euler–Lagrange equation (4.9) with the constraint $J = 1$ substituted in it. We point out that the above equation is not equivalent to the Euler–Lagrange equations, i.e. the constraint cannot be recovered from the Noether current. Notice, also that the Noether currents (5.8) and (5.9) are different due to the difference in the corresponding Lagrangians.

5.2. *Time translation invariance*

Lagrangian densities (2.3) and (4.3) are equivariant with respect to the group $\mathbb{R}$ action on Y, given by $\tau_Y : (x, t, y) \mapsto (x, t + \tau, y)$ for any $\tau \in \mathbb{R}$. This is because the Lagrangians are explicitly time independent. One can readily compute the jet prolongation of the corresponding infinitesimal generator vector field $\zeta_Y = (0, \zeta, 0)$, where $\tau = \exp\zeta$. Then, the pull-back by $j^1\phi$ of the covariant momentum map corresponding to this symmetry, which we denote by $J^t_{\mathcal{L}}$ to distinguish it from expressions in the previous section, is given by the following n-form on X:

$$\begin{aligned}(j^1\phi)^* J^t_{\mathcal{L}}(\zeta) &= \zeta(L(j^1\phi)\,\mathrm{d}^n x_0 - p_a{}^{\mu}(j^1\phi)\dot{\phi}^a\,\mathrm{d}^n x_\mu)\\ &= -\zeta(e(j^1\phi)\,\mathrm{d}^n x_0 + p_a{}^j(j^1\phi)\dot{\phi}^a\,\mathrm{d}^n x_j)(j^1\phi),\end{aligned}$$

where, in the last equality, we have used the definition of the energy density e given by (2.5).

Theorem 5.1 implies that the exterior derivative of this expression will be zero along solutions of the Euler–Lagrange equations. Computing this divergence for an arbitrary ζ recovers the energy continuity equation. For a barotropic fluid, it is given by

$$\dot{e} = -\sqrt{\det[G]}\,\mathrm{DIV}\left(PJ\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)^j_a\dot{\phi}^a\right),$$

while for standard elasticity the equation has the form

$$\dot{e} = \sqrt{\det[G]}\,\mathrm{DIV}(\mathcal{P}_a{}^j\dot{\phi}^a).$$

The expressions for an incompressible fluid and elastic medium are similar.

Alternatively, one can consider the inverse statement and *require* that $\mathrm{d}(J^t_{\mathcal{L}}(\zeta)) = 0$. This forces the energy continuity equation to be satisfied for some arbitrary section ϕ.

6. Concluding remarks and future directions

In this section, we would like to comment on the work in progress and point out general future directions of the multisymplectic program. Some of the aspects discussed here are analyzed in detail in our companion paper [21].

6.1. Other models of continuum mechanics

The formalism set up in this paper naturally includes other models of three-dimensional linear and non-linear elasticity and fluid dynamics, as well as rod and shell models. For elasticity, the choice of the stored energy W determines a particular model with the corresponding Euler–Lagrange equation given by (2.13); this is a PDE to be solved for the deformation field ϕ. Introducing the first Piola–Kirchhoff stress tensor $\mathcal{P}$, the same equation can be written in a compact fully covariant form (2.21). An explicit form of the Euler–Lagrange equations and conservation laws for rod and shell models are not included in this paper but can be easily derived by following the steps outlined above. The constrained director models which are common in such models are handled well by the formulation of constraints that we use in Section 3.

6.2. Constrained multisymplectic theories

The issue of holonomic vs. non-holonomic constraints in classical mechanics has a long history in the literature. Though there are still many open questions, the subject of linear and affine non-holonomic constraints is relatively well understood (see, e.g. [4]). We already mentioned in Section 3 that this topic is wide open for multisymplectic field theories, partly due to the fact that there is simply no well-defined notion of a non-holonomic constraint for such theories — it appears that one needs to distinguish between time and space partial derivatives.

As all of the examples under present consideration are non-relativistic and do not have constraints involving time derivatives, we used the restriction of Hamilton's principle to the space of allowed configurations to derive the equations of motion. Note that this reduces to vakonomic mechanics in the case of an ODE system with non-holonomic constraints, and is thus incorrect. Of course a multisymplectic approach to non-holonomic field theories (such as one elastic body rolling, while deforming, on another, such as a real automobile tire on pavement), would be of considerable interest to develop.

6.3. Multisymplectic form formula and conservation laws

A very important aspect of any multisymplectic field theory is the existence of the multisymplectic form formula (2.11) which is the covariant analog of the fact that the flow of conservative systems consists of symplectic maps. We deliberately avoid here any detailed analysis of the implications of this formula to the multisymplectic continuum mechanics and refer the reader to [21], where it is treated in the context of Euclidean spaces and discretization. Preliminary results indicate, however, that applications of the multisymplectic form formula not only can be linked to some known principles in elasticity (such as the Betti reciprocity principle), but also can produce some new interesting relations which depend on the space–time direction of the first variations $\mathcal{V}$, $\mathcal{W}$ in (2.11). An accurate and consistent discretization of the model then results in so-called *multisymplectic integrators* which preserve the discrete analogs of the multisymplectic form and the conservation laws.

6.4. Discretization

This is another very interesting and important part of our project which is addressed in detail in our companion paper [21], where the approach of finite elements for models in Euclidean spaces is adopted. It is shown that the finite element method for static elasticity is a multisymplectic integrator. Moreover, based on the result in [21], it is shown that the finite elements time-stepping with the Newmark algorithm is a multisymplectic discretization.

As we mentioned in the previous paragraph, a consistent discretization based on the variational principle would preserve the discrete multisymplectic form formula together with the discrete multimomentum maps corresponding to the symmetries of a particular system. Then, the integral form of the discrete Noether's theorem implies that a sum of the values of the discrete momentum map over some set of nodes is zero. One implication of this statement for incompressible fluid dynamics is a discrete version of the vorticity preservation. Such discrete conservations are among the hot topics of the ongoing research.

6.5. Symmetry reduction

In the previous section, we discussed at length the particle relabeling symmetry of ideal homogeneous hydrodynamics and its multisymplectic realization. Reduction by this symmetry takes us from the Lagrangian description in terms of *material* positions and velocities to the Eulerian description in terms of *spatial* velocities. In the compressible case one only reduces by the subgroup of the particle relabeling group that leaves the stored energy function invariant; e.g., if the stored energy function depends on the deformation only through the density and entropy, then this means that one introduces them as dynamic fields in the reduction process, as in Euler–Poincaré theory (see [11]).

In the unconstrained (i.e., defined on the extended jet bundle J^1E) multisymplectic description of ideal incompressible fluids, the multisymplectic reduced space is realized as a fiber bundle Υ over X whose fiber coordinates include the Eulerian velocity u and some extra field corresponding to compressibility. Then, the reduced Lagrangian density determines, by means of a constrained variational principle, the Euler–Lagrange equations which give the evolution of the spatial velocity field $u(x) \in \Upsilon_x$ together with a condition of u being divergence-free. A general Euler–Poincaré-type theorem relates this equation with Eq. (4.11) by relating the corresponding variational principles.

Such a description is a particular example of a general procedure of multisymplectic reduction. The case of a finite-dimensional vertical group action was first considered in [5]. More general cases of an infinite-dimensional group action such as that for incompressible ideal hydrodynamics, electro-magnetic fields and symmetries in complex fluids is planned for a future publication. The reader is also referred to a related work by Fernández et al. [8].

6.6. Vortex methods

One of our ultimate objectives is to further develop, using the multisymplectic approach, some methods and techniques which were derived in the infinite-dimensional framework and which proved to be very useful. One of them is the vortex blob method developed by Chorin [6], which recently has been linked to the so-called averaged Euler equations of ideal fluid (see [25]).

6.7. Higher-order theories

Constraints involving higher than first-order derivatives are beyond the current exposition and should be treated in the context of higher-order multisymplectic field theories defined on J^kY, $k > 1$.

The averaged Euler equations (see [11,22] and references therein) provide an interesting example of a higher-order fluid theory with constraints (depending only on first derivatives of the field) to which the multisymplectic methods can presumably be applied by using the techniques of Kouranbaeva and Shkoller [14]. It would be interesting to carry this out in detail. In the long run, this promises to be an important computational model, so that its formulation as a multisymplectic field theory and the multisymplectic discretization of this theory is of considerable interest.

6.8. Covariant Hamiltonian description

Finally, another very interesting aspect of the project is developing the multi-Hamiltonian description of continuum mechanics along the lines outlined in [24].

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