Fast Iterative Reconstruction of Band-limited Images from Non-uniform Sampling Values

H. G. Feichtinger, T. Strohmer *

University Vienna, Dept. of Mathematics, Strudlhofgasse 4, A-1090 Wien, AUSTRIA

Abstract. We will report on significant improvements of some of the algorithms (cf. [12]) for reconstruction of band-limited signals from non-uniform sampling sets. Although the methods apply in principle to any dimension we shall put special emphasis in this report on the application to band-limited images. Especially the combination of the ideas of the Adaptive Weights Method suggested by Feichtinger and Gröchenig with the conjugate gradient approach is better than the methods described so far in the literature, sometimes by several orders of magnitude. There is also experimental evidence that a combination of the steepest descent method with the use of a suitable relaxation parameter gives very often a highly efficient reconstruction method. In case of product sampled images successive 1D reconstruction performs best. Several examples illustrate the results.

1 Introduction

We deal with the following discrete situation. A band-limited image is considered, i.e. a rectangular matrix (of grey values) of size $n \times m$ with the property that the 2D Fourier transform $y = FFT2(x)$ has non-zero entries over a finite subset Ω of the space frequency domain. We write $I\!B^\Omega$ for the space of all band-limited images with spectrum Ω. $I\!B^\Omega$ is considered as a closed subspace of the Hilbert space $I\!H = \ell^2(I)$, where I is the abstract description of the full index set for matrix entries, with norm $\|x\|_2 := \left(\sum_{k\in I} |x_k|^2\right)^{1/2}$. Furthermore $s_x := (x_{p_j})_{j=1}^r$ is the collection of non-uniform sampling values taken at the points $(p_j)_{j=1}^r$.

The projection $\mathcal{P}$ from $I\!H$ onto $I\!B^\Omega$ is a low pass filtering, i.e. convolution with $g := IFFT2(\mathbf{1}_\Omega)$, the inverse Fourier transform of $\mathbf{1}_\Omega$, the indicator function of the spectrum Ω. Hence the convolution $x * g$ can be efficiently implemented by pointwise multiplication of $FFT2(x)$ by $\mathbf{1}_\Omega$. For any point p in the plane the the shift operator is denoted by T_p. It is a convolution by the Dirac measures δ^p, defined by $\delta^p_p := 1$ and $\delta^p_k := 0$ for $k \neq p$. It acts on images as cyclic shift of rows and columns, where the two coordinates of p determine the amount of those shifts.

In a series of joint papers by Feichtinger and Gröchenig [6, 9, 7] various iterative techniques to recover a band-limited signal sampled at a sequence of

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irregularly spread points have been presented. For the corresponding discrete reconstruction problem it has been shown in [5] and [4] that the so-called *Adaptive Weights method* appears to be most efficient compared to other methods found in the literature. Its main advantages are the computational simplicity combined with robustness.

2 The 2D Adpative Weigths Method

A description of the method of *adaptive weights* as proposed by [6, 4] makes use of the fact that the operator $\boldsymbol{S}_w$, given by

$$\boldsymbol{S}_w x := \sum_{k \in I} w_k \cdot x_{p_k} \cdot T_{p_k} g \quad , \tag{1}$$

delivers a good first approximation to x, which is then improved through iteration on the remainder terms. Here the (adaptive) weight factors w_k are chosen to be essentially equal to the size of the nearest neighborhood of the sampling point coordinate p_k. Especially for highly irregular sampling sets the introduction of adaptive weights results in a significant gain in the speed of convergence compared to the so-called ordinary Frame- or Marvasti method (cf. [16, 1, 18]) which uses a global relaxation parameter γ instead of the weigths w_k in the definition of the operator S above. In contrast to the 1D situation an effective caculation of those weights is not trivial.

We present a method which uses the inverse of the local density as a simple rule for the choice of the weights. It is of course a matter of taste how to define "local densitiy". The method suggested here is meant for sampling sets which do not have too many larger holes. It considers in a cyclic sense neighbors of order one which are direct horizontal or vertical neighbors or of order two being connected to the sampling points through the diagonal.

```
%  WGT2.M    2D version for weight functions
%  xpo  is a 0/1 matrix, indicating the sampling positions as ones

function  w2  = wgt2(xpo,par);

if nargin == 1; par = 1.5; end;   % default solution
[m n ] = size(xpo);               % dimension of image
xpc = ones(m,n) - xpo;            % indicator of complementary set
nbw = zeros(m,n);                 % building a neighborhood matrix
nbw(1,2) = 3; nbw(2,1) = 3; nbw(m,1) = 3; nbw(1,n) = 3;
nbw(m,2) = 2; nbw(2,n) = 2; nbw(2,2) = 2; nbw(m,n) = 2;
% setting weight factors for neighbors of order one and two
xpw =  real(ifft2(fft2(xpo).*fft2(nbw)));
   %  approximate local density
xpw = oneover(xpw).*xpc;
w2 =  real(ifft2(fft2(xpw).*fft2(nbw))).*xpo + xpo;
```

One important feature of the algorithm as described above is the fact that it can be easily adapted to determine "heritage" rules. The basic idea is to try to concentrate the total mass for a full sampling matrix (which is $n \cdot m$) to the sampling positions. Thinking of the matrix positions as a population, with the sampling points being the survivors we only have to set up a rule, how the unit mass of a point in the complement ("a dead point") is distributed to the surviving sampling points (at the time he "dies"). Of course a fair rule distributes this mass either to all close neighbors, proportional to their distance (this is essentially what has been done above). The best discrete version of nearest neighborhood interpolation takes the value of the nearest pixel, and if there are several surviving pixels at the same distance (or more general, in the same neighborhood class), then their average value should be taken. For a fixed order of the neighborhood the mass shifted from "dead" to surving points can be described in the same way as above, by just replacing the matrix nbw above by the indicator function of this neighborhood class. Observe further, that the resulting weight depends on the definition of (equal) neighborhoods, i.e. on the metric choosen. We found that the ℓ^1 distance between points, i.e. $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ is a very reasonable choice.

3 Acceleration of ADPW Method

In this section we will present some methods which allow to accelerate the ordinary Frame algorithm as well as the ADPW method (cf. [12]). Reconstructing a band-limited image x out of its sampling values $(x_{p_j})_{j=1}^r$ can be interpreted as finding the solution for a system of linear equations $Sx = y$, where x is the original image or its 2D Fourier coefficients and y contains information about the sampling values. In this connection we can interprete the ADPW algorithm as classical *Richardson method* in the form

$$\begin{aligned} x^0 &= 0, \\ x^{k+1} &= (I - S)x^k + y, \end{aligned}$$

where S is the operator described in section 2.

3.1 Steepest Descent Method

In our case the so-called *frame operator* S is a hermitean positve definte matrix (cf. [18]), hence we can use the so-called *steepest descent algorithm* (referred to as SD, cf. [13, 14, 15]) to speed up the algorithm described above.

For an image x of size $n \times m$ the correspoding matrix S would be of size $mn \times mn$. Assumed the computer can store such a matrix (which may be hard for $m, n \geq 256$) the multiplication of S with x^k needs $(mn)^2$ complex multiplications, which makes fast reconstruction impossible.

The crucial point is that it is possible to implement the steepest descent method as well as the conjugate gradient method without establishing the matrix S, because the multiplication of S with an image x^k can be done by

- simply sampling the image x^k at the given sampling coordinates $p_i, i = 1, \ldots l$, and for the ADPW method multiplying the value $x^k(p_i)$ with the weigth w_i, which needs only l complex multiplications.
- and then convolving the resulting image with the filter, which is best done by a pair of 2D FFT's.

In [2] the reader can find a good introduction to the SD method, here we give only the algorithm:

$$\begin{aligned} x^0 &\quad \text{is arbitrary,} \\ x^{k+1} &= x^k + \alpha_k r^k \qquad \text{for } k = 0, 1, \ldots \\ r^k &= b - Ax^k \\ \alpha_k &= \frac{\langle r^k, r^k \rangle}{\langle r^k, Ar^k \rangle}. \end{aligned}$$

3.2 SDR - Steepest Descent with Relaxation

For ill-conditioned matrices S, the rate of convergence of the SD method can be very slow. One variant to speed up the SD algorithm is to use SD with *relaxation* (called SDR method). That means that x^{k+1} is now computed by :

$$x^{k+1} = x^k + \gamma \alpha_k \; r^k.$$

A large number of experiments showed that for our class of problems the best choice for the relaxation parameter γ is $\gamma \sim 0.9$.

The improvement of the rate of convergence of SD by using SDR is not restricted to the special case of reconstructing band-limited functions, but gives also good results for the general case of solving a system of linear equations $Ax = b$, where A is a hermitean positve definite matrix.

3.3 Conjugate Gradient Method

While the *Conjugate Gradient method* (referred to as CG) is in principle well known (cf. [14, 15, 13, 2, 12]), its potential for reconstruction of band-limited images has not been yet observed and recognized.

The formulas for the CG method are given by

$$\begin{aligned} x^0 &\quad \text{is arbitrary,} \\ x^{k+1} &= x^k + \alpha_k p^k \qquad \text{for } k = 0, 1, \ldots \\ p^k &= \begin{cases} r^k & \text{for } k = 0 \\ r^k + \beta_k p^{k-1} & \text{for } k = 1, 2, \ldots \end{cases} \\ \beta_k &= \frac{\langle r^k, r^k \rangle}{\langle r^{k-1}, r^{k-1} \rangle} \qquad \text{for } k = 1, 2, \ldots \\ \alpha_k &= \frac{\langle r^k, r^k \rangle}{\langle p^k, Ap^k \rangle} \qquad \text{for } k = 0, 1, \ldots \\ r^k &= r^{k-1} - \alpha_k A p^k \qquad \text{for } k = 1, 2, \ldots \end{aligned}$$

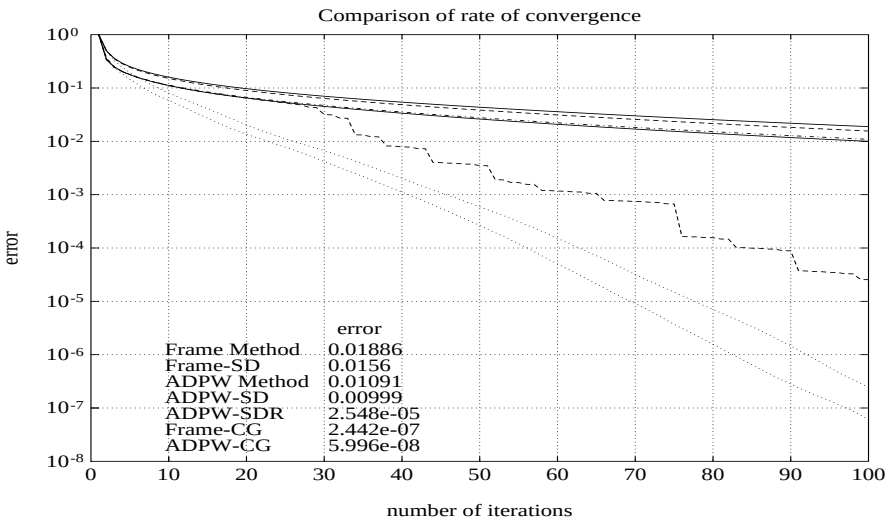


Figure 1:

Fig. 1 shows the behavior of convergence for several methods. Since we know the original image x shown in Fig. 2a, we can measure the error between x and x_k after the k-th iteration in the ℓ^2-norm. Since the sampled image shows large gaps, the Frame method and even the ADPW method would diverge, hence for both methods a relaxation parameter was necessary to provide convergence. The steepest descent variant of both methods converges without relaxation, although the rate of convergence is still very poor. The behavior of convergence for the SDR method typically looks like as shown in Fig. 1, it is much better than the standard SD variants. ADPW-CG needs 10 iterations less than the Frame-CG method to reach a precision of reconstruction in the order of 10^{-6}. Fig. 2b shows the sampled image, from which we started our reconstructions. The approximation of various methods after 25 iterations is demonstrated in Fig. 2c-f. ADPW-CG provides a very good reconstruction of the original image. All experiments were carried out with MATLAB^{TM} using our toolbox IRSATOL (cf. [8]).

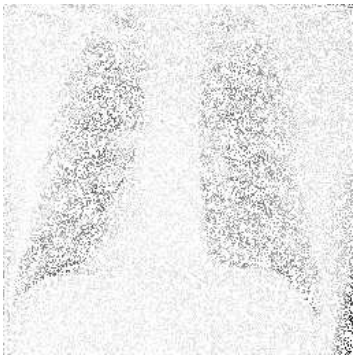
4 Product sampling

A typical application is the reconstruction of a 2D band-limited signal of size $n \times m$ with rectangular spectrum from a sampling set which arises as the product of sampling sets in each of the coordinates. Thus the sampling coordinates are of the form $(k_i, l_j)_{1 \le i \le i_0, 1 \le j \le j_0}$. The restriction of such a band-limited image to a single row or column is again band-limited since the spectrum is just the

(a) Original image of size 256 x 256

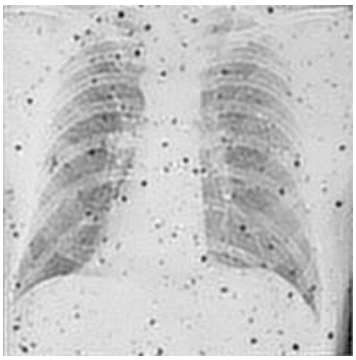
10805 spectral coefficients

(b) Sampled image



22934 sampling values

(c) Marvasti Method



error after 25 iterations = 0.07896

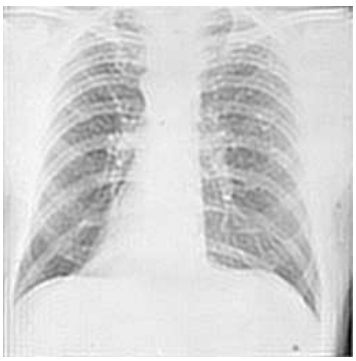
(d) ADPW Method

error after 25 iterations = 0.05366

(e) ADPW-SDR Method

error after 25 iterations = 0.04942

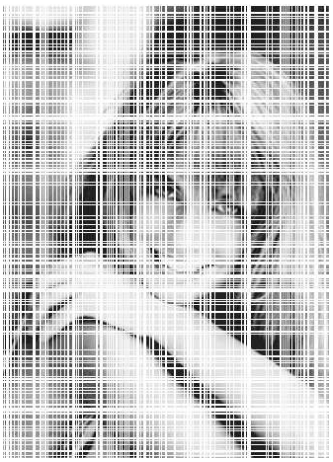
(f) ADPW-CG Method



error after 25 iterations = 0.006959

Figure 2:

(a) Sampled Image



70850 sampling points

(b) Reconstructed Image

error = 1.588e-05

Figure 3:

projected rectangle on the corresponding axis. Hence we can reconstruct the rows $(l_j)_{1 \leq j \leq j_0}$ from the corresponding sampling values using a 1D reconstruction method. Having done this we have to reconstruct m columns from their values at the coordinates $(k_i)_{1 \leq i \leq i_0}$. Obviously all those sections have the *same* spectrum and have to be reconstructed from their sampling values over a *fixed* geometry. To make advantage of this fact we apply a matrix inversion method (cf. [5]) for the successive 1D reconstruction.

Fig. 3a shows an image of size 330×460 where a number of vertical and horicontal lines have been removed. Fig. 3b shows a quite good reconstruction out of Fig. 3a.

5 Conclusion

We have demonstrated that the ADPW method and its accelerated variants especially ADPW-SDR and ADPW-CG are fast and efficient methods for reconstruction of band-limited images. The use of adaptive weights reduces the number of iterations compared to the ordinary Frame method. For product sampled images we have presented a new fast method, which makes use of successive 1D reconstruction.

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