Inexpensive Gabor Decompositions

Sigang Qiu, Hans G. Feichtinger and Thomas Strohmer
Department of Mathematics, University of Vienna
Strudlhofgasse 4, A-1090, Wien, Austria

ABSTRACT

This is a paper on the discrete Gabor transform. We discuss the calculation of dual and tight Gabor atoms, for Gabor atoms which satisfy certain support restrictions related to the relevant time- and frequency lattice constants. These conditions imply that the Gabor frame operator is just a pointwise multiplication operator and therefore computing the inverse or the square root of the inverse frame operator are computationally inexpensive.

1 Introduction

The theory of Gabor expansions of signals and images has obtained refreshed interest in the last few years. Despite the very natural interpretation of Gabor coefficients as a time-frequency distribution of the signal their practical use has been hindered mostly by the computational complexity required to obtain the Gabor coefficients for a given signal.

This presentation is concerned with the theory of discrete Gabor transforms, finite and infinite. In contrast to the general situation, where the determination of the coefficients for the non-orthogonal expansion is as a computationally intensive task, there are some situations where the determination of dual Gabor wavelets, and therefore the calculation of the Gabor coefficient is rather "cheap" to obtain. One is the case of critical sampling (with $ab = n$), which however leads to rather instable expansion. Much more interesting for practical situations is the case of modestly oversampled Gabor families. As we will show, if the (cyclic) support of the Gabor atom is small, then the dual Gabor and tight Gabor wavelets can be computed by simple pointwise multiplication.

The by now standard approach to Gabor transforms goes back to the method introduced for a special case by Bastiaans ([1,2]) and consists of two steps. In a first (and usually more expensive) step one determines the so-called "dual Gabor" wavelet depending only on the given Gabor atom g, and lattice constants (a, b). In the second step the Gabor coefficients are obtained by taking the short time Fourier Transform (STFT) of the signal x with respect to the dual Gabor atom as a window over a time-frequency lattice with lattice constants (a, b). Sometimes, e.g. for the design of TF-localization operators it is also desirable to find "tight Gabor frames", i.e., to exhibit for a given pair (a, b), Gabor atoms which coincide with its dual atoms, in other words, for which the Gabor family acts like an (pseudo-) orthonormal system in terms of expanding general N-periodic signals (cf. the well-known paper[6] for the role of tight Gabor frames).

Efficient methods to obtain the dual atom are only known for a few special cases, e.g. by means of the Zak transform for the case $ab = N$ which is considered as the "critical" case. However, it is known that in this case rather bad (i.e. highly singular and badly concentrated) dual atoms arise even for nice Gabor atoms. This corresponds to the Balian-Low phenomenon in the continuous setting.

Whereas the Zak transform is a tool which in few cases allows to efficiently determine the dual Gabor window (without specific restrictions on the atom), we describe situations, where combined conditions on TF-lattice and the atom (support conditions) ensure "cheap" determination of the dual atom.

For the discrete setting we follow mostly[4,14]. Signals $x = \{x(k)\}_{k=0}^{N-1}$ is considered as N-dimensional row vector in the complex space $\mathbb{C}^N$. For convenience we also write $x_k = x(k)$. The inner product of two signals is given as $\langle x, y\rangle = \sum_{j=0}^{N-1} x_j \overline{y}_k$. Here and throughout the paper we extend finite sequences in $\mathbb{C}^N$ to infinite sequences of period N by setting $x(n + kN) = x(N)$ for $n = 0, 1, \ldots, N-1, k \in \mathbb{Z}$.

The time translation operator T_k corresponds to the cyclic rotation:

$$T_k(x) := (x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0, \ldots, x_{N-k-1}) \ .$$

The modulation operator M_l – i.e. the frequency translation operator – is given by

$$M_l(x) := (x_0, x_1 w, x_2 w^2, \ldots, x_{N-1} w^{N-1})$$

where

$$w = e^{-2\pi i l/N} \qquad \text{for} \quad l \in \mathbb{Z}.$$

We will make use of following well-known properties of the Fourier transform $\mathcal{F}$.

$$\mathcal{F}(T_k x) = M_{-k}(\mathcal{F}x) \quad \text{and} \quad \mathcal{F}(M_k x) = T_k(\mathcal{F}x) \ .$$

For any sequence x we denote by $supp(x)$ the set of "non-zero" coordinates, i.e. $supp(x) = \{k, 0 \le k \le N, |x(k)| > 0\}$. We denote by p the length of the support of a signal x, i.e. the minimal length of a cyclic interval I containing $supp(x)$. We will call such a (cyclic) interval a *supporting interval* for the signal x.

For a signal $g \in \mathbb{C}^N$ we call (g, a, b) a *Gabor triple* if a and b are divisors of N. The constants a and b are usually called *the time* and *frequency gap* and form a pair of *lattice constants*. For convenience we write $\tilde{a} = N/a$ and $\tilde{b} = N/b$.

A *Gabor expansion* with respect to the Gabor triple (g, a, b) is a series expansion of the form

$$x = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} c_{nm} g_{nm} \tag{1}$$

$$\text{with} \quad g_{nm} := M_{mb} T_{na} g \ . \tag{2}$$

We call g the *Gabor atom*, *Gabor window* or *Gabor wavelet*. The family $\{g_{nm}\}$ is known as the *Weyl-Heisenberg wavelet system*[16] (WH-system for short)and any family of complex numbers $\{c_{nm}\}$ giving rise to such a series expansion is referred to as *Gabor coefficients*. Most often we shall use the "standard" method, based on the concept of dual frames.

The *Gabor operator* S associated to (g, a, b) is defined as

$$Sx = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, g_{nm}\rangle g_{nm} \ . \tag{3}$$

We shall deal with Gabor expansions in the following two cases. 1.) *Critical Sampling*: $ab = N$, where the number of elements of the WH-system is exactly N. If arbitrary signals can be represented by the elements g_{nm} as in Formula 1, the coefficients c_{nm} are necessarily uniquely determined.

2.) We are mainly interested in *Oversampling*: $ab < N$, the number of representing functions is larger than the length of the signal, i.e. the WH-system is necessarily linear dependent, and Gabor coefficients are not determined uniquely.

If the functions $\{g_{nm}\}$ constitute a *frame*[7] (with frame bounds A, B), g is called a *Gabor frame atom*. In this case S is clearly the *frame operator*. Observe that $\{g_{nm}\}$ is a frame[5,11] if and only if S is invertible.

Defining the *dual Gabor atom* (or dual Gabor wavelet) $\tilde{g}$ by $\tilde{g} := S^{-1}g$ and using the property[5] that

$$\tilde{g}_{nm} := S^{-1}(g_{nm}) \ ,$$

any signal $x \in \mathbb{C}^N$ can be represented as

$$x = S(S^{-1}x) = \sum_{n=0}^{\bar{a}-1} \sum_{m=0}^{\bar{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm} \tag{4}$$

or

$$x = S^{-1}(Sx) = \sum_{n=0}^{\bar{a}-1} \sum_{m=0}^{\bar{b}-1} \langle x, g_{nm} \rangle \tilde{g}_{mn} \ . \tag{5}$$

The Gabor coefficients arising in this way are known to have minimal l^2-norm. In[14] one can find an efficient method to calculate the dual Gabor atom.

If $A = B$ a frame is called *tight frame*. One of the advantages of tight frames is that the frame operator equals $S = AI$ – where I is the identity operator – and thus for the Gabor scheme x can be easily reconstructed from the coefficients $\langle x, g_{nm} \rangle$ by the formula $x = A^{-1} \sum_{n,m} \langle x, g_{nm} \rangle g_{nm}$. One way to obtain a tight frame for a given Gabor triple (g, a, b) is to calculate

$$g_t := S^{-\frac{1}{2}} g \ . \tag{6}$$

Using this g_t as new Gabor atom, the corresponding representation functions $\{(g_t)_{nm}\}$ as defined by (2) constitute a tight frame.[5] In general the computation of $S^{-\frac{1}{2}}$ needs much computational effort, therefore we address the problem of finding simple sufficient conditions under which this task is computational inexpensive.

For an arbitrary Gabor triple (g, a, b), we define $GAB_{(g,a,b)}$ as the following $M \times N$ matrix ($M = \tilde{a}\tilde{b}$), containing the time-frequency shifted version of the atom g as row vectors:

$$GAB_{(g,a,b)} = \begin{pmatrix} g \\ T_a g \\ \vdots \\ T_{(\tilde{a}-1)a} g \\ M_b g \\ M_b T_a g \\ \vdots \\ M_b T_{(\tilde{a}-1)a} g \\ \vdots \\ M_{mb} T_{na} g \\ \vdots \\ M_{(\tilde{b}-1)b} T_{(\tilde{a}-1)a} g \end{pmatrix} \ . \tag{7}$$

Following the MATLAB-related notations, we prefer to identify linear mappings (operators) on the signal space by matrix multiplication from the right: $x \mapsto x * A$. The adjoint operator is thus represented by matrix multiplication from the right by the transposed and conjugated matrix A'.

The action of the Gabor operator S on a signal of length N (presented as row vector) can be expressed in terms of matrix algebra as

$$Sx = x * \mathbf{S}, \qquad \text{where} \quad \mathbf{S} = GAB'_{(g,a,b)} * GAB_{(g,a,b)}$$

We call $\mathbf{S}$ the associated *Gabor matrix*. In the following, we shall regard the Gabor frame operator and the associated Gabor frame matrix as the same, and state the mathematical results for the Gabor frame matrix.

We have to mention that only after the preparation of the first draft of this paper it came to our attention that some of the ideas have been described already in the literature for the continuous case in papers by J. Benedetto and coauthors.[3] Also G. Kaiser (cf. his book[9] for details) was using a similar assumption in his 1978 proposal on "local Fourier analysis" to the NSF (private communication to H.G. Feichtinger).

2 The Mathematical Results

THEOREM 2.1. *Given an arbitrary Gabor triple (g, a, b), suppose that p – the length of the support of g – satisfies the condition $p \leq \tilde{b}$, then the Gabor operator S is simply a multiplication operator with respect to some a-periodic and non-negative multiplier d. Conversely, if g is non-negative, and has strictly positive coordinates within a cyclic interval I of length $p > n/b$ then the Gabor operator cannot be just be a multiplication operator, in other words, the corresponding matrix contains at least some non-zero entries outside the main diagonal.*

Proof. We derive the result utilizing the general structure of Gabor matrices.[14] Let $\mathbf{S}$ be the Gabor matrix associated to (g, a, b), then the general entry $\mathbf{S}_{k,j}$ $(j, k = 0, 1, \ldots, N-1)$ of $\mathbf{S}$ can be calculated via following formula[14]

$$\mathbf{S}_{k,j} = \begin{cases} \tilde{b} \sum_{n=0}^{\tilde{a}-1} T_{na}\overline{g(k)} T_{na} g(j) & \text{if } |k-j| \text{ is divided by } \tilde{b} \\ 0 & \text{otherwise} \quad . \end{cases} \tag{8}$$

Using the support condition that $p \leq \tilde{b}$, we assume without loss of generality that the support of g is located at the first consecutive integers $\{0, 1, \ldots, p-1\}$, i.e., $g(s) = 0$ for $p \leq s \leq N-1$. Thus

$$\sum_{n=0}^{\tilde{a}-1} T_{na}\overline{g(k)}\ T_{na} g(j) = 0, \qquad \text{for } |j-k| \geq \tilde{b} \geq p-1\,,$$

and consequently

$$\mathbf{S}_{kj} = 0 \quad \text{for } |j-k| \geq \tilde{b}\,. \tag{9}$$

Combining (8) and (9), we see that

$$\mathbf{S}_{kj} = 0, \quad \text{for } k, j = 0, 1, \ldots, N-1 \quad \text{and} \quad k \neq j\,.$$

Therefore nonzero elements of $\mathbf{S}$ can only found in the main diagonal. Setting $j = k$ in (8), we have

$$d_k := \mathbf{S}_{k,k} = \tilde{b} \cdot \sum_{n=0}^{\tilde{a}-1} |T_{na} g(k)|^2 \text{ for } k = 0, 1, \ldots, N-1\,. \tag{10}$$

Hence, $\mathbf{S}$ is non-negative and satisfies $\mathbf{S}_{k,k} = \mathbf{S}_{k+a,k+a}$ for $k = 0, 1, \ldots, N-1$.

In order to verify the last part of the statement just go back to the beginning of this proof, in order to see that at least for some pair (k, j) with $k \neq j$ some entry $\mathbf{S}_{k,j}$ will be strictly positive. □

REMARK: The second part of the above theorem is of course not true without the non-negativity assumption concerning the atom g. Indeed, there are tight frames using atoms which do not satisfy the support condition $p < n/b$, but as a consequence of the above statement we may conclude that in this case either the frame operator is not just a multiplier or that g has also negative values.

The following two corollaries provide conditions for the sequence $\{g_{nm}\}$ to constitute a frame. The conditions given in Corollary 2.2 and Corollary 2.3 are sharp.

COROLLARY 2.2. *Under the assumptions in Theorem 2.1, the sequence $\{g_{nm}\}$ is a frame, if the number of consecutive nonzero coordinates of g is at least a. Moreover, g is real (positive, non-negative) if and only if $\tilde{g}$ or g_t respectively is real (positive, non-negative).*

Proof. The general entry $\mathbf{S}_{k,j}$ of the Gabor matrix $\mathbf{S}$ is given by (10), with the condition that there are at least a consecutive nonzero entries of g. For any $k = 0, 1, \ldots, N-1$, there exists a number n_0 with $0 \leq n_0 \leq \tilde{a}-1$ such that $T_{n_0 a}\, g(k) \neq 0$, which yields that $\mathbf{S}_{k,k} > 0 \ \forall k$. Hence $\mathbf{S}$ is invertible, which implies that the sequence $\{g_{nm}\}$ is a frame.[5]

The last conclusion in the Corollary is obvious. □

COROLLARY 2.3. *If $p < a$, the sequence $\{g_{nm}\}$ cannot be a frame.*

Proof. Suppose that there are only $p < a$ nonzero coordinates $g(r_1)$, $g(r_2)$, ..., $g(r_p)$ of g, then there exists a number s such that $s \neq r_k \pmod a$, for $k = 1, 2, \ldots, p$, which implies that $g(s + na) = 0$ for $n = 0, 1, \ldots, \tilde{a} - 1$. Therefore, the s-th entry in the main diagonal of $\mathbf{S}$ is $\sum_{n=0}^{\tilde{a}-1} |g(s+na)|^2 = 0$. This implies that $\mathbf{S}$ is not invertible, since $\mathbf{S}$ is a diagonal matrix. Hence $\{g_{nm}\}$ cannot be a frame. □

Summing up we have obtained that $\{g_{nm}\}$ constitutes a frame if and only if there exist $0 \leq r_1, r_2, \ldots, r_a \leq N-1$ satisfying

$$|r_k - r_j| \neq 0 \mod a \quad \text{for } k, j = 0, 1, \ldots, a$$

such that the coordinates $g(r_s) \neq 0$ for $s = 1, 2, \ldots, a$.

Using Corollary 2.2 and Corollary 2.3, we can easily draw the following conclusion for the critically sampled case.

COROLLARY 2.4. *Assume that $p < n/b$ and $ab = N$. Then $\{g_{nm}\}$ is a frame if and only if*

$$p = a \qquad \text{and} \qquad g(k) \neq 0 \quad \text{for } k \in I \supseteq \mathrm{supp}(g).$$

COROLLARY 2.5. *If $\{g_{nm}\}$ constitutes a frame, then $\tilde{g}$ and g_t have the same support as g. Set $d_k = \mathbf{S}_{k,k}$, then $\tilde{g}$ and g_t can be calculated by*

$$\tilde{g}(k) = d_k^{-1} g(k) \qquad \text{and} \qquad g_t = d_k^{-\frac{1}{2}} g(k) \quad \text{for } k = 0, \ldots, N-1 .$$

COROLLARY 2.6. *Assume we are given two Gabor atoms g, g' with the same support (i.e., $\mathrm{supp}(g) = \mathrm{supp}(g') := I$), but different signal lengths N and N' with $N' > N$. Let p denote the length of I. If $g(k) = g'(k)$ for $k \in I$ and $p \leq N/b$, then $\{g_{nm}\}$ is a frame if and only if $\{g'_{nm}\}$ is a frame. Moreover the following relation holds for the associated dual Gabor atoms $\tilde{g}$ and $\tilde{g}'$*

$$\tilde{g}' = \frac{N}{N'} (\tilde{g}, \underbrace{0, \ldots, 0}_{N'-N}).$$

COROLLARY 2.7. *Assume that the sequence $\{g_{nm}\}$ is a frame for a given Gabor atom g of length N and lattice constants a, b. Let g' a Gabor atom of length $N' > N$ with*

$$g' = (g, \underbrace{0, \ldots, 0}_{N'-N}) .$$

Suppose that the length p of the support of g' satisfies $p \leq N'/b$, then $\{g'_{nm}\}$ constitutes a frame and the associated dual Gabor atom $\tilde{g}'$ can be expressed as

$$\tilde{g}' = \frac{N}{N'}(g, \underbrace{0, \ldots, 0}_{N'-N}) .$$

Corollary 2.6 can be proved by using equation (10). Corollary 2.7 can be deduced from the characterization of the Gabor matrix structure and the relevant consequences stated in.[14]

REMARK: Suppose that g_t is the tight Gabor atom associated to (g, a, b) in the sense of equation 6. Then the Gabor atom g'_t associated with (g', a, b) can be obtained in a similar way. More precisely,

$$g'_t = \sqrt{\frac{N_1}{N_2}}\ (g_t, \underbrace{0, \ldots, 0}_{N_2-N_1}) .$$

COROLLARY 2.8. *Under the assumptions of Theorem 2.1, let N be the signal length of g, $\tilde{g}$ and g_t are the associated dual and tight Gabor atoms, respectively. We assume that for some positive integer q divided by a and $\tilde{b}$ assume that $b_0 = \frac{N+q}{\tilde{b}}$ and g' fulfills $g' = (g, \underbrace{0, \ldots, 0}_{q})$. Then the family $\{g'_{nm}\}$ is a frame. Further $\tilde{g}'$ and g'_t are given by*

$$\tilde{g}' = (\tilde{g}, \underbrace{0, \ldots, 0}_{q}) \quad \textit{and} \quad g'_t = (g_t, \underbrace{0, \ldots, 0}_{q}) .$$

REMARK: It follows from the formula in Corollary 2.8, that the nonzero entries of $\tilde{g}'$ are independent from the signal length. This also leads to a discrete version of the Gabor expansion for the infinite case. We give the relevant algorithms in the next section.

3 Numerical Algorithms and Experimental Results

In this section we present the algorithm for computing $\mathbf{S}, \tilde{g}$ and g_t for a given Gabor atom g and lattice constants a and b, assuming that g satisfies the support condition of 2.1 and that the family $\{g_{nm}\}$ is a frame.

ALGORITHM:

1. *Frame operator S:* Calculate the first a elements of the main diagonal of $\mathbf{S}$ by using Formula (2):

$$d_k := \mathbf{S}_{k,k} = \tilde{b} \sum_{n=0}^{\tilde{a}-1} |T_{na} g(k)|^2 \quad \text{for} \quad k = 0, 1, \ldots, a-1 .$$

   By Theorem 2.1 the other non-zero entries of $\mathbf{S}$ are given by $\mathbf{S}_{k,k} = \mathbf{S}_{k+a,k+a}$ for $k = 0, 1, \ldots, N-1$.

2. *Inverse frame operator S^{-1}:* Compute the first a entries of the main diagonal of $\mathbf{S}^{-1}$ by

$$\mathbf{S}^{-1}_{k,k} = d_k^{-1} \quad \text{for } k = 0, 1, \ldots, a-1 .$$

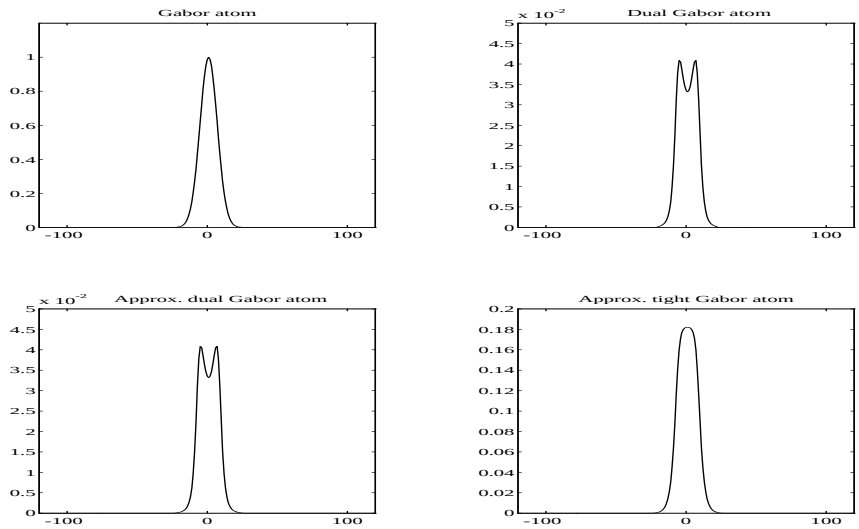

Figure 1: Gabor atom g, the dual Gabor atom $\tilde{g}$, the approximated dual and tight Gabor atom

Similar to item 2, we can get all the other entries of $\mathbf{S}^{-1}$.

3. *Dual Gabor atom* $\tilde{g}$: For any $j = ra + k$, where $r = 0, 1, \ldots, \tilde{a} - 1$ and $k = 0, 1, \ldots, a - 1$

$$\tilde{g}(j) = d_k^{-1} \cdot g(j) \ .$$

If the support condition $p < n/b$ is satisfied, only p such multiplications have to be carried out, because the other coordinates are all zero.

4. *Tight Gabor atom* g_t: For any $j = ra + k$, where $r = 0, 1, \ldots, \tilde{a} - 1$ and $k = 0, 1, \ldots, a - 1$

$$g_t(j) = d_k^{-\frac{1}{2}} \cdot g(j) \ .$$

Again, the support condition implies that only p multiplications have to be carried out.

5. *Discrete Gabor Expansion:* With the calculated $\tilde{g}$ and g_t the signal $x \in \mathbb{C}^N$ can be easily reconstructed from the coefficients $\langle x, \tilde{g}_{nm} \rangle$ or $\langle x, (g_t)_{nm} \rangle$ respectively by

$$x(k) = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm}(k) \tag{11}$$

and

$$x(k) = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} \langle x, (g_t)_{nm} \rangle (g_t)_{nm}(k) \ . \tag{12}$$

The expansion formula (12) can be seen as the pseudo-orthogonal Gabor expansion with the tight Gabor atom g_t.

6. *Infinite Discrete Gabor expansion:* By Corollary 2.8 and the cited formulas in,[13] any infinite signal $x \in l^2(\mathbb{Z})$ has the Gabor expansion

$$x(k) = \sum_{n=-\bar{a}}^{\infty} \sum_{m=0}^{\bar{b}-1} \langle x, \tilde{g}_{nm} \rangle g_{nm}(k)$$

and the pseudo-orthogonal expansion utilizing the tight Gabor atom g_t

$$x(k) = \sum_{n=-\bar{a}}^{\infty} \sum_{m=0}^{\bar{b}-1} \langle x, (g_t)_{nm} \rangle (g_t)_{mn}(k)$$

where g_{nm} is defined as

$$g_{nm}(k) = g(k - na)w^k \qquad \text{for } k \in \mathbb{Z} \text{ and } w = e^{-2\pi i/\bar{b}} .$$

REMARKS: If the support condition of Theorem 2.1 is not satisfied in the strict sense, but only approximately, one can still use the above algorithm to calculate an approximate dual/tight Gabor atom for Gabor expansion and synthesis. In such a case the Gabor frame matrix is strongly diagonal dominant, hence we can use the above algorithm to obtain a good approximate Gabor transform.

Figure 1 shows the Gabor atom of signal length $N = 240$, the associated dual Gabor atom which is calculated with the efficient algorithm obtained in[14] and the approximate dual Gabor atom which is computed with the algorithm presented here. It took about 26,974 FLOPs computing the precise dual Gabor atom $\tilde{g}$ and it took only 865 FLOPs calculating the approximated dual Gabor atom $\tilde{g}_a$ with the above algorithm. The error $\|\tilde{g} - \tilde{g}_a\|/\|\tilde{g}\|$ is around 0.0039. We can reconstruct a signal x from its coefficients $\langle x, g_{nm} \rangle$ according to Formula 11 using $\tilde{g}$ up to an error which is close to the machine precision. Using $\tilde{g}_a$ instead of $\tilde{g}$ for an approximation of x results in a relative error smaller than 0.4%, which is still acceptable for practical purposes.

4 Conclusion

We have described an "inexpensive" Gabor transforms for the finite discrete setting, assuming that the Gabor atom fulfills appropriate support conditions. The numerical implementations have the following nice properties:

1. Fast and efficient: Since the relevant algorithms presented in Algorithm 1 is very simple and only needs few computations (for the dual Gabor atom, it needs only about $2N + a$ multiplications and $2N$ additions), the implementations is extremely "cheap" and thus also fast.

2. Wide applicability: There are no further conditions on the Gabor frame atoms besides the imposed support-conditions. In contrast to the general case the signal length is of no importance in this case.

3. Error-free reconstruction: The reconstruction error with the calculated dual Gabor or tight Gabor atom is usually around 10^{-15} which can be regarded as error-free reconstruction if the Gabor atom satisfies the conditions in Theorem 2.1.

4. Since in general the determination of dual Gabor atoms can be reduced iterated $1D$ Gabor problems, if on is using a separable Gabor atom, the above method to generate dual or tight Gabor frames extends to the $2D$-case in a very natural way.

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