A KACZMARZ-BASED APPROACH TO NONPERIODIC SAMPLING ON UNIONS OF RECTANGULAR LATTICES

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ABSTRACT

We consider the reconstruction of a band-limited signal from a nonuniform sampling set having a special geometry. The sampling set consists of two or more uniform sampling lattices with possibly different sampling rate (each sampling rate being below Nyquist rate). We propose a block-Kaczmarz based method for signal reconstruction. The blocks are of Toeplitz type, their inverses can be calculated inexpensively applying fast Toeplitz solvers. Moreover computing the inverses in the Gohberg-Semencul form enables both economical storage and fast matrix-vector multiplications via FFT's. These facts allows to apply the proposed method to large-scaled real data problems. Furthermore the approach can easily be generalized to higher dimensions. We also focus on the behavior of the ACT method. Numerical experiments and comparisons to standard methods complete this note.

1. Introduction

One of the principal problems in signal analysis is the reconstruction or approximation of a signal from its discrete samples. In many applications one can assume some maximal frequency in the signal. One may thus consider the sampling problem for band-limited functions. In the ideal case of equally spaced samples the reconstruction is routine and can be carried out explicitly by one of the many variations of the Shannon-Whittaker-Kotel'nikov sampling theorem [1].

In many applications, for instance in astronomy, seismology, tomography and physics, one is forced to sample signals at nonuniformly spaced points. This problem has received much attention in the past years, see [2, 3, 4, 5] for a theoretical treatment of the problem and [5, 6, 7, 8] for efficient numerical solutions.

A special case of nonuniform sampling is nonperiodic sampling on unions of rectangular lattices. In other words a signal is sampled by two or more sensors with perhaps different sampling rate. Such a situation can arise e.g. in tomography [9]. Walnut investigated such sampling sets in connection with the multisensor deconvolution problem [10].

It is the purpose of this paper to present a numerical efficient solution for reconstructing a signal sampled at such sampling sets. The proposed method is based on a modification of Kaczmarz's algorithm.

The paper is organized as follows: In this section we give a detailed introduction to the problem. Section 2 is dedicated to a detailed description of the proposed algorithm. Discussion of the behavior of the ACT method and numerical experiments are presented in Section 3. In Section 4 we generalize our results to higher dimensions.

In any application the input for an algorithm must consist only of a finite number of data. Therefore we have to set up a finite-dimensional model for the originally infinite-dimensional problem. For the finite-dimensional model we follow [11, 6].

A discrete signal x of length N is a finite sequence $x(0), x(1), \ldots, x(N-1) \in \mathbb{C}^N$. Its ℓ^2-norm is

$$\|x\| = \left(\sum_{n=0}^{N-1} |x(n)|^2 \right)^{1/2} .$$

The discrete Fourier transform of x is given by

$$\hat{x}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-2\pi i k n/N} \qquad k = 0, \ldots, N-1 .$$

The space of band-limited discrete signals with bandwidth $M < N/2$ is defined by

$$\mathcal{B}_M = \left\{ x \in \mathbb{C}^N | \hat{x}(k) = 0 \ \text{ for } |k \bmod N| > M \right\} .$$

The number $\nu := \frac{N}{2M+1}$ is called the *Nyquist interval*, $T_l x(k) = x(k-l)$ denotes the *translation operator*.

Finite sequences in $\mathbb{C}^N$ are extended to infinite sequences of period N by $x(n+lN) = x(n)$ and $\hat{x}(k+lN) = \hat{x}(k)$ for $n, k = 0, \ldots, N-1, l \in \mathbb{Z}$. In other

words we identify the index set $\{0, 1, \ldots, N-1\}$ with the cyclic group $\mathbb{Z}_N$. Let D_M be the Dirichlet kernel

$$D_M(t) = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} e^{2\pi i k t/N} = \frac{\sin(2M+1)\pi t/N}{\sqrt{N}\sin \pi t/N} .$$

Further we use the notation $[q]$ for the largest integer $\leq q \in \mathbb{R}$.

We consider sampling sets $\{t_j\}$ which are unions of l different lattices. These lattices may have a different number of sampling points and different sampling rate. To be more precise, $\{t_j\}_{j=1}^{p} := \bigcup_{k=1}^{l} \Lambda_k$ where

$$\Lambda_k := \{\alpha_k n + \beta_k\}_{n=1}^{c_k} \tag{1}$$

with $p = \sum_{k=1}^{l} c_k; 1 \leq c_k \leq N, \alpha_k, \beta_k, c_k \in \mathbb{Z}_N$. If some $c_k \geq \frac{N}{2M+1}$, then the corresponding lattice Λ_k contains sufficient points such that x can be reconstructed form its samples at Λ_k alone. This case is closely related to signal extrapolation [12], where one has full information of x in one region and no information in the other region. If we do not mention it separately we assume $c_k < [\frac{N}{2M+1}]$. We do not require that α_k divides N. In case of $N \bmod \alpha_k = 0 \ \forall k$, the set $\{t_j\} = \bigcup_{k=1}^{l} \Lambda_k$ consists of l (translated) subgroups of $\mathbb{Z}_N$, this is also referred to as *bunched sampling* or *periodic sampling* [13]. In principle the proposed algorithm can be applied to periodic sampling [14], but for a more efficient solution we refer the reader to [15].

2. Block-Kaczmarz reconstruction

In this section and in Section 4 we show that the efficiency of Kaczmarz's method can be significantly increased when properly utilizing the special structure of the sampling geometry.

Frames are meanwhile frequently used in signal processing in connection with sampling, cf. [16, 3, 5]. We also use the concept of frames as starting point for the presentation of our results.

Recall that a system of elements $\{f_j\}_{j \in I}$ in a Hilbert space $\mathcal{H}$ is called a frame, if there exist two constants $0 < A \leq B$, such that

$$A\|x\|^2 \leq \sum_{j \in I} |\langle x, f_j \rangle|^2 \leq B\|x\|^2 \qquad \forall x \in \mathcal{H} . \tag{2}$$

The constants A, B are called *frame bounds*. The *frame operator* S is defined by

$$Sx = \sum_{j \in I} \langle x, f_j \rangle f_j . \tag{3}$$

The frame operator is a positive invertible operator [16] on $\mathcal{H}$.

One standard approach to prove irregular sampling theorems is to show that the family $\{\sqrt{w_j} T_{t_j} D_M\}_{j=1}^{p}$ constitutes a frame for $\mathbb{B}_M$ (where the w_j are adaptive weights which should compensate for the irregularities in the sampling set [5]).

It follows then immediately that the signal x can be reconstructed from its samples $x(t_j)$, since the frame operator S is invertible on $\mathbb{B}_M$. See the articles [3, 5] for theoretical results, and [5, 11, 6] for both theoretical and practical results.

In our finite-dimensional model a sufficient condition for $\{\sqrt{w_j} T_{t_j} D_M\}_{j=1}^{p}$ to be a frame for $\mathbb{B}_M$ is $p \leq 2M+1$. A numerically stable reconstruction is guaranteed if two successive sampling points satisfy $\max_j |t_j - t_{j-1}| \leq [\frac{N}{2M+1}]$, cf. [6].

For clearity of notation we choose in the sequel $w_j = 1, \forall j$, i.e. we consider the frame $\{T_j D_M\}_{j=1}^{p}$. We define the operator $R : \mathbb{B}_M \to \ell_2$ by: $x \mapsto \{\langle x, T_j D_M \rangle\}_{j=1}^{p}$. R can be represented as $p \times N$-matrix having $T_j D_M$ as j-th row, since D_M is a reproducing kernel for $\mathbb{B}_M$, i.e. $x(t_j) = \langle x, T_j D_M \rangle$. Note that the frame operator S can be expressed as $S = R^* R$, where R^* denotes the conjugate transpose of R. Setting $y_k := x(t_k), k = 1, \ldots, p$, we can formulate the reconstruction problem as linear system of equations $Rx = y$. Establishing the system of normal equations $R^* R x = R^* y$ in the frequency domain leads to the ACT method [6], the most efficient reconstruction method for general irregular sampling problems. In the next section we discuss the performance of ACT for the given sampling geometry.

In applications a direct solution of the system $Rx = y$ has to be left out of consideration due to the size of R, which is often in the order of $10^4 - 10^6$ or higher. Linear algebra provides several iterative techniques for solving linear systems of equations.

One of these algorithms is Kaczmarz's algorithm, also known as POCS method (Projection onto Convex Sets, cf. [17]) or ART (Algebraic Reconstruction Technique, cf. [18]) in signal processing and computer tomography respectively.

The standard Kaczmarz procedure – also referred to as single step Kaczmarz method – for the system $Rx = y$ is given by the following iteration:

$$x^{(n+1)} = x^{(n)} + \sum_{k=1}^{N} \frac{r_K^* (y_k - r_k x^{(n)})}{r_k r_k^*} , \tag{4}$$

where r_k denotes the k-th row of R. If the rows r_k are nearly orthogonal we can expect a fast rate of convergence. However in general one single iteration step does little, since it handles only one equation of the system. Therefore we consider following partitioning

of R and y:

$$R = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_L \end{bmatrix}, \quad y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_L \end{bmatrix} \tag{5}$$

with $1 \le L \le N$. We do not assume that all B_k have equal size. Kaczmarz's method can be generalized by modifying Formula 4 as follows:

$$x^{(n+1)} = x^{(n)} + \sum_{k=1}^{p} B_k^*(B_k B_k^*)^{-1}(Y_k - B_k x^{(n)}) . \tag{6}$$

This modification is known as block-Kaczmarz method (BK for short). Note that in the spcial case $L = N$, $B_k = r_k$; setting $L = 1$ is equivalent to a direct solution of the system $Rx = y$.

Consult [19] for statements about the convergence of Kaczmarz's algorithm and its modifications. For numerical results in context with irregular sampling see [20].

In the sequel we describe how to compute efficiently multiplication of the $N \times m$ matrix B_k^* by a vector z, as it appears in equation (6). Let $\overline{R}$ be the $N \times N$ matrix having $T_j D_M$ as its j-th row. Then $\overline{R}^*$ is a circulant matrix and it is well-known that multiplication of a circulant matrix with a vector can be computed via FFT's of length N. We expand the vector z of length m to a vector $\overline{z}$ of length N by defining

$$\overline{z}_j = \begin{cases} z_j & \text{if } T_j D_M \in B_k \\ 0 & \text{if } T_j D_M \notin B_k . \end{cases} \tag{7}$$

Then $B_k^* z = \overline{R}^* \overline{z}$.

An important question for an optimal performance of BK is the proper choice of the block-size. If the block-size is too large, inversion of $B_k B_k^*$ may be quite time-consuming. On the other hand choosing the blocks B_k too small may result in a slow rate of convergence. Moreover the blocks do not have to be of the same size.

For the sampling geometry we investigate in this paper, there is a natural optimal way of partitioning R, enabling both efficient inversion of $B_k B_k^*$ and fast convergence.

We partition R into l blocks in the following sense: The k-th block B_k has $T_{\{\Lambda_k\}} D_M$ as row-entries, i.e. all $T_j D_M$ with $t_j \in \Lambda_k$.

The reason for this natural partitioning is that the matrices $B_k B_k^*$ are Toeplitz, since

$$(B_k B_k^*)_{jl} = (B_k B_k^*)_{j-l} = D_M(t_j - t_l) . \tag{8}$$

Hermitian Toeplitz matrices of size $n \times n$ can be stored by keeping only the first row in memory. Further $n \times n$ Toeplitz matrices can be inverted by $\mathcal{O}(n \log n)$ operations [21].

Clearly for the block-Kaczmarz algorithm the inverse matrices $(B_k B_k^*)^{-1}$ have to be computed only once at the beginning of the algorithm. Unfortunately the inverse A^{-1} of a Toeplitz matrix A is not Toepltiz any more. The question arises how to cheaply store $(B_k B_k^*)^{-1}$ and additionally how to compute efficiently the multiplication of $(B_k B_k^*)^{-1}$ with a vector, as required in Formula (6).

Note that the inverse of a Toeplitz matrix A is completely determined by solving

$$Az = e_1 \qquad \text{where } e_1 = (1, 0, \ldots, 0)^T . \tag{9}$$

Specifically

$$A^{-1} = \frac{1}{z_1}(L^* L - G G^*) \tag{10}$$

where

$$L = \begin{bmatrix} z_1 & 0 & \ldots & 0 \\ z_2 & z_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ z_n & z_{n-1} & \ldots & z_1 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & z_n & \ldots & z_2 \\ 0 & 0 & \ldots & z_3 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$

This formula for inverse Toeplitz matrices is known as the Gohberg-Semencul formulation. Since G and L are Toeplitz matrices, we can carry out the multiplication of A^{-1} via formula (10) by FFT's.

Obviously if $\Lambda_k = \Lambda_j + \beta$, then $B_k B_k^* = B_j B_j^*$. Further if $N \bmod \alpha_k = 0$ (periodic sampling), $B_k B_k^*$ is even a circulant matrix [14].

The problem of solving equation (9) can be attacked best by applying the conjugate gradient method [22]. Initialize $r_0 = b$ and $q_0 = r_0$, then the formulas of CG for a system $Az = b$ are given by

$$\begin{aligned} \alpha_k &= \frac{\langle r_{k-1}, r_{k-1} \rangle}{\langle p_{k-1}, A p_{k-1} \rangle} \\ z_k &= z_{k-1} + \alpha_k p_{k-1} \\ r_k &= r_{k-1} - \alpha_k A p_{k-1} \\ \beta_k &= \frac{\langle r_k, r_k \rangle}{\langle r_{k-1}, r_{k-1} \rangle} \\ q_k &= r_k + \beta_k p_{k-1} . \end{aligned}$$

It is well-known that CG delivers a high rate of convergence, if the eigenvalues of A are clustered [23]. This is the reason why CG is the method of choice to solve

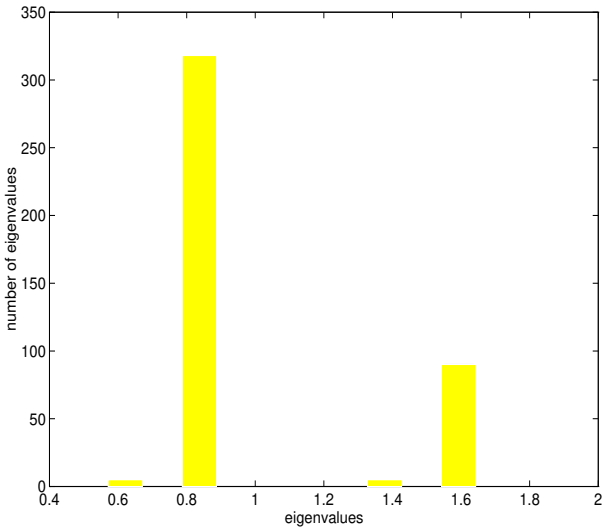

Figure 1: Distribution of eigenvalues of RR^*

equation (9). The eigenvalues of the Gram-matrices $B_k B_k^*$ are always strongly clustered.

Figure 1 illustrates this observation. We have computed the eigenvalues of the matrix RR^* for $N = 4096$ and $M = 250$, the lattice Λ has $\alpha = 10, \beta = 0, n = 1, \ldots [\frac{N}{\alpha}]$. In our example – which is typical in its tendency for all experiments – there are only 4 cluster points. Nearly all eigenvalues cluster around 0.8192 and 1.6.

We attribute this clustering tendency to the relation of sampling on the lattice Λ with two other special cases of sampling, both mentioned in the beginning of this section. The first case is when Λ_k is a subgroup of $\mathbb{Z}_N$. Then it can be shown that the eigenvalues of RR^* are always clustered [15]. The second case is the analogy to signal extrapolation, where the eigenvalues of the *prolate matrix* RR^* are clustered exponentially to the limits zero and one, the transition region consists of only $\mathcal{O}(\log p)$ eigenvalues, cf. [24, 12].

3. Numerical results

We consider a synthetic signal of length 16384 with bandwidth $M = 3500$ and randomly generated Fourier coefficients. The sampling set consists of 8193 sampling points and is built of three lattices having the same sampling rate (each below Nyquist rate). Since we know the actual signal x, we can measure the error $\|x - x^{(n)}\|/\|x\|$ between x and the approximation $x^{(n)}$ of the n-th iteration.

We apply the single-step Kaczmarz algorithm, the block-Kaczmarz algorithm and the ACT method and compare the rate of convergence as well as the computational effort of these methods.

Solving $R^*Rx = R^*y$ by CG is equivalent to ACT in terms of convergence rate (assuming exact arithmetics). We do not include this method – referrred to as *adaptive weights conjugate gradient acceleration* – in our comparisons here. Instaed we refer the reader to [6] for a detailed discussion.

In Figure 2 one can see that single-step Kaczmarz needs significantly more iterations (about 30% more) than block-Kaczmarz to reach a mean square error of 10^{-12}. ACT provides a nearly error-free reconstruction after only 16 iterations for a problem with about 7000 unknowns. We attribute this remarkable result to the observation that the eigenvalues of the corresponding Toeplitz matrix are strongly clustered for such kinds of sampling sets.

We compare the number of floating point operations (flops) needed to reach a mean square error of about 10^{-12} (note the logarithmic scaling of the y-axis). Single-step Kaczmarz requires about 58 times more flops than the proposed block-procedure. The improvement of BK is significant in terms of convergence rates (recall Figure 2), but it is dramatically in terms of flops. However it cannot compete with ACT, which needs more than 20 times less flops than the proposed algorithm. These observations are illustrated in Figure 3.

Summing up the standard Kaczmarz method (remind that it is similar to POCS and ART) can be accelerated by several orders of magnitude by using the proposed procedure. The improvement in terms of floating point operations is drastically.

The block-Kaczmarz method can compete with ACT only if $M > N/2$ and if the subspaces spanned by $\{T_{\{\Lambda_k\}} D_M\}$ are mutually nearly orthogonal. In all other cases ACT is significantly superior in terms of convergence rate and in floating point operations.

It is obvious that the proposed algorithm can be highly parallelized. The matrices $B_k B_k^*$ can be inverted independently from each other. And clearly the summation in equation (6) can be carried out in a straightforward parallel computation.

4. Generalization to higher dimensions

The proposed procedure can easily be generalized to higher dimensions by means of the Kronecker product. We briefly discuss the 2-D case, because of its relevance in tomography. The blocks arising from partitioning the system matrix corresponding to each sampling lattice are of block-Toeplitz type with Toeplitz blocks (BTTB for short). Analogous to the 1-D case multiplication of a BTTB-matrix by a vector can be

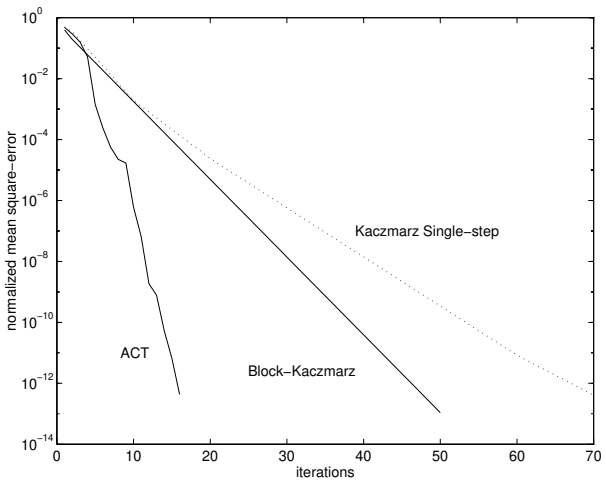

Figure 2: Comparison of convergence rate

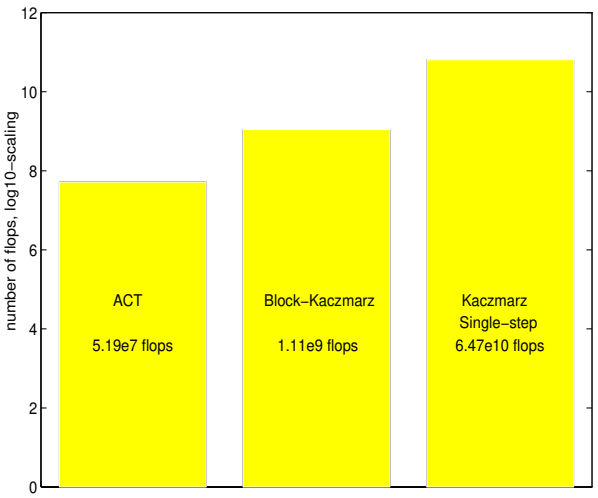

Figure 3: Comparison of FLOPS

computed via 2-D FFT's and block-circulant matrices.

In tomography it is known that the use of three or more detectors leads to efficient sampling schemes [9, 25]. Sampling on coarse grids – where each coarse grid is associated with one detector – result in a sampling geometry which is a generalization of definition (1) for 2-D.

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