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Analysis of a Singular Hyperbolic System of Conservation Laws

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We solve the Riemann and Cauchy problems globally for a singular system of n hyperbolic conservation laws. The system, which arises in the study of oil reservoir simulation, has only two wave speeds, and these coincide on a surface of codimension one in state space. The analysis uses the random choice method of Glimm (*Comm. Pure Appl. Math.* **18** (1965), 697–715).

1. INTRODUCTION

In the study of enhanced oil recovery, the scalar Buckley–Leverett equation

$$s_t + (sg)_x = 0, \tag{1.1}$$

models the water flooding of an oil reservoir [6]. Water flooding involves the injection of water, which is immiscible with oil, into certain wells of the reservoir to force oil out at others. In this case, $s = s(x, t)$ is the saturation of water (i.e., the volume fraction of water in the total fluid, $0 \leqslant s \leqslant 1$), and $g = g(s)$ is the particle velocity of the water.

Consequently, since the total volumetric flow rate is fixed [2, 6, 7], the fraction of the flow rate associated with the water is $sg(s) \equiv f(s)$. In this problem, the fractional flow curve $f(\cdot)$ is nonconvex: f increases from 0 to

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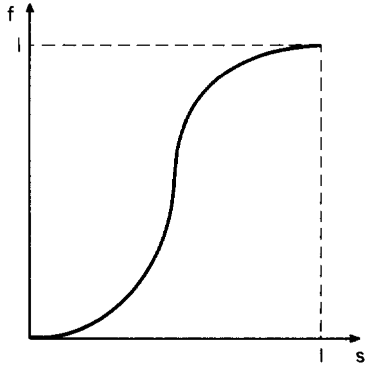

FIG. 1. A typical S-shaped curve corresponding to $f(\cdot)$ in system (1.1) and $f(\cdot, \mathbf{c})$ in systems (1.2), (1.3).

1, $f'(0) = f'(1) = 0$, and f has one inflection point. We say f is S-shaped. (See Fig. 1.)

The 2×2 system of nonlinear conservation laws

$$\begin{aligned} s_t + (sg)_x &= 0 \\ (sc)_t + (scg)_x &= 0 \end{aligned} \tag{1.2}$$

models the polymer flooding of an oil reservoir [2, 7]. (See also [3].) In a polymer flood, water thickened with polymer is injected into the reservoir. A polymer is a solute of water which inhibits its flow, and thus improves the oil displacement. In this case, $s = s(x, t)$ is the saturation of the aqueous phase (the solution of polymer and water), $c = c(x, t)$ is the concentration of polymer in the water, $g = g(s, sc)$ is the particle velocity of the aqueous phase, and $f(s, c) \equiv sg(s, sc)$ is the fraction of the flow rate associated with the equeous phase. For fixed c, the fractional flow curve $f(\cdot, c)$ has the same qualitative properties as in the Buckley–Leverett problem.

In the present paper, we study a natural generalization of Eqs. (1.1) and (1.2) given by the $(n+1) \times (n+1)$ system

$$U_t + (gU)_x = 0, \tag{1.3}$$

where $U = (s, s\mathbf{c})$ is the vector of conserved quantities, $\mathbf{c} = (c_1, ..., c_n)$, $g = g(s, s\mathbf{c})$, and $\{(s, \mathbf{c})\colon 0 \leqslant s \leqslant 1,\ 0 \leqslant c_i \leqslant 1\}$ is the state space. Again, the function $f(s, \mathbf{c}) \equiv sg(s, s\mathbf{c})$ is important for our analysis. Our only assumption for system (1.3) is that f is smooth and the curves $f(\cdot, \mathbf{c})$ have the S-shape of the Buckley–Leverett problem for each fixed $\mathbf{c}$. Our main results are:

THEOREM A. *The Riemann problem for* (1.3) *has a unique solution for arbitrary initial data* $(s_L, \mathbf{c}_L)$ *and* $(s_R, \mathbf{c}_R)$. *Moreover, for every positive* t, *the solution* $(s(x, t), \mathbf{c}(x, t))$ *is in* L^1_{loc} *with respect to* x *and depends continuously on* t.

THEOREM B. *There is a singular transformation*

$$\Psi: (s, \mathbf{c}) \to W \equiv (z, \mathbf{c})$$

such that, if the initial data $(s_0(x), \mathbf{c}_0(x))$ *satisfies* $TV(z_0(\cdot), \mathbf{c}_0(\cdot)) < \infty$, *then there exists a global weak solution* $U(x, t)$ *of the Cauchy problem for* (1.3). *Moreover,* $U(x, t)$ *is in* L^1_{loc} *with respect to* x *and depends continuously on* t.

To prove Theorem A we construct the solutions explicitly, and to prove Theorem B we use the random choice method (RCM) of Glimm [1].

System (1.3) can be viewed as a model for the flooding of a reservoir by water containing n additives. Our present assumptions relax the requirement that the additives inhibit the water flow as in [2, 7]. This system is of interest in the study of conservation laws because of the new phenomena which arise. The system is nowhere strictly hyperbolic and, in fact, has at most two distinct eigenvalues which coincide on a surface of codimension one.

There are two new features in the present analysis. First, we analyze the Riemann problem entirely in the 2-dimensional sf-plane, and not the $(n+1)$-dimensional state space. This enables us to handle the higher dimensional problem. It also makes unnecessary the assumption that $f(s, \cdot)$ be monotone for each fixed s. Second, unbounded variation occurs in approximate solutions of the RCM for systems (1.2), (1.3). As in [7], we prove convergence by bounding the variation in s as measured under a singular transformation. Here we must incorporate a supplemental bound on the total variation in $\mathbf{c}$ because extra degrees of freedom occur in (1.3) that do not occur in (1.2). Also, as in [7], it is necessary to randomize in space as well as time to handle the singularity.

2. THE RIEMANN PROBLEM

2.1. *Preliminaries*

In this section we consider the Riemann problem; i.e., the initial value problem

$$\begin{aligned} U_t + (gU)_x = 0, \\ U(x, 0) = U_0(x), \end{aligned} \tag{2.1}$$

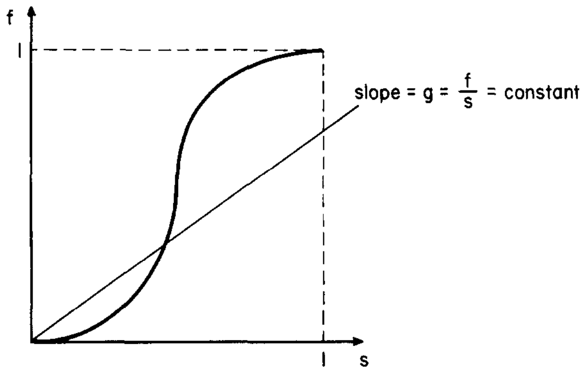

FIG. 2. In system (1.2) and (1.3) the surface $g = g_0 \equiv$ const is given by all points $(s, \mathbf{c})$ such that $(s, f(s, \mathbf{c}))$ lies on the line $\{(s, f): f = g_0 s\}$.

where

$$U_0(x) = \begin{cases} U_L, & x \leqslant 0 \\ U_R, & x > 0. \end{cases}$$

Here, $U = (s, \mathbf{b})$, $\mathbf{b} = s\mathbf{c}$, $0 \leqslant s \leqslant 1$, $0 \leqslant b_i \leqslant s$, and $f = sg$ satisfies the constitutive assumptions of system (1.3). (See Figs. 1, 2.) Here, U is the vector of conserved quantities. Let

$$U = \Phi(V), \qquad V \equiv (s, \mathbf{c}).$$

We let "state space" refer either to the space of allowed U's or the corresponding values of V. The map Φ is contractive, one-to-one, and regular except at $s = 0$. We solve the Riemann problem for arbitrary values of V_L and V_R. We note that when $s = 0$, the solution is not unique in the U-variables, but specifying a value of $\mathbf{c}$ determines a unique solution in the V-variables.

2.2. *Notation*

First, because our analysis to follow uses V-space instead of U-space, we take g to be a function of $V = (s, \mathbf{c})$; i.e., $g(V) \equiv g \circ \Phi(V)$. The set of points where $f_s = f/s$ is crucial in our analysis. At these transition points the system (1.3) fails to be strictly hyperbolic. Let $s_T(\mathbf{c})$ denote the unique positive value of s for which $f(s, \mathbf{c})/s = f_s(s, \mathbf{c})$. The existence and uniqueness of s_T for each $\mathbf{c}$ follow from the fact that $f(\cdot, \mathbf{c})$ is S-shaped (see Fig. 3). The transition surface $\mathscr{T}$ is the subset of state space given by

$$\mathscr{T} = \{(s, \mathbf{c}): s = s_T(\mathbf{c})\}.$$

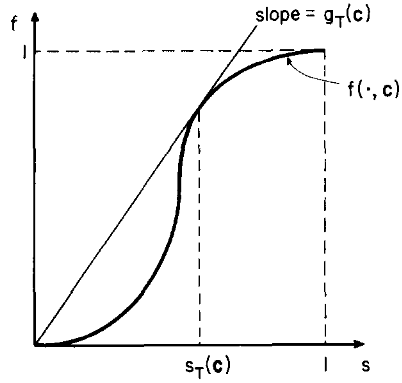

FIG. 3. Construction of $s_T(\mathbf{c})$ and $g_T(\mathbf{c})$ from the graph of $f(\cdot, \mathbf{c})$.

We also define the region

$$\mathscr{L} = \{(s, \mathbf{c}), 0 \leqslant s \leqslant s_T(\mathbf{c})\}$$

to the left of $\mathscr{T}$ and the region

$$\mathscr{R} = \{(s, \mathbf{c}): s_T(\mathbf{c}) \leqslant s \leqslant 1\}$$

to the right of $\mathscr{T}$. For each $\mathbf{c}$, the unique state on the transition surface is

$$V_T(\mathbf{c}) \equiv (s_T(\mathbf{c}), \mathbf{c}).$$

For each $\mathbf{c}$ this gives a unique g-value given by (see Fig. 3)

$$g_T(\mathbf{c}) = g(V_T(\mathbf{c})).$$

2.3. *S-waves and C-waves*

By differentiating (1.3), we obtain the system

$$U_t + A(U)\, U_x = 0,$$

where

$$A(U) \equiv A(s, \mathbf{b}) = gI + \begin{pmatrix} s \\ \mathbf{b} \end{pmatrix} (g_s, g_{\mathbf{b}}),$$
$$g_{\mathbf{b}} \equiv (g_{b_1}, ..., g_{b_n}).$$

The following lemma is a consequence of the fact that A is a rank one perturbation of a scalar matrix.

LEMMA 2.1. *The eigenvalues of A are $g = f/s$ and $g + sg_s + \mathbf{b} \cdot g_{\mathbf{b}} = f_s$. Off the transition surface $\mathscr{T}$, the eigenvalue g has multiplicity n, and f_s is simple. The eigenvectors corresponding to the eigenvalue g span the tangent space to the surface $g = \text{const}$. The eigenvector corresponding to the eigenvalue f_s is $u = (s, \mathbf{b})$. On $\mathscr{T}$ there are only n independent eigenvectors.*

From this lemma we obtain

THEOREM 2.2. *The integral manifolds of the eigenvectors corresponding to the eigenvalue g are the connected components of the surfaces $g = \text{const}$. The integral curves of the eigenvector corresponding to the eigenvalue f_s are the lines $\mathbf{c} = \text{const}$.*

The Hugoniot locus [4] of a state U_L consists of those states U_R for which the discontinuous function

$$U(x, t) = \begin{cases} U_L & x \leqslant \sigma t, \\ U_R & x > \sigma t, \end{cases}$$

is a weak solution of (2.1). An equivalent statement is that $U = U_R$ satisfy the Rankine-Hugoniot jump condition:

$$g(U)\,U - g(U_L)\,U_L = \sigma(U - U_L)$$

for some number σ. A straightforward calculation yields

THEOREM 2.3. *The Hugoniot locus of a state U_L is the set*

$$\{U\colon g(U) = g(U_L)\} \cup \{U\colon \mathbf{c} = \mathbf{c}_L\}.$$

From Theorems (2.2) and (2.3), it is clear that there are two families of elementary waves: C-waves (concentration waves) consist of contact discontinuities joining two states with equal g-values; S-waves (saturation waves) solve the scalar Riemann problem that is obtained from (2.1) when $\mathbf{c}$ is constant. To insure the uniqueness of solutions to the Riemann problem, we assume that the elementary waves of each family satisfy an entropy condition. We require that the S-waves satisfy the standard entropy condition for a scalar conservation law. Our entropy condition for the C-waves is that C-waves can join two states only if both states are in $\mathscr{L}$ or both are in $\mathscr{R}$. This is equivalent to the generalization of the Lax entropy conditions given by Keyfitz and Kranzer [3].

We let $V_L \rightarrow^S V_R$ (resp. $V_I \rightarrow^C V_R$) indicate that V_L on the left can be connected to V_R on the right by an S-wave (resp. C-wave). Two consecutive arrows (e.g., $V_L \rightarrow^S V_I \rightarrow^C V_R$) indicate that the waves can be

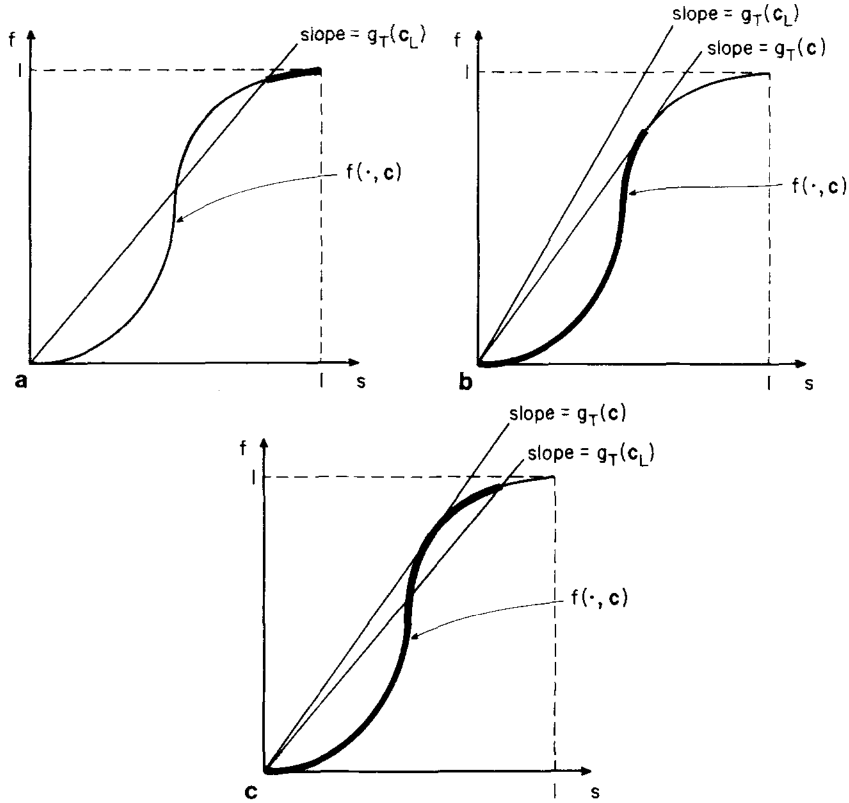

FIG. 4. (a) For each $\mathbf{c}$, $(s, \mathbf{c}) \in \mathscr{R}_1$ iff $(s, f(s, \mathbf{c}))$ lies on the shaded segment of $f(\cdot, \mathbf{c})$). (That is, $(s, \mathbf{c}) \in \mathscr{R}$ and $(s, f(s, \mathbf{c}))$ lies below the line with slope $g_T(\mathbf{c}_L)$.) (b) For each $\mathbf{c}$, $(s, \mathbf{c}) \in \mathscr{R}_2$ iff $g_T(\mathbf{c}) \leqslant g_T(\mathbf{c}_T)$ and $(s, f(s, \mathbf{c}))$ lies on the shade segment of $f(\cdot, \mathbf{c})$. (c) For each $\mathbf{c}$, $(s, \mathbf{c}) \in \mathscr{R}_3$ iff $g_T(\mathbf{c}) \geqslant g_T(\mathbf{c}_L)$ and $(s, f(s, \mathbf{c}))$ lies on the shaded segment of $f(\cdot, \mathbf{c})$.

composed; i.e., the wave speeds increase from left to right, and we say that the speeds are compatible.

2.4. *Solution of the Riemann Problem*

We present the solution in six cases:

(I) $V_L \in \mathscr{R}$ and V_R in one of (see Fig. 4)

(i) $\mathscr{R}_1 \equiv \mathscr{R} \cap \{V: g(V) \leqslant g_T(\mathbf{c}_L)\}$

(ii) $\mathscr{R}_2 \equiv \mathscr{L} \cap \{V: g_T(\mathbf{c}) \leqslant g_T(\mathbf{c}_L)\}$,

(iii) $\mathscr{R}_3 \equiv (\mathscr{R}_1 \cup \mathscr{R}_2)^c = \{V: g_T(\mathbf{c}) \geqslant g_T(\mathbf{c}_L)\} \backslash \mathscr{R}_1$.

(II) $V_L \in \mathscr{L}$ and V_R in one of

(i) $\mathscr{L}_1 \equiv \mathscr{R} \cap \{V: g(V) \leqslant g(V_L)\}$,

(ii) $\mathscr{L}_2 \equiv \mathscr{L} \cap \{V: g_T(\mathbf{c}) \leqslant g(V_L)\}$,

(iii) $\mathscr{L}_3 \equiv (\mathscr{L}_1 \cup \mathscr{L}_2)^c = \{V: g_T(\mathbf{c}) \geqslant g(V_L)\} \backslash \mathscr{L}_1$.

These cover all possibilities. As the regions have common boundaries, one must verify that the solutions presented for initial data in the overlap actually agree. In fact, two such solutions $V(x, t)$ agree in L^1 for each t. Finally, the observations referred to below are listed in Section 2.5 and follow from the assumption that $f(\cdot, \mathbf{c})$ is S-shaped. That solutions depend continuously on the data in L^1_{loc} is a consequence of the continuity in each region, together with the agreement of solutions at the boundaries of regions:

(I) $V_L = (s_L, \mathbf{c}_L) \in \mathscr{R}$.

Case (i). $V_R \in \mathscr{R}_1$. Solution: $V_L \to^S V_I \to^C V_R$. The intermediate state V_I is given by

$$V_I = (s_I, \mathbf{c}_L)$$

where s_I is chosen so that $g(V_I) = g(V_R)$, and $V_I \in \mathscr{R}$. State V_I exists by observation (O1). The wave speeds are compatible by observation (O3) (given below).

Case (ii). $V_R \in \mathscr{R}_2$. Solution: $V_L \to^S V_I \to^C V_{II} \to^S V_R$. The intermediate states V_I and V_{II} are given by

$$V_I = (s_I, \mathbf{c}_L),$$
$$V_{II} = (s_{II}, \mathbf{c}_R)$$

where s_I is chosen so that $g(V_I) = g_T(\mathbf{c}_R)$ and $s_{II} = s_T(\mathbf{c}_R)$. State V_I exists by observation (O1). The wave speeds are compatible by observations (O3) and (O4).

Case (iii). $V_R \in \mathscr{R}_3$. Solution: $V_L \to^S V_I \to^C V_{II} \to^S V_R$. The intermediate states V_I and V_{II} are given by

$$V_I = (s_I, \mathbf{c}_L)$$
$$V_{II} = (s_{II}, \mathbf{c}_R)$$

where $s_I = s_T(\mathbf{c}_L)$ and s_{II} is chosen so that $g(V_{II}) = g_T(\mathbf{c}_L)$. State V_{II} exists by observation (O2). The wave speeds are compatible by observations (O3) and (O5).

(II) $V_L = (s_L, \mathbf{c}_L) \in \mathscr{L}$.

Case (i). $V_R \in \mathscr{L}_1$. Solution: $V_L \to^S V_I \to^C V_R$. The intermediate state V_I is given by

$$V_I = (s_I, \mathbf{c}_L)$$

where s_I is chosen so that $g(V_I) = g(V_R)$ and $V_I \in \mathscr{R}$. State V_I exists by observation (O1). The wave speeds are compatible by observation (O6).

Case (ii). $V_R \in \mathscr{L}_2$. Solution: $V_L \to^S V_I \to^C V_{II} \to^S V_R$. The intermediate states V_I and V_{II} are given by

$$V_I = (s_I, \mathbf{c}_L)$$

$$V_{II} = (s_{II}, \mathbf{c}_R),$$

where s_I is chosen so that $g(V_I) = g_T(\mathbf{c}_R)$ and $s_{II} = s_T(\mathbf{c}_R)$. State V_I exists by observation (O1). The wave speeds are compatible by observations (O6) and (O4).

Case (iii). $V_R \in \mathscr{L}_3$. Solution: $V_L \to^C V_I \to^S V_R$. The intermediate state V_I is given by

$$V_I = (s_I, \mathbf{c}_R),$$

where s_I is chosen so that $g(V_I) = g(V_L)$. State V_I exists by observation (O2). The wave speeds are compatible by observation (O5).

2.5. *Observations in the sf-plane*

In the observations below, we let $l(g)$ be the line in the sf-plane with slope g passing through the origin. We say $l(g_0)$ intersects the curve $f = f(\cdot, \mathbf{c}_0)$ at the point $V_0 = (s_0, \mathbf{c}_0)$ if $g(V_0) = g_0$. (In the sf-plane, the two curves actually intersect at $(s_0, f(V_0))$. E.g., see Fig. 2.) Also, the wave speeds in observations (O3) through (O6) refer to those in the scalar Buckley–Leverett equation:

(O1) If $l(g_0)$ intersects $f = f(\cdot, \mathbf{c}_0)$ at a point in $\mathscr{R}$, and if $l(g_1)$ intersects $f = f(\cdot, \mathbf{c}_1)$ with $g_1 \geqslant g_0$, then $l(g_0)$ also intersects $f = f(\cdot, \mathbf{c}_1)$ at a point in $\mathscr{R}$.

(O2) If $0 < g_0 \leqslant g_T(\mathbf{c})$, then $l(g_0)$ intersects $f = f(\cdot, \mathbf{c})$ in $\mathscr{L}$.

(O3) The wave speeds in $V_1 \to^S V_2$ are at most $g(V_2)$ if $V_1, V_2 \in \mathscr{R}$.

(O4) The wave speeds in $V_1 \to^S V_2$ are at least $g(V_1)$ if $V_1 \in \mathscr{T}$ and $V_2 \in \mathscr{L}$.

(O5) The wave speeds in $V_1 \to^S V_2$ are at least $g(V_1)$ if $V_1 \in \mathscr{L}$ and either (a) $V_2 \in \mathscr{L}$ or (b) $V_2 \in \mathscr{R}$ with $g(V_2) \geqslant g(V_1)$.

(O6) The wave speeds in $V_1 \to^S V_2$ are at most $g(V_2)$ if $V_1 \in \mathscr{L}$, $V_2 \in \mathscr{R}$, and $g(V_1) \geqslant g(V_2)$.

3. Solution of the Cauchy Problem

3.1. *Preliminaries*

In this section we use the random choice method to solve the Cauchy problem (2.1) with general initial data. First define the variable z by

$$z \equiv z(V) = \begin{cases} g_{\mathrm{T}}(\mathbf{c}) - g(V) & \text{if } V \in \mathscr{R}, \\ g(V) - g_{\mathrm{T}}(\mathbf{c}) & \text{if } V \in \mathscr{L}. \end{cases} \tag{3.1}$$

Let Ψ: $V \rightarrow W$ be defined by

$$W = (z(V), \mathbf{c}). \tag{3.2}$$

The transformation Ψ is one-to-one and regular except at the transition surface $\mathscr{T}$. We define the strength of S- and C-waves, as well as the strengths of waves in the variable z, as follows. Let S, C denote arbitrary S- and C-waves, respectively. Let V_{L} and V_{R} denote the left and right states of the wave, respectively. Then define

$$\begin{aligned} |S| &= |V_{\mathrm{L}} - V_{\mathrm{R}}| \equiv |s_{\mathrm{L}} - s_{\mathrm{R}}| + |\mathbf{c}_{\mathrm{L}} - \mathbf{c}_{\mathrm{R}}| \\ &= |s_{\mathrm{L}} - s_{\mathrm{R}}| + |c_{1_{\mathrm{L}}} - c_{1_{\mathrm{R}}}| + \cdots + |c_{n_{\mathrm{L}}} - c_{n_{\mathrm{R}}}|, \\ |C| &= |V_{\mathrm{L}} - V_{\mathrm{R}}|, \\ |C|_c &= |\mathbf{c}_{\mathrm{L}} - \mathbf{c}_{\mathrm{R}}|, \\ |S|_z &= |z_{\mathrm{L}} - z_{\mathrm{R}}|, \\ |C|_z &= \begin{cases} 4\, |z_{\mathrm{L}} - z_{\mathrm{R}}| & \text{if (3.4) holds}, \\ 2\, |z_{\mathrm{L}} - z_{\mathrm{R}}| & \text{if (3.4) does not hold}, \end{cases} \end{aligned} \tag{3.3}$$

where the condition is

$$\begin{aligned} & V_{\mathrm{L}}, V_{\mathrm{R}} \in \mathscr{L} \quad \text{and} \quad g_{\mathrm{T}}(V_{\mathrm{L}}) \geqslant g_{\mathrm{T}}(V_{\mathrm{R}}), \\ \text{or} \quad & V_{\mathrm{L}}, V_{\mathrm{R}} \in \mathscr{R} \quad \text{and} \quad g_{\mathrm{T}}(V_{\mathrm{L}}) \leqslant g_{\mathrm{T}}(V_{\mathrm{R}}). \end{aligned} \tag{3.4}$$

The general Riemann problem has a solution of the form $S^{\mathrm{L}}CS^{\mathrm{R}}$; i.e., two S-waves separated by a C-wave with the convention that $S^{\mathrm{R}} = 0$ if $C = 0$. We let $\gamma = S^{\mathrm{L}}CS^{\mathrm{R}}$ and set

$$\begin{aligned} |\gamma| &= |S^{\mathrm{L}}| + |C| + |S^{\mathrm{R}}|, \\ |\gamma|_z &= |S^{\mathrm{L}}|_z + |C|_z + |S^{\mathrm{R}}|_z. \end{aligned}$$

The following lemma is crucial in the convergence proof of Section 4.

LEMMA 3.1. *For every $\varepsilon > 0$ there exists a constant $M(\varepsilon)$ depending only on ε such that*

$$\text{if} \quad |S| > \varepsilon, \qquad \text{then} \qquad |S| \leqslant M(\varepsilon)\,|S|_z, \tag{3.5}$$

$$\text{if} \quad |C| > \varepsilon, \qquad \text{then} \qquad |C| \leqslant M(\varepsilon)\{|C|_z + |C|_{\mathbf{c}}\}. \tag{3.6}$$

Proof. Suppose $|S| > \varepsilon$. Then $|s_q - s_T(\mathbf{c}_q)| > \varepsilon/2$ for $q = \mathrm{L}$ or R. The map $(z, \mathbf{c}) \to (s, \mathbf{c})$ is a one-to-one regular map off the transition surface $\mathscr{T}$. Therefore the Jacobian is bounded off any neighborhood of $\mathscr{T}$. Consequently, there is a constant $M_1(\varepsilon)$ so that

$$|s_{\mathrm{L}} - s_{\mathrm{R}}| \leqslant M_1(\varepsilon)\,|z_{\mathrm{L}} - z_{\mathrm{R}}|. \tag{3.7}$$

To prove (3.6), we use the fact that s_T is a differentiable function of $\mathbf{c}$. Thus,

$$|s_T(\mathbf{c}_{\mathrm{L}}) - s_T(\mathbf{c}_{\mathrm{R}})| \leqslant \gamma\,|\mathbf{c}_{\mathrm{L}} - \mathbf{c}_{\mathrm{R}}|$$

for some number γ. (We may assume $\gamma > 1$.)

Now suppose $|C| > \varepsilon$. If $|C|_{\mathbf{c}} > \varepsilon/2\gamma$, then we can take $M_2(\varepsilon) = 2\gamma(n+1)/\varepsilon$ since $|C| \leqslant n+1$. On the other hand, if $|C|_{\mathbf{c}} \leqslant \varepsilon/2\gamma$, then

$$\begin{aligned}(1 - 1/2\gamma)\varepsilon < |C| - |C|_{\mathbf{c}} &= |s_{\mathrm{L}} - s_{\mathrm{R}}| \\ &\leqslant |s_{\mathrm{L}} - s_T(\mathbf{c}_{\mathrm{L}})| + |s_T(\mathbf{c}_{\mathrm{L}}) - s_T(\mathbf{c}_{\mathrm{R}})| + |s_T(\mathbf{c}_{\mathrm{R}}) - s_{\mathrm{R}}| \\ &\leqslant |s_{\mathrm{L}} - s_T(\mathbf{c}_{\mathrm{L}})| + \gamma\,|C|_{\mathbf{c}} + |s_{\mathrm{R}} - s_T(\mathbf{c}_{\mathrm{R}})| \\ &\leqslant |s_{\mathrm{L}} - s_T(\mathbf{c}_{\mathrm{L}})| + \varepsilon/2 + |s_{\mathrm{R}} - s_T(\mathbf{c}_{\mathrm{R}})|.\end{aligned}$$

Rearranging gives $|s_q - s_T(\mathbf{c}_q)| > (\gamma - 1)\varepsilon/4\gamma$ for $q = L$ or R. Therefore, we can use (3.7) again to obtain

$$\begin{aligned}|C| &= |s_{\mathrm{L}} - s_{\mathrm{R}}| + |C|_{\mathbf{c}} \\ &\leqslant M_1((\gamma - 1)\varepsilon/2\gamma)\,|z_{\mathrm{L}} - z_{\mathrm{R}}| + |C|_{\mathbf{c}} \\ &\leqslant M_1((\gamma - 1)\varepsilon/2\gamma)\{|S|_z + |C|_{\mathbf{c}}\}\end{aligned}$$

provided $M_1 \geqslant 1$.

3.2. *The Random Choice Method*

We first define the approximate solutions generated by the RCM. Let h, k be mesh lengths in x, t, respectively. Let $k \equiv k(h)$ be chosen so that the Courant–Friedrichs–Lewy condition holds:

$$\frac{k}{h} = \max_{\substack{0 \leqslant s \leqslant 1 \\ 0 \leqslant c_i \leqslant 1}} \left\{ f_s, \frac{f}{s} \right\}. \tag{3.8}$$

For every integer i and nonnegative integer j, define the mesh points $x_i \equiv ih$, $t_j \equiv jk$ in the xt-plane. Let $\mathbf{a}$ denote a sequence

$$\mathbf{a} \equiv \{a_{ij}\}, \tag{3.9}$$

where $0 \leqslant a_{ij} \leqslant 1$; i.e., an element of the measure space

$$A = \prod_{\substack{i \in Z \\ j \in Z^+}} [0, 1]_{ij} \tag{3.10}$$

with Lebesgue measure denoted by m. For each h, $\mathbf{a}$, and initial data $V_0(x)$, we define the approximate solution

$$V_h(x, t) \equiv V_h(x, t; \mathbf{a})$$

generated by the RCM, by induction on j as follows: Let

$$V_h(x, 0) = V_0(x_i + a_{i0}h) \qquad \text{for} \quad x_i < x \leqslant x_{i+1}, \tag{3.11}$$

and for $0 \leqslant t < t_1$, let $V_h(x, t)$ be the solution of (3.1) obtained by solving the Riemann problems posed in (3.11). Here $V_h(x, t)$ is well defined for $t < t_1$ by (3.8).

Now assume $V_h(x, t)$ has been defined for times $t < t_j$. Then define V_h for $t = t_j$ by setting

$$V_h(x, t_j) = V_h(x_i + a_{ij}h, t_j) \qquad \text{for} \quad x_i < x \leqslant x_{i+1}, \tag{3.12}$$

and define V_h for $t_j < t < t_{j+1}$ by solving the corresponding Riemann problems with "initial data" $V_h(x, t_j)$. Again, $V_h(x, t)$ is defined for all time by (3.8). Let

$$W_h(x, t) \equiv W_h(x, t; \mathbf{a}) = \Psi(V_h(x, t; \mathbf{a})), \tag{3.13}$$

$$U_h(x, t) \equiv U_h(x, t; \mathbf{a}) = \Phi(V_h(x, t; \mathbf{a})), \tag{3.14}$$

$$W_0(x) = \Psi(V_0(x)), \tag{3.15}$$

$$U_0(x) = \Phi(V_0(x)). \tag{3.16}$$

Let $\gamma_{ij} = S^{\mathrm{L}}_{ij} C_{ij} S^{\mathrm{R}}_{ij}$ be the waves appearing in the solution of the Riemann problem centered at (x_i, t_j). Note that all waves have nonnegative speeds. Define

$$F(j) = \sum_i \{|\gamma_{ij}|_z + |C_{ij}|_{\mathrm{c}}\}. \tag{3.17}$$

Notation. In the following lemma and throughout the remainder of the paper, we let M denote a generic constant depending only on the function f and any explicit arguments.

LEMMA 3.2. *If* $TV\{W_0(\cdot)\} < M_0 < \infty$, *then*

$$F(j+1) \leqslant F(j) \leqslant F(0) \leqslant MM_0, \tag{3.18}$$

$$TV\{W_h(\cdot, t)\} \leqslant MM_0. \tag{3.19}$$

Proof. Inequality (3.18) holds because both $F_{\mathbf{c}}(j) \equiv \sum_i |C_{ij}|_{\mathbf{c}}$ and $F_z(j) \equiv \sum |\gamma_{ij}|_z$ are nonincreasing functions of j. This can be proven by induction on j. That $F_{\mathbf{c}}$ is nonincreasing follows easily from the fact that the change in $\mathbf{c}$ in any Riemann problem is due to a single jump; i.e., the jump in $\mathbf{c}$ across the C-wave. That F_z is nonincreasing follows from the fact that the sum of the wave strengths in z for any sequence of elementary waves connecting a given left and right state is minimized by the waves in the Riemann problem solution (cf. [7]). Inequality (3.19) follows immediately from (3.18) together with the definition of $W = \Psi(V)$, since for $t_j \leqslant t < t_{j+1}$,

$$TV\{W_h(\cdot, t)\} \leqslant F(j) \leqslant F(0) \leqslant MM_0.$$

Let the index set $\mathscr{K}(\varepsilon, \mathbf{a})$ be defined by

$$\mathscr{K} \equiv \mathscr{K}(\varepsilon, \mathbf{a}) \equiv \{(i, j): |S_{ij}^{\mathrm{L}}| \leqslant \varepsilon \text{ and } |S_{ij}^{\mathrm{R}}| \leqslant \varepsilon \text{ and } |C_{ij}| \leqslant \varepsilon\}. \tag{3.20}$$

LEMMA 3.3. *If* $(i, j) \notin \mathscr{K}(\varepsilon, \mathbf{a})$, *then*

$$|\gamma_{ij}| \leqslant M(\varepsilon)\{|\gamma_{ij}|_z + |C_{ij}|_{\mathbf{c}}\}. \tag{3.21}$$

Proof. If $(i, j) \notin \mathscr{K}(\varepsilon, \mathbf{a})$, then either $|S_{ij}^{\mathrm{L}}| > \varepsilon$ or $|S_{ij}^{\mathrm{R}}| > \varepsilon$ or $|C_{ij}| > \varepsilon$. If $|C_{ij}| > \varepsilon$, then by (3.6)

$$\varepsilon \leqslant |C_{ij}| \leqslant M(\varepsilon)\{|C_{ij}|_z + |C_{ij}|_{\mathbf{c}}\}.$$

Since $|S_{ij}^q| \leqslant 1$ for $q = \mathrm{L}$ or R, we obtain

$$\begin{aligned} |\gamma_{ij}| &= |S_{ij}^{\mathrm{L}}| + |S_{ij}^{\mathrm{R}}| + |C_{ij}| \\ &\leqslant 2 + M(\varepsilon)\{|C_{ij}|_z + |C_{ij}|_{\mathbf{c}}\} \\ &\leqslant (2/\varepsilon + 1)\, M(\varepsilon)\{|C_{ij}|_z + |C_{ij}|_{\mathbf{c}}\} \\ &\leqslant M(\varepsilon)\{|\gamma_{ij}|_z + |C_{ij}|_{\mathbf{c}}\} \end{aligned}$$

where $M(\varepsilon)$ is generic. The case $|S_{ij}^q| > \varepsilon$, $q = \mathrm{L}$ or R, follows from (3.5) in the same manner.

3.3. *Convergence of the Approximate Solution*

We prove the following result.

THEOREM 3.4. *For every $\mathbf{a} \in A$ and sequence of mesh lengths $h \to 0$, a subsequence U_h of approximate solutions converges uniformly in L^1_{loc} to a function $U(x, t) \equiv U(x, t; \mathbf{a})$; i.e., $U_h(\cdot, t)$ converges to $U(\cdot, t)$ in $L^1[-M, M]$, uniformly on $0 \leqslant t \leqslant T$ for any positive M, T. Moreover, $\|u(\cdot, t)\|_{L^1_{\mathrm{loc}}}$ is a continuous function of t.*

We use the following result of Oleinik [1]:

LEMMA 3.5. *Let h denote a positive sequence tending to zero, and let W_h: $(x, t) \to R^n$ be a sequence of functions satisfying*

$$TV\{W_h(\cdot, t)\} < M_1 < \infty \qquad \text{for all } h,\ t, \tag{3.23}$$

$$\|W_h(\cdot, t) - W_h(\cdot, \tau)\|_{L^1} \leqslant M\{|t - \tau| + h\}. \tag{3.24}$$

Then a subsequence of the functions W_h converges uniformly in L^1_{loc} to a function $W(\cdot, t)$. Moreover, $W(\cdot, t)$ satisfies

$$\|W(\cdot, t) - W(\cdot, \tau)\|_{L^1} \leqslant M|t - \tau|. \tag{3.25}$$

Proof of Theorem 3.4. The approximate solutions $W_h(x, t) \equiv W_h(x, t; \mathbf{a})$ satisfy the conditions of Lemma 3.5 for all $\mathbf{a} \in A$. The first condition follows from Lemma 3.2 with $M_1 = MM_0$. To see that the second condition is satisfied, let t_{j1} and t_{j2} be the times with

$$\tau - k \leqslant t_{j1} \leqslant \tau < t \leqslant t_{j2} \leqslant t + k.$$

Then for $x \in (x_i, x_{i+1}]$,

$$|W_h(x, t) - W_h(x, \tau)| \leqslant \sum_{j=j1}^{j2} \{|\gamma_{ij}|_z + |C_{ij}|_c\}.$$

Therefore,

$$\int_{-\infty}^{\infty} |W_h(x, t) - W_h(x, \tau)|\, dx$$

$$\leqslant \sum_{i=-\infty}^{\infty} h \sum_{j=j_1}^{j2} \{|\gamma_{ij}|_z + |C_{ij}|_c\}$$

$$= h \sum_{j=j1}^{j2} \left\{ \sum_i |\gamma_{ij}|_z + \sum_i |C_{ij}|_c \right\}$$

$$\leqslant h \sum_{j=j1}^{j2} \{F(j) + M_0\} \leqslant MM_0 h(j2 - j1)$$

$$\leqslant MM_0\{|t-\tau| + h\}.$$

Thus (3.24) holds for the approximate solutions.

We now show that V_h also converges to V uniformly in L^1_{loc}. Note that Ψ^{-1} is continuous with compact domain; so it is uniformly continuous. Suppose $T > 0$. For $0 \leqslant t \leqslant T$, let

$$E \equiv E(M, h, t, \mu)$$
$$= \{x \in [-M, M]: |W_h(x, t) - W(x, t)| \leqslant \mu\}.$$

Then

$$\int_{-M}^{M} |V_h - V|\, dx = \int_E + \int_{[-M,M]\setminus E}.$$

But $\int_E \leqslant 2M\delta(\mu)$, which is uniformly small in $[0, T]$. (Here, δ is the modulus of continuity of Ψ.) Also

$$\int_{[-M,M]\setminus E} |V_h - V| \leqslant 2(n+1)\, m\{[-M, M]\setminus E\}$$

$$\leqslant \frac{2(n+1)}{\mu} \int_{-M}^{M} |W_h - W|\, dx. \tag{3.26}$$

The last member of (3.26) is uniformly small for h small because $W_h \to W$ uniformly in L^1_{loc}. Consequently the convergence of V_h to V is uniform in L^1_{loc}. Since Φ is also uniformly continuous, U_h also converges to U uniformly in L^1_{loc}. The uniform continuity of Ψ and Φ together with (3.25) imply that $\|U(\cdot, t)\|_{L^1_{loc}}$ is a continuous function of time. This completes the proof of Theorem (3.4).

3.4. *The Weak Solution*

We say that $U(x, t)$ is a weak solution of (1.3) if it satisfies

$$\mathscr{D}(U, \phi) \equiv \iint_{\substack{-\infty < x < +\infty \\ t \geqslant 0}} U\phi_t + gU\phi_x\, dx\, dt + \int_{-\infty}^{\infty} U(x, 0)\, \phi(x, 0)\, dx = 0 \tag{3.27}$$

for every smooth ϕ with compact support. Let

$$D(h, \mathbf{a}, \phi) \equiv \mathscr{D}(U_h(\cdot, \cdot; \mathbf{a}), \phi). \tag{3.28}$$

THEOREM 3.6. *Suppose $U_0(x)$ is initial data satisfying*

$$U_0(x) = \Phi(V_0(x)) \tag{3.29}$$

and

$$TV\{W_0(\cdot)\} < M_0 < \infty. \tag{3.30}$$

Then there exists a set $N \subseteq A$, $m\{N\} = 0$, such that if $\mathbf{a} \in A \backslash N$, then the function $U(x, t; \mathbf{a})$ of Theorem (3.5) satisfies (3.27) for all ϕ of compact support.

Proof of Theorem 3.6. Let $h \to 0$ be a sequence of mesh lengths such that $U_h(x, t; \mathbf{a}) \to U(x, t; \mathbf{a})$ in the sense of Theorem 3.4. We will show that there exists a set N of measure zero in A such that if $\mathbf{a} \in A \backslash N$, then

$$D(h, \mathbf{a}, \phi) \to 0$$

for all smooth ϕ of compact support. This completes the proof because U_h converges to U uniformly in L^1_{loc}.

Since U_h is an exact solution on $[t_j, t_{j+1})$, we can integrate by parts in (3.27) to write

$$D(h, \mathbf{a}, \phi) = \sum_{i,j} D_{ij}(h, \mathbf{a}, \phi) \tag{3.31}$$

where

$$D_{ij}(h, \mathbf{a}, \phi) = \int_{x_i}^{x_i + h} [U(x, t_j+) - U(x, t_j-)]\, \phi(x, t_j)\, dx. \tag{3.32}$$

Thus

$$D_{ij}(h, \mathbf{a}, \phi) \leqslant h \|\phi\|_\infty |\gamma_{ij}|. \tag{3.33}$$

Let ϕ_{ij} be the constant function $\phi \equiv \phi(x_i, t_j)$. Then

$$\begin{aligned} D(h, \mathbf{a}, \phi) &= \sum_{ij} D_{ij}(h, \mathbf{a}, \phi) \\ &= \sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij}) + \sum_{ij} D_{ij}(h, \mathbf{a}, \phi - \phi_{ij}). \end{aligned} \tag{3.34}$$

But

$$\left|\sum_{ij} D(h, \mathbf{a}, \phi - \phi_{ij})\right| \leqslant \sum_{j \in \mathscr{J}} \|\nabla\phi\|_{\infty} h^2 \sum_{i \in \mathscr{I}} |\gamma_{ij}| \tag{3.35}$$

where

$$\mathscr{I} = \{i\colon \exists j \text{ with } [x_i - h, x_i + h] \times [t_j, t_j + k] \cap \operatorname{supp}(\phi) \text{ not empty}\},$$

$$\mathscr{J} = \{j\colon \exists i \text{ with } [x_i - h, x_i + h] \times [t_j, t_j + k] \cap \operatorname{supp}(\phi) \text{ not empty}\}.$$

Also, set

$$\mathscr{S} = \mathscr{I} \times \mathscr{J}.$$

By Lemma 3.3, with

$$\mathscr{K}_j = \{i\colon (i, j) \in \mathscr{K}\},$$

we obtain

$$\sum_{i \in \mathscr{I}} |\gamma_{ij}| = \sum_{i \in \mathscr{I} \cap \mathscr{K}_j} + \sum_{i \in \mathscr{I} \cap \mathscr{K}_j^c} \leqslant 3\varepsilon M h^{-1} + M(\varepsilon) F(j) \leqslant 3\varepsilon M h^{-1} + M(\varepsilon)$$

where we have applied Lemma 3.2. (We now allow the generic constant M to depend on the test function ϕ.) Thus by (3.35),

$$\left|\sum_{ij} D(h, \mathbf{a}, \phi - \phi_{ij})\right| \leqslant M \|\nabla\phi\|_{\infty}\{3\varepsilon + M(\varepsilon) h\}.$$

This implies that

$$\left|\sum_{ij} D(h, \mathbf{a}, \phi - \phi_{ij})\right| \to 0 \tag{3.36}$$

as $h \to 0$ as follows: for any $\delta > 0$, choose $\varepsilon < \delta/\{6M \|\nabla\phi\|_{\infty}\}$. Then for all $h < \delta/\{6M(\varepsilon) M \|\nabla\phi\|_{\infty}\}$ we have

$$\left|\sum_{ij} D(h, \mathbf{a}, \phi - \phi_{ij})\right| \leqslant \delta.$$

Consider now the first sum in the right-hand side of (3.34). We have

$$\begin{aligned} &\int_A \left(\textstyle\sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij})\right)^2 d\mathbf{a} \\ &\quad = \sum_{\substack{ij \\ kl}} \int_A D_{ij}(h, \mathbf{a}, \phi_{ij})\, D_{kl}(h, \mathbf{a}, \phi_{kl})\, d\mathbf{a} \\ &\quad = \sum_{ij} \int_A D_{ij}(h, \mathbf{a}, \phi_{ij})^2\, d\mathbf{a} = \int_A \sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij})^2\, d\mathbf{a} \end{aligned} \tag{3.37}$$

since

$$\int_A D_{ij} D_{kl}\, d\mathbf{a} = 0 \qquad \text{for} \quad (i, j) \neq (k, l),$$

by the independence of sampling points. But for any $\mathbf{a} \in A$,

$$\begin{aligned}
&\sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij})^2 \\
&\quad \leqslant \sum_{(i,j) \in \mathscr{S}} \|\phi\|_\infty^2 \, h^2 |\gamma_{ij}|^2 \\
&\quad = \sum_{(i,j) \in \mathscr{S} \cap \mathscr{K}} + \sum_{(i,j) \in \mathscr{S} \cap \mathscr{K}^c} \\
&\quad \leqslant \sum_{(i,j) \in \mathscr{S}} \|\phi\|_\infty^2 \, h^2 3^2 \varepsilon^2 + \sum_{(i,j) \in \mathscr{S}} \|\phi\|_\infty^2 \, h^2 M(\varepsilon)^2 (|\gamma_{ij}|_z + |C_{ij}|_{\mathbf{c}})^2 \qquad (3.38)
\end{aligned}$$

where we have applied Lemma 3.3. But with $\varepsilon < 1$ we obtain

$$\sum_{(i,j) \in \mathscr{S}} \|\phi\|_\infty^2 \, h^2 3^2 \varepsilon^2 \leqslant M\varepsilon,$$

and

$$\begin{aligned}
&\sum_{(i,j) \in \mathscr{S}} \|\phi\|_\infty^2 \, h^2 M(\varepsilon)^2 (|\gamma_{ij}|_z + |C_{ij}|_{\mathbf{c}})^2 \\
&\quad \leqslant \sum_{j \in \mathscr{J}} M(\varepsilon)\, h^2 \sum_i (|\gamma_{ij}|_z + |C_{ij}|_{\mathbf{c}})^2 \\
&\quad \leqslant \sum_{j \in \mathscr{J}} M(\varepsilon)\, h^2 \left\{ \sum_i (|\gamma_{ij}|_z + |C_{ij}|_{\mathbf{c}}) \right\}^2 \\
&\quad \leqslant \sum_{j \in \mathscr{J}} M(\varepsilon)\, h^2 F(j)^2 \leqslant M(\varepsilon) h.
\end{aligned}$$

Thus from (3.38),

$$\sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij})^2 \leqslant M\varepsilon + M(\varepsilon) h.$$

Using this estimate in (3.37) yields

$$\int_A \left(\sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij}) \right)^2 \leqslant M\varepsilon + M(\varepsilon) h. \qquad (3.39)$$

Now as in (3.36), (3.39) implies that

$$\int_A \left(\sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij})\right)^2 d\mathbf{a} \to 0 \tag{3.40}$$

as $h \to 0$. Therefore there exists a set N of measure zero in A and a subsequence $h \to 0$ such that, if $\mathbf{a} \in A \backslash N$, then

$$\sum_{ij} D_{ij}(h, \mathbf{a}, \phi_{ij}) \to 0 \tag{3.41}$$

as $h \to 0$. Therefore

$$D(h, \mathbf{a}, \phi) \to 0$$

as $h \to 0$ by (3.36), (3.41), and (3.34). This achieves the desired result for a particular ϕ, since N depends on ϕ. To extend this to arbitrary $\phi \in C_0^1$, we can apply the diagonal process in a direct manner as in [1, 7]. This completes the proof.

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