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Sup-norm Estimates in Glimm's Method*

BLAKE TEMPLE

Department of Mathematics, University of California, Davis, Davis, California 95616

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We give a proof that Total Variation $\{u_0(\cdot)\} \ll 1$ can be replaced by $\text{Sup}\{u_0(\cdot)\} \ll 1$ in Glimm's method whenever a coordinate system of Riemann invariants is present. The argument is somewhat simpler but in the same spirit as that given by Glimm in his celebrated paper of 1965.

We consider the Cauchy problem

$$u_t + F(u)_x = 0, \tag{1}$$

$$u(x, 0) = u_0(x), \tag{2}$$

where (1) denotes a strictly hyperbolic system of two conservation laws, $u = (u_1, u_2)$, $F = (f, g)$. Let λ_p, R_p, $p = 1, 2$ be the eigenvalues and corresponding eigenvector fields associated with the matrix ∇F, $\lambda_1 < \lambda_2$. Assume that $\mathscr{U}$ is a neighborhood of a state $\bar{u}$ in u-space in which each characteristic field is either genuinely nonlinear ($\nabla\lambda_p \cdot R_p > 0$) or else linearly degenerate ($\nabla\lambda_p \cdot R_p \equiv 0$), and such that $\lambda_1(u) < \lambda_2(v)$ for all $u, v \in \mathscr{U}$[4]. Without loss of generality, assume $\bar{u} = 0$. In this note we give a simplified proof of the following sup-norm estimate which is contained in the results of Glimm [2] and which is required for the proof in [12]. (A stronger result also follows from the analysis in [3] which, however, involves the theory of approximate characteristics and is much more technical.) We comment that a corresponding sup-norm estimate for more than two equations is all that is required in order to extend the results in [12, 13] to systems of more than two equations. We anticipate that the ideas here will help in the proof of such an estimate. Let $u(x, t)$ denote a weak solution of (1), (2) which is a limit of approximate solutions generated by the random choice method of Glimm.

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THEOREM 1. *For every $V_0 > 0$ there exists a small constant $\delta > 0$ and a large constant $G > 0$ such that, if*

$$TV\{u_0(\cdot)\} < V_0, \tag{3}$$

and

$$\|u_0(\cdot)\|_S < \delta, \tag{4}$$

then

$$\|u(\cdot, t\|_S \leq G\,\|u_0(\cdot)\|_S \tag{5}$$

for all $t > 0$. Here

$$\|u_0(\cdot)\|_S \equiv \text{Sup}\{u_0(\cdot)\}. \tag{6}$$

It suffices to verify Theorem 1 for any approximate solution u^h generated by the random choice method. Recall that there exists a coordinate system of Riemann invariants for (1) in a neighborhood of $u = 0$. (Indeed, Theorem 1 and the proof to follow apply to any system satisfying the above assumptions and for which there exists a coordinate system of Riemann invariants [2].) Moreover, the Riemann problem is uniquely solvable in a neighborhood of $u = 0$ by the method of Lax [4]. Such a solution consists of a 1-wave γ^1 followed by a 2-wave γ^2 each of which is either a shock wave or a rarefaction wave. Assume that $\mathscr{U}$ is a neighborhood of $u = 0$ satisfying all of the above conditions. Define the strength of a wave $|\gamma^p|$ to be the absolute value of the change in the opposite Riemann invariant between the left and right states of the wave. Finally, in order to set notation, we briefly review the construction of the random choice method approximate u^h.

Let h be a mesh length in x, and let

$$k = Ch$$

be the corresponding mesh length in t, $C > \text{Sup}_{p,\, u \in \mathscr{U}}\ \{|\lambda_p(u)|\}$. For $i, j \in Z$, $j \geqslant 0$, let $x_i \equiv ih$, $t_j \equiv jk$. Let $\mathbf{a}$ be a simple sequence, $\mathbf{a} \equiv \{a_j\}_{j=1}^{\infty}$, $0 < a_j < 1$. For given initial data $u_0(\cdot) \subset \mathscr{U}$, define the random choice method approximate solution $u^h(x, t) \equiv u^h(x, t; \mathbf{a})$ by induction on j as follows: First, for $x_i \leq x < x_{i+1}$, define

$$u^h(x, 0) \equiv u_0^h(x) = u_0\left(x_i + \frac{h}{2}\right).$$

Next, assume for induction that $u^h(x, t)$ has been defined for $t < t_j$. Define

$$u^h(x, t_j) \equiv u^h(x_i + a_j h, t_j -),$$

and for $t_j < t < t_{j+1}$, define $u^h(x, t)$ to be the solution of the Riemann problem posed in (3.3) at time t_j. By (3.1), u^h is well defined so long as $u^h(x, t_j) \subseteq \mathscr{U}$ for all t_j. Let γ_{ij}^p denote the p-wave that appears in the solution of the Riemann problem that is posed at (x_i, t_j) in the approximate solution u^h. Recall that the quadratic functional associated with u^h is defined by

$$Q(t) \equiv \sum |\gamma_{ij}^p| \, |\gamma_{i'j}^{p'}|, \tag{7}$$

where the sum is over all waves that approach at time t_j, and $t_j \leqslant t < t_{j+1}$. Let Δ_{ij} denote the interaction diamond centered at (x_i, t_j), and let D_{ij} denote the products of approaching waves that enter Δ_{ij} [12]. We use the following notation:

$$V_j \equiv \sum_{p,\, i} |\gamma_{ij}^p|, \tag{8}$$

$$Q_j \equiv Q(t_j) = Q(t_j+), \tag{9}$$

$$D_j \equiv \sum_i D_{ij} \tag{10}$$

$$F_j \equiv V_j + Q_j \tag{11}$$

$$S_j \equiv \|u^h(\cdot, t_j)\|_S. \tag{12}$$

Note that V_j estimates the total variation of $u^h(\cdot, t_j)$ and that (7), (9) give immediately that

$$Q_j \leqslant V_j^2. \tag{13}$$

We show that Theorem 1 is a consequence of the following lemma which is a restatement of results in [2];

LEMMA 1. *There exists a constant $G_0 > 1$ depending only on F such that, if $u^h(x, t) \in \mathscr{U}$ for all $t \leq t_j$, then*

$$V_{j+1} - V_j \leqslant G_0 S_j D_j, \tag{14}$$

$$S_{j+1} - S_j \leq G_0 S_j D_j, \tag{15}$$

$$Q_{j+1} - Q_j \leq \{G_0 S_j V_j - 1\} \, D_j. \tag{16}$$

Proof of Theorem 1. Fix $V_0 > 1$. Choose

$$S_0 < \{(8GV_0^2 \, e^{2G_0 V_0^2})\}^{-1} = \delta, \tag{17}$$

where G_0 is large enough so that

$$\{u: |u| < G_0^{-1}\} \subseteq \mathscr{U}. \tag{18}$$

We show by induction that (17) implies

$$S_j \leq (e^{2G_0 V_0^2})\, S_0, \tag{19}$$

and

$$G_0 S_j V_j \leqslant \tfrac{1}{4} \tag{20}$$

for all $j \geqslant 0$. Note that (19) gives (5) with $G = \exp(2G_0 V_0^2)$. Also note that (17), (19), and (20) imply

$$G_0 S_j \leqslant \tfrac{1}{4}. \tag{21}$$

Thus, since (14) and (16) give

$$Q_{j+1} - Q_j \leq \{G_0 S_j V_j - 1\}\, D_j, \tag{22}$$

$$F_{j+1} - F_j \leq \{G_0 S_j + G_0 S_j V_j - 1\}\, D_j, \tag{23}$$

estimates (19) and (20) also imply that

$$Q_{j+1} - Q_j \leq -\tfrac{1}{2} D_j, \tag{24}$$

$$F_{j+1} - F_j \leq -\tfrac{1}{2} D_j. \tag{25}$$

We now verify (19) and (20) by induction. The idea here is that (20) guarantees that both $\{Q_j\}$ and $\{F_j\}$ are decreasing. The decreasing of Q_j controls S_j at the induction step, while the decreasing of $\{F_j\}$ maintains (20) at the induction step, since then $V_j \leq F_j \leq F_0$.

First, when $j = 0$, (19) and (20) follow from (17). So assume (19), (20) hold for $j' \leqslant j$. We verify (19), (20) for $j' = j+1$. By (24) and (15),

$$S_{k+1} - S_k \leq 2G_0 S_k [Q_k - Q_{k+1}],$$

or

$$S_{k+1} \leq \{1 + 2G_0[Q_k - Q_{k+1}]\}\, S_k, \tag{26}$$

for $k \leq j$. Thus by (26)

$$S_{j+1} \leq \prod_{k=0}^{j} \{1 + 2G_0[Q_k - Q_{k+1}]\}\, S_0. \tag{27}$$

But one can easily verify that the maximum of

$$\prod_{k=0}^{j} (1+a_k)$$

over all nonnegative sequences $\{a_k\}_{k=0}^{j}$ satisfying $\sum_{k=0}^{j} a_k \le M$ is attained when $a_k = M/(j+1)$ for all k. Thus by (13) and (24),

$$\sum_{k=0}^{j} [Q_k - Q_{k+1}] \le Q_0 \le V_0^2, \tag{28}$$

so

$$\prod_{k=0}^{j} \{1 + 2G_0[Q_k - Q_{k+1}]\} \le \left\{1 + \frac{2G_0 V_0^2}{j+1}\right\}^{j+1} \le e^{2G_0 V_0^2}. \tag{29}$$

Therefore by (27),

$$S_{j+1} \le (e^{2G_0 V_0^2})\, S_0,$$

which verifies (19) at $j+1$. Moreover, (25) implies

$$V_{j+1} \le F_{j+1} \le F_0 \le 2V_0^2.$$

Thus by (17),

$$G_0 S_{j+1} V_{j+1} \le 2Ge^{2G_0 V_0^2}\, V_0^2 \le \tfrac{1}{4},$$

which verifies (20) at $j+1$. This completes the proof of Theorem 1.

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