SECTION-11
Numerical Approximations and Entropy

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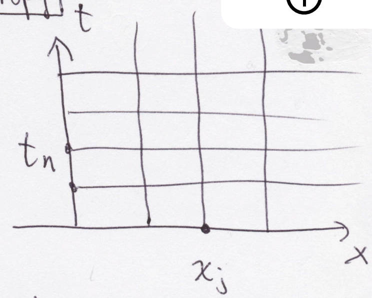
Math-280: A Mathematical Introduction to Shock Waves

Blake Temple, UC-Davis

Numerical Approx to (CL) & Entropy

- Approx soln on grid

$$x_j = jh$$
$$t_n = nk$$



Let $u(x,t)$ be exact soln to

$$u_t + f(u)_x = 0$$

$U(x,t)$ be approx soln.

$$u^n_j = u(x_n, t_j)$$
$$U^n_j = U(x_n, t_j)$$

Assume at start an $n \times n$ conr law $u_t + f(u)_x = 0$ with convex entropy $\eta_t + \psi_x = 0$

Really: $h_\ell, k_\ell \quad \ell \to \infty, \; h_\ell, k_\ell \to 0$.

U^n = vector of approx values $(U^n_j)_{j=-\infty}^{\infty}$

- Explicit schemes:

$$U^{n+1} = \mathcal{H}_k(U^n)$$
$$U^{n+1}_j = \mathcal{H}_k(U^n; j)$$

Ex: Upwind Scheme for $u_t + u u_x = 0$

$$U^{n+1}_j = U^n_j - \frac{k}{h} U^n_j (U^n_j - U^n_{j-1})$$

(very bad)

Ex: Lax-Friedrichs Scheme for $u_t + f(u)_x = 0$:

$$\underbrace{\frac{U^{n+1}_j - \frac{U^n_{j-1} + U^n_{j+1}}{2}}{k}}_{\text{approx } u_t} + \underbrace{\frac{f(U^n_{j+1}) - f(U^n_{j-1})}{2h}}_{\text{approx } u_x} = 0$$

③

(LaFr) Solve for U_j^{n+1}:

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{k}{2h}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

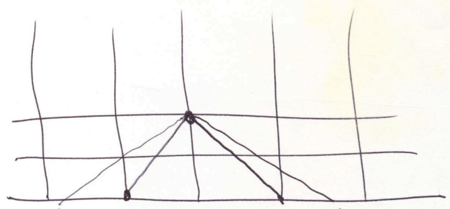
Stencil 

- CFL condition (1928) (Courant-Friedrichs-Levy)

A FDS is unstable if the domain of dependence of FDS does not contain the domain of dep of the PDE.

<u>Ex</u> (LaF)

"Chng i-data & get no chng in approx"



Domain of dep of $u_t + f(u)_x = 0$ det's by $\max|\lambda_i|$

Need: $\frac{h}{k} > \max|\lambda_i|$

④

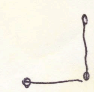
Q: When does a FDS compute the shock discontinuities correctly?

Q1: Do the FDS approximations converge to a weak soln of CL?

Q2: Does the weak soln satis the entropy condt?

<u>Ex</u>: Not a mute point –

Consider Nonlinear upwind scheme

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n \left(U_j^n - U_{j-1}^n\right)$$ 

Compute soln to $u_t + uu_x = 0$ starting from i-data

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Numerical Soln: $U_j^{n+1} = U_j^n = \cdots = U_j^0 = 0 \quad j > 0$

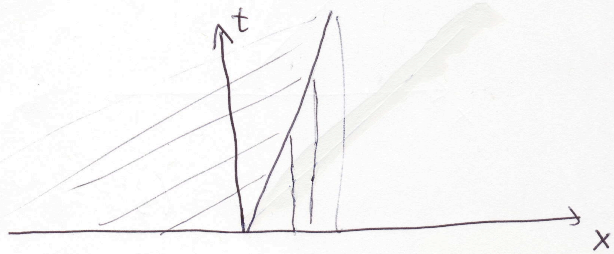
$U_j^{n+1} = U_j^n = \cdots = U_j^0 = 1 \quad j < 0$

⑤

• True Soln: $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ $\quad f(u) = \frac{1}{2}u^2$

$u_R = 0,\ u_L = 1 \Rightarrow$ shock of speed

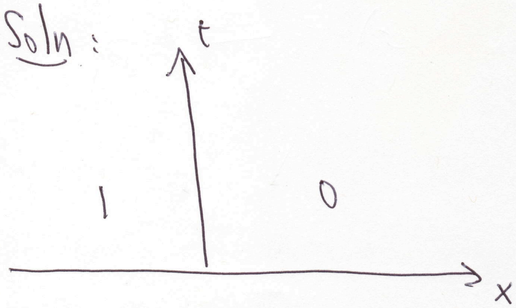
$$\frac{f(1)-f(0)}{1} = \frac{1}{2}$$



$u \in [0,1] \Rightarrow 0 \le \lambda = f'(u) = u \le 1$

∴ CFL met if $\frac{h}{k} > 1$

• Numerical Soln:



$\Rightarrow$ wrong shock speed $\Rightarrow$ does not conv.

⑥

• Consider (LaFr)

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{k}{2h}\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

Rewrite as:

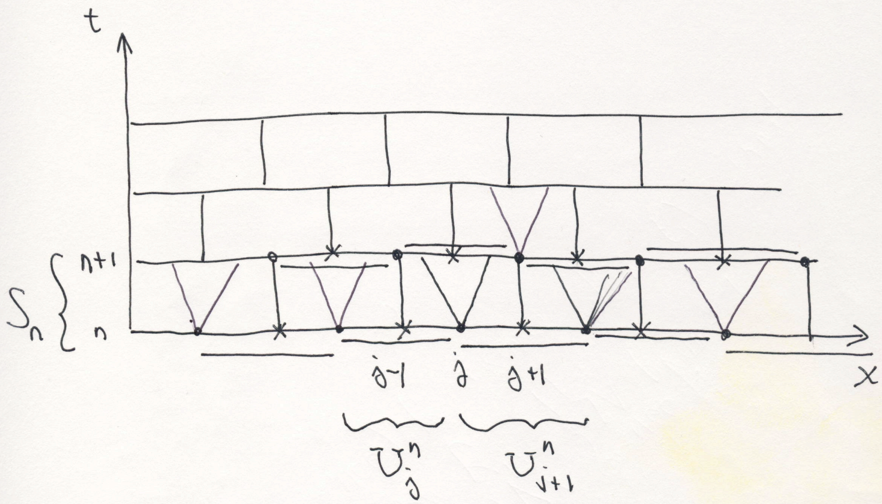
$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{f(U_{j+1}^n) - f(U_{j-1}^n)}{2h} = \underbrace{\frac{1}{2k}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)}_{(*)}$$

$$(*) = \frac{h}{2k} \cdot \underbrace{\left\{ \frac{\frac{U_{j+1}^n - U_j^n}{h} - \frac{U_j^n - U_{j-1}^n}{h}}{h} \right\}}_{\text{An approx to } u_{xx}} h$$

$\approx$ solves $\quad u_t + f_x = \frac{h}{2k} u_{xx} \cdot h$

Might expect as $h \to 0$ get weak soln's that are limits of visiosity method $\Rightarrow$ expect to solve entropy condt.

• Claim: The LaFr scheme results by "solving RP's on a staggered grid"



• Assume pw constant states at time t_n

Approx Soln: $U(x,t_n) = U^n_{j-1}$ for $x_{j-2} < x \le x_j$

$U(x,t_n) = U^n_{j+1}$ for $x_j < x \le x_{j+2}$

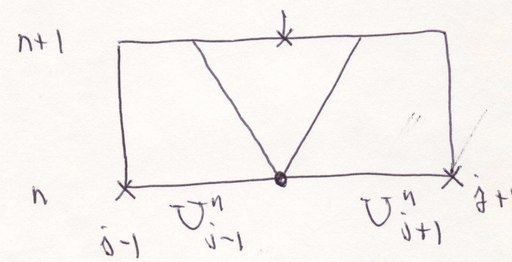
$U(x,t_{n+1}) = U_j^{n+1}$ for $x_{j-1} < x \le x_{j+1}$

That is: U_j^n defined only for $n+j$ even centered on the x's

• Fill in the rectangles by solving the RP's posed at the dot's to get $U(x,t)$ for $t_n < t < t_{n+1}$

• Assume CFL in form $\frac{h}{k} > 2\,\mathrm{Max}|\lambda_j|$ so waves never hit sides of rectangles @ each time step

• Consider rectangle R_{jn} ≡ bottom center (x_j, t_n)



• Now $U(x,t)$ is a weak soln of (CL) $u_t + f(u)_x = 0$ in R_{jn}

- Assume first that $U(x,t)$ is a smooth soln in R_{jn}

$$\Rightarrow 0 = \iint_{R_{jn}} U_t + f(U)_x \, dx\,dt = \iint_{R_{jn}} \operatorname{div}(U, f(U))\, dx\,dt$$

$$= \int_{\partial R_{jn}} (U, f(U)) \cdot \vec{n}\, ds$$

$$= \underbrace{\int_{x_{j-1}}^{x_{j+1}} U(x,t_{n+1})\,dx}_{\text{top}} - \underbrace{\int_{x_{j-1}}^{x_{j+1}} U(x,t_n)\,dx}_{\text{bottom}}$$

$$+ \underbrace{\int_{t_n}^{t_{n+1}} f(U(x_{j+1},t))\,dt}_{\text{R-side}} - \underbrace{\int_{t_n}^{t_{n+1}} f(U(x_{j-1},t))\,dt}_{\text{L-side}}$$

But $U(x,t) = U^n_{j+1}$ on R-side, U^n_{j-1} on L-side

$$\int_{x_{j-1}}^{x_{j+1}} U(x,t_{n+1})\,dx = h(U^n_{j-1} + U^n_{j+1})$$

$$+ k\left(f(U^n_{j+1}) - f(U^n_{j-1})\right)$$

Assuming U^n_j generated by LaFr scheme $\Rightarrow$

$$U^{n+1}_j = \frac{1}{h}\int_{x_{j-1}}^{x_{j+1}} U(x,t_{n+1})\,dx \qquad (*)$$

$\Rightarrow$ "Filling in the LaFr n+j even with RP's in each strip is consistent."

HW. Show that $(*)$ holds so long as RP inside R_{jn} consists of pw smooth soln separated by finite # shocks that satisfy RH jump condts. (Assume char's are st lines emanating from center, break up into smooth integrals, & apply RH at ∂'s)

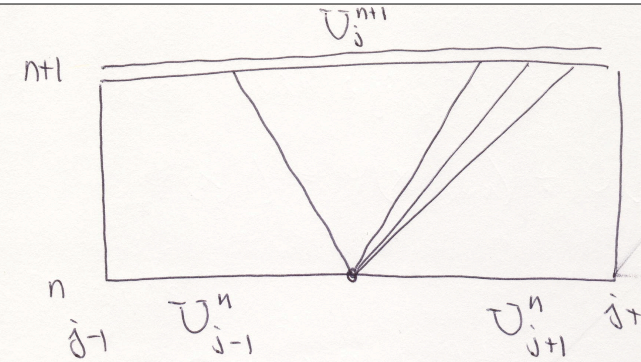
(11)

<u>Theorem</u>: Assume $\bar{U}(x,t)$ generated by LaFr with $n+j$ even as above. Assume (CL) ~~has~~ is strictly hyp & gen nonlinear or linearly degen in each char field in a nbhd $\mathcal{U}$ of u-space & assume Lax's soln of RP applies in $\mathcal{U}$. Assume that all states U_j^n in $\bar{U}(x,t)$ lie in $\mathcal{U}$. Then (LaFr) approx soln satisfies the entropy inequality

$$\eta(\bar{U})_t + \psi(\bar{U})_x \le O(h)$$

in the weak sense.

(12)



Need following assumptions about stability of approx soln's:

① $|\bar{U}(x,t_{n+1}+) - \bar{U}(x,t_{n+1}-)| \le O(1)|\bar{U}_{j+1}^n - \bar{U}_{j-1}^n|$

$\forall \quad x_{j-1} < x < x_{j+1}$

(This follows from stability of Lax RP)

② Total Variation Bound $\left(\begin{matrix}\text{Not yet proven}\\ \text{in general}\end{matrix}\right)$

$$\sum_{\substack{j=-\infty\\ n+j \text{ even}}}^{\infty} |\bar{U}_{j+1}^n - \bar{U}_{j-1}^n| < \bar{V} < \infty \quad \forall\, n > 0$$

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Proof: Let ϕ be a smooth pos test fn. of compact supp in $t>0$, $x\in\mathbb{R}$. We prove

$$-\iint_{\substack{t>0\\ -\infty<x<\infty}} \eta(\bar{U})\phi_t + \psi(\bar{U})\phi_x \,dx\,dt \le 0.$$

- $\bar{U}(x,t)$ is an exact weak soln in each strip $t_n<t<t_{n+1}$, thus

$$-\iint_{\substack{t>0\\ -\infty<x<\infty}} \eta(\bar{U})\phi_t + \psi(\bar{U})\phi_x \,dx\,dt$$

$$= -\sum_{n=0}^{\infty} \iint_{S_n} \eta(\bar{U})\phi_t + \psi(\bar{U})\phi_x \,dx\,dt$$

$$= +\sum_{n=0}^{\infty} \underbrace{\iint_{S_n} \eta(\bar{U})_t\phi + \psi(\bar{U})_x\phi \,dx\,dt}_{\le 0 \,\sim\, \text{as before in HW}} +$$

=0 if exact str soln in S_n, ≤ 0 ow

$$+\sum_n \int_{-\infty}^{\infty} [\eta(\bar{U})(x,t_n+) - \eta(\bar{U})(x,t_n-)]\,\phi(x,t)\,dx$$

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$$+\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\left[\eta(\bar{U}(x,t_{n+1}+)) - \eta(\bar{U}(x,t_{n+1}-))\right]\phi(x,t)\,dx$$

$$\le \sum_{\substack{n\\ n+j \text{ even}}}\sum_{j} \underbrace{\int_{x_{j-1}}^{x_{j+1}} [\eta(\bar{U}(x,t_{n+1}+)) - \eta(\bar{U}(x,t_{n+1}-))]\phi(x,t)\,dx}_{(*)}$$

Now:

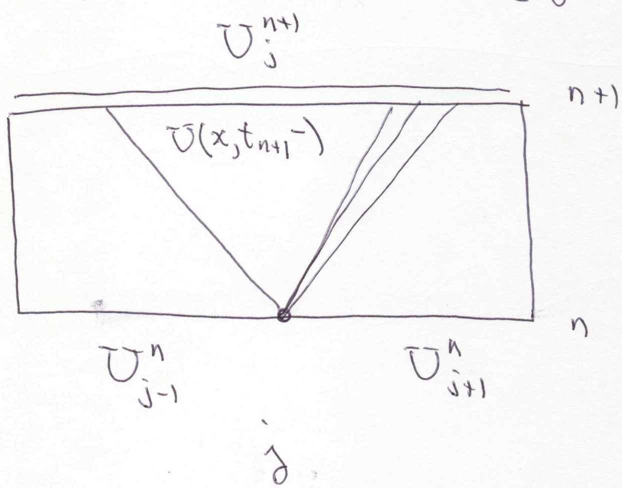
$$\int_{x_{j-1}}^{x_{j+1}} \eta(\bar{U}(x,t_{n+1}+))\,dx = 2h\,\eta\left(\frac{1}{2h}\int_{x_{j-1}}^{x_{j+1}} \bar{U}(x,t_{n+1}-)\,dx\right)$$

$$\le \cancel{2h}\,\frac{1}{\cancel{2h}}\int_{x_{j-1}}^{x_{j+1}} \eta(\bar{U}(x,t_{n+1}-))\,dx \qquad (A)$$

Jensons –
$$\eta\left(\frac{1}{b-a}\int_a^b u(x)\,dx\right) \le \frac{1}{b-a}\int_a^b \eta(u(x))\,dx$$

Conclude: If $\varphi(x,t)$ were constant on Δ_{jn} we'd have:

$$\int_{x_{j-1}}^{x_{j+1}} \left[\eta(\bar U(x,t_{n+1})) - \eta(\bar U(x,t_{n+1}^-))\right]\varphi(x,t)\,dx \le 0$$



- To take advantage, write

$$\varphi(x,t) = \varphi(x_j,t_n) + O(h)$$

$\Rightarrow$ Conclude:

$$+\iint_{\substack{t>0\\ -\infty<x<\infty}} \eta(\bar U)\,\varphi_t + \psi(\bar U)\,\varphi_x \,dx\,dt$$

$$\le + \sum_{\substack{n \\ j+n\text{ even}}} \sum_j \int_{x_{j-1}}^{x_{j+1}} \left[\eta(\bar U(x,t_{n+1}^+)) - \eta(\bar U(x,t_{n+1}^-))\right] \cdot \left(\varphi(x_j,t_n) + O(h)\right) dx$$

$$\le (\text{neg}) + O(1)\, h \underbrace{\sum_n \sum_j}_{\text{in Supp}\,\varphi} \int_{x_{j-1}}^{x_{j+1}} \left|\eta(\bar U(x,t_{n+1}^+)) - \eta(\bar U(x,t_{n+1}^-))\right| dx$$

$$\le O(1)\, h \underbrace{\sum_n \sum_j}_{\text{in Supp}\,\varphi} 2h \,\mathrm{Max}\,|\nabla\eta|\; \mathrm{Max}\,|\bar U_j^{n+1} - \bar U(x,t_{n+1}^-)|$$

$$\le O(1)\, h^2 \underbrace{\sum_n \sum_j}_{\text{in Supp}\,\varphi} \underbrace{|\bar U_{j+1}^n - \bar U_{j-1}^n|}_{\le V} \le O(1)\, h^2 \cdot \frac{1}{h} V$$

$$= O(h) \checkmark$$

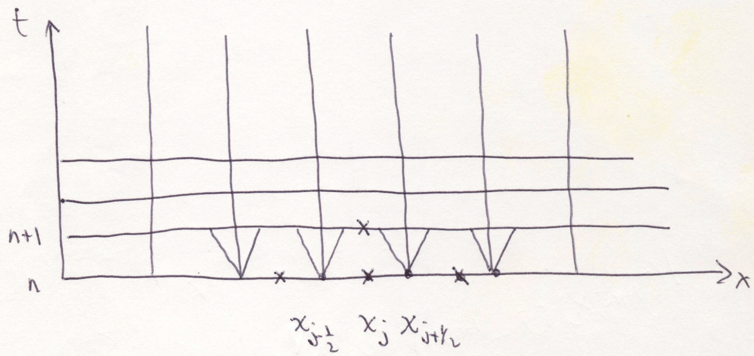
Numerical methods based on Riemann Problems

- Lax-Friedrichs:

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) + \frac{k}{2h}\left(F(U_{j+1}^n) - F(U_{j-1}^n)\right)$$

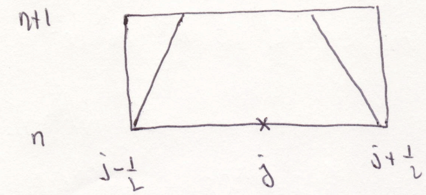
Can view U_j^{n+1} as average of soln obtained by RP's on a staggered grid. No need to solve RP!

- Godunov: U_j^{n+1} as average of soln obtained by RP's on an unstaggered grid



U_j^n = value of numerical approx $U(x,t_n)$, $x_{j-\frac{1}{2}} < x < x_{j+1/2}$

Extend to $t_n < t < t_{n+1}$ by solving RP's $\Rightarrow U(x,t)$



$$\text{"}\; 0 = \iint_{R_{jn}} U_t + f(U)_x \, dx\,dt \;\text{"}$$

$$= \underbrace{\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} U(x,t_{n+1})\,dx}_{\text{top}} - \underbrace{\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} U(x,t_n)\,dx}_{\text{bottom}}$$

$$+ \underbrace{\int_{t_n}^{t_{n+1}} f(U(x_{j+\frac{1}{2}},t))\,dt}_{\text{R-side}} - \underbrace{\int_{t_n}^{t_{n+1}} f(U(x_{j-\frac{1}{2}},t))\,dt}_{\text{L-side}}$$

$$= hU_j^{n+1} - hU_j^n + kf(u_*(U_j^n, U_{j+1}^n)) - kf(u_*(U_{j-1}^n, U_j^n))$$

$u_*(u_L, u_R)$ = state that propagates at speed $\frac{dx}{dt} = 0$ in soln of RP $[u_L, u_R]$

$$\boxed{U_j^{n+1} = U_j^n + \frac{k}{h}\left[f(u_*(U_j^n, U_{j+1}^n)) - f(u_*(U_{j-1}^n, U_j^n))\right] \quad \text{(GOD)}}$$

- Note: (God) requires soln of RP, (LaFr) does not!
- Modified (God) use some approx to soln of RP to approx u_*
- <u>Glimm's</u> <u>Method</u>: Sample instead of average (staggered or unstaggered)

- Not conservative
- Only method proven to converge