SECTION-2
Introduction to Fluid Mechanics

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Math-280: A Mathematical Introduction to Shock Waves

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FLUID MECHANICS

REF: HUGH'S & MARSDEN - "Short course in Fl. Mech" Pr
COURANT-FRIEDRICH'S "Supersonic Flow & Shock Waves"

- Assume that a fluid is moving in $\mathbb{R}^3$
Assume that particles fixed wrt fluid move with a given smooth velocity field $u(x,t)$, $x \in \mathbb{R}^3$, $u \in \mathbb{R}^3$

- If a particle is at position $a = (a_1, a_2, a_3)$ at time $t=0$, let $x(a,t) \equiv x_t(a)$ be its position at time t. Call $x(a,\cdot)$ the trajectory of the initial particle a. Thus

(ODE) $$\frac{\partial}{\partial t} x(a,t) = u(x,t) \quad , \quad x(a,0) = a$$

I.e. "particle trajectories solve ODE."

THM① If u is smooth, then (ODE) has unique soln for some t-interval

DEFN: We call a the Lagrangian coordinate of the fluid particle

DEFN: We call $x \equiv x(a,t)$ the Eulerian coordinate of fluid particles

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- Consider a scalar function $f(x,t)$ (Think of f as being the density of the fluid). Then f is a fn of (a,t) also:

$$f[a,t] = f(x(a,t),t)$$

Lagrangian coordinate — Eulerian coordinate

Moreover: $\frac{\partial f}{\partial t}[a,t]$ is the rate at which f changes in a frame fixed wrt the particle

$$\frac{\partial f}{\partial t}[a,t] = \frac{\partial f}{\partial t}(x(a,t),t) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t}$$

$$= \nabla f \cdot u + f_t \equiv \frac{Df}{Dt}$$

$\nabla f \equiv (f_{x_1}, f_{x_2}, f_{x_3})(x,t)$ — f in Eulerian coordinate

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DEFN. the material derivative of $f(x,t)$ is defined to be

$$\dot{f} \equiv \frac{Df}{Dt} \equiv f_t + \nabla f \cdot u$$

" $\dot{f}$ is the rate of change of f along particle path"

H-W#1 SHOW $\dot{(fg)} = \dot{f}g + f\dot{g}$

$$\dot{\overline{h(f(x,t),g(x,t))}} = h_f \dot{f} + h_g \dot{g}$$

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Let $\Omega(t)$ be a 3-D open set moving with the fluid

$$\Omega(t) \equiv \{x \in \mathbb{R}^3 : x = x(a,t) \text{ for some } a \in \Omega(0)\}$$

We wish to calculate

$$\frac{d}{dt} \int_{\Omega(t)} f(x,t)\,dx^3$$

PROBLEM: we can't pass the deriv. thru the $\int$ sign because the region is changing with time

THEOREM ② (REYNOLDS TRANSPORT THEOREM)

$$\frac{d}{dt} \int_{\Omega(t)} f(x,t)\,dx^3 = \int_{\Omega(t)} f_t + \operatorname{div}(fu)\,dx^3$$

BASIC IDENTITY: Let $J(a,t) = \det\left|\frac{\partial x}{\partial a}\right|$

(J measures the local volume change from $t=0$ to $t=t$)

Then

$$(J) \qquad \dot{J} = \frac{\partial}{\partial t} J(a,t) = J \operatorname{div} u$$

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CONCLUDE: the rate and which volumes are changing is proportional to div u.

HW #2 Verify (J)

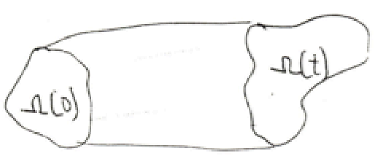
HINT: differentiate the determinant using the multilinearity of det plus the fact that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial a}x = \frac{\partial}{\partial a}\frac{\partial}{\partial t}x = \frac{\partial}{\partial a}u$$

Proof of Reynolds Transport:

$$\int_{\Omega(t)} f(x,t)\,dx^3 = \int_{\Omega(0)} f(x(a,t),t)\underbrace{J(a,t)\,da}$$

$dx = \left|\frac{\partial x}{\partial a}\right| da$ gives the volume change.



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$$\therefore \frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx = \frac{d}{dt}\int_{\Omega(0)} f(x(a,t),t)\,J(a,t)\,da$$

$$= \int_{\Omega(0)} \frac{\partial}{\partial t}\left\{ f(x(a,t),t)\,J(a,t)\right\} da$$

$$= \int_{\Omega(0)} \frac{Df}{Dt}J + f\frac{\partial}{\partial t}J(a,t)\;da$$

$$= \int_{\Omega(0)} (f_t + \nabla f\cdot u)J + (f\,\mathrm{div}\,u)J\;da$$

$$= \int_{\Omega(t)} f_t + \underbrace{\nabla f\cdot u + f\,\mathrm{div}\,u}_{\mathrm{div}(fu)}\;dx$$

$$= \int_{\Omega(t)} f_t + \mathrm{div}\,fu\;dx \quad \checkmark$$

Conservation of Mass: (Continuity Equation)

- Assume the fluid has a density

$$\rho(x,t) = \frac{\text{mass}}{\text{volume}}$$

Conservation of mass reads:

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\,dx = \int_{\Omega(t)} \rho_t + \operatorname{div}(\rho u)\,dx = 0$$

(MA) $\rho_t + \operatorname{div}(\rho u) = 0 \quad \Leftrightarrow$

Homework: Assume (MA) holds. Prove ~~the~~ the identity

(HW)#3) $$(\rho f)_t + \operatorname{div}(\rho f u) = \rho \frac{Df}{Dt}$$

hold for all smooth $f \equiv f(x,t)$.

Cor: $$\frac{d}{dt}\int_{\Omega(t)} f\rho\,dx = \int_{\Omega(t)} \rho \frac{Df}{Dt}\,dx$$

Balance of momentum

Assume: $\exists$ a stress tensor $\sigma \equiv [\sigma^{ij}] \equiv (\sigma_{ij})$ such that the force on $\partial\Omega(t)$ is given by

$$F_{\partial\Omega} = \int_{\partial\Omega} \sigma\cdot n\,ds \qquad (\sigma\cdot n = \sigma^{ij} n_j)$$

$$\left([\sigma] \sim \frac{\text{force}}{\text{area}}\right)$$

$n \equiv$ outward normal to $\partial\Omega$

(we don't say what σ depends on — this determines the equations)

Defn: $F \equiv \sigma\cdot n$ equals the force per area exerted by fluid on an area element oriented by $\vec{n}$ (The force exerted inward so $\sigma = -pI$ for perfect fluid)

- "Conservation of Momentum" means that the time rate of change of the momentum in $\Omega(t)$ equals the total force acting on $\partial\Omega$:

3 eqs
$$\frac{d}{dt}\int_{\Omega(t)} \rho u \, dx = \int_{\partial\Omega(t)} \sigma \cdot n \, ds = \int_{\Omega(t)} \operatorname{div} \sigma \, dx$$

$$\int_{\Omega(t)} \left\{ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div} \sigma \right\} dx = 0$$

(MO) $$(\rho u)_t + \operatorname{div}\left\{ (\rho u \otimes u) - \sigma \right\} = 0$$

$$u \otimes u = u \cdot u^t = \left[u^i u^j \right]$$

$(MO)^i$ $$(\rho u^i)_t + \operatorname{div}\left\{ \rho u^i u - \sigma^i \right\} = 0.$$

ith row of σ

- If (MA) holds, then (MO) can be rewritten as:

$(MO)'$ $$\rho \frac{Du}{Dt} = \operatorname{div} \sigma$$

infinitessimal version of Newton's Laws

"mass × acc" = "force"

HW #4 Derive $(MO)'$

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- **Theorem (3)** Assume (MA) & (MO) hold. Then σ is symmetric ($\sigma^{ij} = \sigma^{ji}$) iff the balance of angular momentum holds:

$$\text{(AM)} \quad \frac{d}{dt}\int_{\Omega(t)} \rho (x \times u)\, dx = \int_{\partial\Omega(t)} (x \times \sigma \cdot n)\, dA$$

I.e., $\int_{\Omega(t)} \rho (x \times u)\, dx$ is the angular momentum in $\Omega(t)$, and this changes at a rate given by the moment of the forces on the $\partial\Omega$

(PROOF OMITTED - see Hughs & Marsden)

Ex: $F = m\ddot{x}$, $\frac{d}{dt} m(x \times \dot{x}) = \underbrace{m(\cancel{\dot{x} \times \dot{x}})}_{0} + m(x \times \ddot{x})$

one particle case

$= x \times F$

- We always assume σ symmetric.

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Conservation of Energy

Assume: $E \equiv$ Energy / volume in fluid

$E = \rho e + \frac{1}{2}\rho |u|^2$

e = specific internal energy $\equiv \frac{\text{energy}}{\text{mass}}$

$e \equiv$ "the energy per mass stored in the vibrations of molecules"

$\frac{1}{2}\rho |u|^2 \equiv$ kinetic energy

Work $= \int F \cdot ds$
$= \int F \cdot u\, dt \Rightarrow \frac{d(\text{work})}{d(\text{time})}$
$= F \cdot u = \frac{\text{force} \times \text{dist}}{\text{time}}$ ✓

$\sigma^{ij} u^j n^i = (\sigma u) \cdot n$

$$\text{(En)} \quad \frac{d}{dt}\int_{\Omega(t)} E\, dx = \underbrace{\int_{\partial\Omega(t)} F \cdot u\, ds}_{\text{Rate Work done on } \Omega \text{ by boundary forces}} - \underbrace{\int_{\partial\Omega(t)} q \cdot n\, ds}_{\text{Heat flux through } \partial\Omega}$$

q = heat flux vector.

Fourier's Law: $q = -k \nabla T$; $k \equiv$ conductivity

$T \equiv$ temperature

$$\frac{d}{dt}\int_{\Omega(t)} E\,dx = \int_{\Omega(t)} \mathrm{div}(\sigma\cdot u)\,dx + \int_{\Omega(t)} k\,\mathrm{div}\,\nabla T\,dx$$

$$\int_{\Omega(t)} E_t + \mathrm{div}(Eu) - \mathrm{div}(\sigma\cdot u) - k\Delta T\,dx = 0$$

$$E_t + \mathrm{div}(Eu)$$
$$= (\rho e + \tfrac{1}{2}\rho u^2)_t + \mathrm{div}[\ldots\cdot u]$$
$$= \rho\frac{De}{Dt} + \frac{1}{2}\rho\frac{Du^2}{Dt}$$

$$\text{(En)}\quad E_t + \mathrm{div}(Eu - \sigma u) = k\Delta T$$

<u>Theorem</u> 4 Assume (MA), (MO) hold and that σ is symmetric. Then

$$\frac{d}{dt}K = \underbrace{-\int_{\Omega(t)} \sigma\bullet D\,dx}_{\substack{\text{work per time}\\ \text{generated by interior}\\ \text{stresses}}} + \underbrace{\int_{\partial\Omega(t)} F\cdot u\,ds}_{\substack{\text{work per time}\\ \text{generated by stresses}\\ \text{acting on } \partial\Omega}}$$

<u>Here</u>: $K = \int_{\Omega(t)} \frac{1}{2}\rho u^2\,dx \equiv$ kinetic energy in $\Omega(t)$

$D \equiv$ symmetric part of velocity gradient:

I.e., $u = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$ velocity

$$\nabla u = \left[\partial u^i / \partial x^j\right] = \begin{bmatrix} \nabla u^1 \\ \nabla u^2 \\ \nabla u^3 \end{bmatrix}$$

$$D = \frac{1}{2}\left[\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right] \equiv \text{symmetric part of } \nabla u$$

$$A = \frac{1}{2}\left[\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right] \equiv \text{antisymmetric part of } \nabla u$$

$$D + A = \nabla u.$$

$$\sigma\bullet D = \sum_{i,j} \sigma_{ij} D_{ij}$$

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Proof: By H.W. #4, (MO)' $\Rightarrow$

$$\rho \frac{Du}{Dt} = \operatorname{div} \sigma$$

HW #5 derive formula for integration by parts $\int_\Omega f_{x_i} g = -\int_\Omega f g_{x_i} + \int_{\partial\Omega} f g n_i \, ds$ from the divergence theorem

Thus, $\frac{d}{dt} K = \frac{d}{dt} \frac{1}{2} \int_{\Omega(t)} \rho u^2 dx \overset{\text{HW\#3}}{=} \frac{1}{2} \int_{\Omega(t)} \rho \frac{Du^2}{Dt} dx$

$$= \frac{1}{2} \int_{\Omega(t)} 2 \rho \frac{Du}{Dt} \cdot u \, dx = \int_{\Omega(t)} \operatorname{div} \sigma \cdot u \, dx$$

$$= \int_{\Omega(t)} \sigma^{ij}_{\ ,j} u^i dx \underset{\text{integrate by parts}}{=} - \int_{\Omega(t)} \sigma^{ij} u^i_{\ ,j} dx + \int_{\partial\Omega(t)} \sigma^{ij} u^i n_j ds$$

sum on j, i

$$= - \int_{\Omega(t)} \sigma^{ij} (D_{ij} + A_{ij}) dx + \int_{\partial\Omega} F \cdot u \, ds$$

(FIP)

$\sum_{ij} \sigma^{ij} A_{ij} = 0$ since A antisymmetric, D symmetric.

$\Rightarrow$ Answer.

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- **Note**: since σ, D are symmetric,

$$\sigma \cdot D \equiv \sum_{ij} \sigma^{ij} D_{ij} \equiv \sigma^{ij} D_{ij}$$

dot product

Summation convention - sum repeated indices

$$tr[\sigma \cdot D] = tr[\sigma^{ik} D_{kj}] = \sigma^{ik} D_{ki} = \sigma^{ij} D_{ij}$$

matrix product

symmetry of D or σ

$$\therefore \frac{d}{dt} K = \frac{d}{dt} \int_{\Omega(t)} \frac{1}{2} \rho u^2 dx = \int_{\Omega(t)} -tr(\sigma \cdot D) + \operatorname{div}(\sigma u) dx$$

(KE) $\frac{1}{2} \rho \frac{Du^2}{Dt} = -tr(\sigma \cdot D) + \operatorname{div}(\sigma u)$

(En) $\rho \frac{De}{Dt} + \frac{1}{2} \rho \frac{Du^2}{Dt} = \operatorname{div}(\sigma u) + k \Delta T$

(KE)+(En) $\Rightarrow$

(En)' $\rho \frac{De}{Dt} = tr(\sigma \cdot D) + k \Delta T$

Note We obtain (En)' by substitituting a formula for (KE) in terms of stress obtained from (MO). In fact, the integration by parts is simply a derivation for a vector identity:

$$\frac{1}{2}\rho\frac{Du^2}{Dt} = \rho u\cdot\frac{Du}{Dt} \underset{\uparrow}{=} u\cdot\operatorname{div}\sigma = \operatorname{div}(\sigma u) - tr(\sigma\cdot D)$$

(MO) $\rho\frac{Du}{Dt} = \operatorname{div}\sigma$

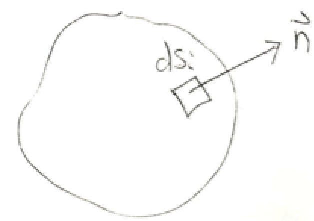
$$u\cdot\operatorname{div}\sigma = \operatorname{div}(\sigma u) - \underbrace{\nabla u\cdot\sigma}_{(D+A)\cdot\sigma = D\sigma}$$

Example ①: Perfect Fluid

Defn: A fluid is perfect if it can exert no tangential forces; Ie

$$[\sigma_{ij}] = -p[\delta_{ij}] = -pI$$

where p is a scalar called the pressure. Note:



$$F_\Omega = \int_{\partial\Omega}\sigma\cdot n\,ds \approx \sum_i F_{dS_i}$$

$$F_{dS_i} \underset{\uparrow}{=} -p\vec{n} = -p[\delta_{ij}]\vec{n}$$

"Force on Ω"

$$\Rightarrow \sigma_{ij} = -p\delta_{ij}$$

In Case of Perfect Fluid:

$$(MA)_p \quad \rho_t + \operatorname{div}(\rho u) = 0$$

$$(Mo)^i_p \quad (\rho u^i)_t + \operatorname{div}(\rho u^i u + \underbrace{p e^i}) = 0$$

$$\operatorname{div}\sigma = \operatorname{div}[-p\delta_{ij}] = -\begin{bmatrix} \operatorname{div} p e_1 \\ \operatorname{div} p e_2 \\ \operatorname{div} p e_3 \end{bmatrix} = -\nabla p$$

$e^i = e_i = (0 \cdots 1 \cdots 0)$ ← ith slot

$e^1 = e_1 = (1,0,0)$

$e^2 = e_2 = (0,1,0)$

$e^3 = e_3 = (0,0,1)$

$$(Mo)_p \quad (\rho u)_t + \operatorname{div}[\rho u \otimes u + pI] = 0$$

↑ 3x3 identity matrix

$$(Mo)'_p \quad \rho \frac{Du}{Dt} = -\nabla p \qquad \operatorname{div}(pI) = \nabla p$$

"force on a fluid element is given by the gradient of the pressure"

$$(En)_p \quad E_t + \operatorname{div}[(E+p)u] = k\Delta T$$

$$-\sigma u = pIu = pu$$

$$(KE)_p \quad \frac{1}{2}\rho \frac{Du^2}{Dt} = \underbrace{-\nabla p \cdot u}$$

$$-\operatorname{tr}(\sigma \cdot D) + \operatorname{div}(\sigma u)$$

$$\underbrace{+ p\operatorname{tr}D - \operatorname{div}(pu)}_{-\nabla p \cdot u}$$

(Changes in KE are due to accelerations caused by ∇p)

$$(En)'_p \quad \rho \frac{De}{dt} = \underbrace{-p\operatorname{div}u} + k\Delta T$$

$$\operatorname{tr}(\sigma \cdot D) = -p\operatorname{div}u$$

(Changes in internal energy are due to the compressive action of pressure forces and transfers due to heat conduction

Note ① $\dot{J} = J \operatorname{div} u$, thus

$$\operatorname{div} u = \frac{\dot{J}}{J} \equiv \frac{\text{rate at which local volumes change}}{\text{per volume}}$$

$$\operatorname{div} u \sim \frac{\Delta[\text{Vol}]}{\text{Vol}\ \Delta t}$$

$$\therefore -P \operatorname{div} u \sim \frac{\text{force}}{\Delta \text{area}} \cdot \frac{\Delta \text{Vol}}{\text{Vol} \cdot \Delta t} \sim \frac{\text{force} \cdot \Delta \text{dist}}{\text{Vol} \quad \Delta t}$$

$$\sim \frac{\Delta \text{work}}{\Delta t \ \ \text{Vol}} \sim \text{work per time}$$

done on fluid element (per volume!) due to compression by pressure forces

- We can make this precise:

Let $v = 1/\rho \equiv$ specific volume $=$ "$\frac{\text{volume}}{\text{mass}}$"

$$(MA)' \Rightarrow \frac{D\rho}{Dt} = -\rho \operatorname{div} u$$

$$(Vol) \quad \frac{Dv}{Dt} = \frac{D(1/\rho)}{Dt} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}\operatorname{div} u$$

$$\therefore \quad -p \operatorname{div} u = -\rho p \frac{Dv}{Dt}$$

and we obtain

$$(En)''_p \quad \rho \frac{De}{Dt} = -\rho p \frac{Dv}{Dt} + k \Delta T$$

- $k \Delta T \equiv$ rate at which energy is transferred into the fluid element by heat conduction due to the temperature gradient. If $k = 0$, then $(En)''$ says

$$\frac{De}{Dt} = -p \frac{Dv}{Dt}$$

– Thus we can interpret $(En)'$

$$\rho \frac{De}{Dt} = -p \,\mathrm{div}\, u + k \Delta T$$

rate at which internal energy changes in fluid element

rate at which work is done on fluid element due to compression

rate at which heat enters fluid element due to heat conduction.

Note ②: If we assume $k = 0$, then $(En)'$ says that all of the work done locally by the compressive action of pressure forces* is stored locally in the internal energy $\Rightarrow$ None of the work is converted into heat and diffused $\Rightarrow$ no viscosity or heat conduction

* (The part that doesn't chng KE)

– The five conservation laws which result when we assume $\sigma \equiv -pI$ and $k = 0$ are called the Compressible **Euler Equations**:

(MA) $\rho_t + \mathrm{div}\,\rho u = 0$

$(Mo)^i$ $(\rho u^i)_t + \mathrm{div}[\rho u^i u + p e^i] = 0$

(En) $E_t + \mathrm{div}[(E+P)u] = 0 \qquad E = \rho e + \frac{1}{2}\rho u^2$

$\Leftrightarrow$ No viscosity and no heat conduction.

5 equations

6 unknowns: $\rho, u^1, u^2, u^3, p, e$

To close system we need a constitutive relation between ρ, p, e.

Thermo: any two of ρ, p, e, T determine other two: we return to this.

Note ③ The equations are closed by giving

$$e = e(v,s), \quad (e, v, s \text{ specific quantities} = \text{per mass})$$

Then 2nd Law Thermo says

$$de = T\,ds - p\,dv \Rightarrow \frac{\partial e}{\partial s} = T, \quad \frac{\partial e}{\partial v} = -p < 0$$

defines T and p. Modelling real gases $\Rightarrow$ assumptions

$$\boxed{p_v < 0, \ p_{vv} > 0, \ e_v < 0}$$

The polytropic eqn of state

$$e(s,v) = C_v \frac{1}{v^{\gamma-1}} \exp\left(\frac{s}{C_v}\right)$$

satisfies these conditions

Theorem: If we assume 2nd Law

$$de = T\,ds - p\,dv,$$

then (MA), (MO) & (En) are equivalent to (MA), (MO) & (S) where

$$(S) \qquad S_t + \operatorname{div}(Su) = 0 \qquad\qquad S = \rho s = \frac{\text{entropy}}{\text{vol}}$$

Proof: First note that

$$S_t + \operatorname{div} Su = (\rho s)_t + \operatorname{div}(\rho s u) = \frac{Ds}{Dt},$$ so that (S) is equivalent to

$$(s) \qquad \frac{Ds}{Dt} = 0.$$

Now (HW) 2nd Law $\Rightarrow$ $\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{Dv}{Dt}$.

Since we have shown that if we assume (MA) & (MO), then (K) holds, and so then

(En) holds iff (En)$'$ holds

$$(En') \qquad \frac{De}{Dt} = -p\frac{Dv}{Dt}$$

Conclude: from 2nd Law that (En$'$) holds iff $\frac{Ds}{Dt} = 0$ holds ✓

Note 4 (FIP) By Note 2 we expect that the flow is reversible in a thermodynamical sense. To this end, note that if p is independent of u, and p is independent of t except through the variables ρ and e, then if $\rho(x,t)$, $e(x,t)$, $u(x,t)$ solve (MA), (MO) and (En), then so does $\rho(x,-t)$, $-u(x,-t)$, $e(x,-t)$ $\Rightarrow$ the flow is reversible. Thus in (En)'

$$(En)' \quad \rho \frac{De}{Dt} = -p \operatorname{div} u$$

it makes sense to call $-p \operatorname{div} u$ the <u>reversible</u> <u>work</u>.

EXAMPLE 2: Compressible Navier Stokes

(Stress tensor accounts for viscosity)

- Assume: $\sigma = -pI + \tilde{\sigma}(D)$

where $\tilde{\sigma}$ depends only on the symmetric part of velocity gradient D. (NLOG as we assume σ is symmetric)

Theorem: Assume $\tilde{\sigma}$ satisfies the following assumptions:

1. $\tilde{\sigma}$ is a smooth function of D
2. $\tilde{\sigma}(0) = 0$
3. $\tilde{\sigma}$ is isotropic (i.e., rotationally invariant)

$$\tilde{\sigma}(UDU^{-1}) = U\,\tilde{\sigma}(D)\,U^{-1} \quad \forall\, U \text{ orthog: } U^tU = id$$

4. $\tilde{\sigma}$ is linear in D.

Then

$$\tilde{\sigma} = \lambda \operatorname{div} u\, I + 2\mu D$$

for some constants λ, μ

If we take

(*) $\sigma = -pI + \lambda \operatorname{div} u\, I + 2\mu D$

then we obtain compressible Naviev Stokes:

(MA) $\rho_t + \operatorname{div} \rho u = 0$

$(Mo)_{NS}$ $(\rho u)_t + \operatorname{div}(\rho u \otimes u + pI) = (\lambda + \mu)\nabla \operatorname{div} u + \mu \Delta u$

μ called dynamic/shear viscosity

$\lambda + \mu \equiv$ bulk viscosity

k = thermal conductivity

2nd order linear in u $\approx$ dissipation term if you see this as $\approx u_t = \Delta u$

H.W. #6 Verify that (*) implies $(Mo)_{NS}$

Note: assume $\mu, \lambda \geq 0$ so RHS represents dissipation

Note: for incompressible Navier-Stokes $\rho = \text{const}$, $\operatorname{div} u = 0$ so RHS reduces to $\mu \Delta u$

(FIP)

$(En)'$ $\rho \frac{De}{Dt} = tr(\sigma \cdot D) + k\Delta T$

$\sigma = -pI + \lambda \operatorname{div} u\, I + 2\mu D$

$\sigma \cdot D = [-p + \lambda \operatorname{div} u] \cdot D + 2\mu D^2$

$tr(\sigma \cdot D) = -p \operatorname{div} u + \lambda (\operatorname{div} u)^2 + 2\mu\, tr D^2$

$(En)_{NS}$ $\rho \frac{De}{Dt} = -\rho p \frac{Dv}{Dt} + \lambda \left[\rho \frac{Dv}{Dt}\right]^2 + 2\mu |\nabla u|^2$

or

$(En)_{NS}$ $\frac{De}{Dt} = -p \frac{Dv}{Dt} + \frac{\lambda}{\rho}\left[\rho \frac{Dv}{Dt}\right]^2 + 2\frac{\mu}{\rho}|\nabla u|^2$

Defn: $-p\frac{Dv}{Dt}$ is called the reversible work (per mass), because we have reversibility when $\lambda = \mu = 0$.

Defn: $\frac{\lambda}{\rho}\left[\rho \frac{Dv}{Dt}\right]^2 + 2\frac{\mu}{\rho}|\nabla u|^2 \geq 0$ is called the irreversible work because it is always positive, and hense it represents viscous work due to stresses being turned into heat

- Using 2nd Law & $(En)_{NS}$ we can see that evolution by Navier Stokes is irreversible:

By $(En)_{NS}$,

$$\frac{De}{Dt} = -p\frac{Dv}{Dt} + \frac{\lambda}{\rho}\left[\rho\frac{Dv}{Dt}\right]^2 + 2\frac{\mu}{\rho}|\nabla u|^2$$

& by 2nd Law

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{Dv}{Dt}$$

so

$$\frac{Ds}{Dt} = \frac{1}{T}\frac{De}{Dt} + \frac{p}{T}\frac{Dv}{Dt} = \frac{1}{T}\left\{\cancel{-p\frac{Dv}{Dt}} + \frac{\lambda}{\rho}\left[\rho\frac{Dv}{Dt}\right]^2 + 2\frac{\mu}{\rho}|\nabla u|^2\right\} + \cancel{\frac{p}{T}\frac{Dv}{Dt}}$$

$$\boxed{\frac{Ds}{Dt} = \frac{\lambda}{\rho T}\left[\rho\frac{Dv}{Dt}\right]^2 + \frac{2\mu}{\rho T}|\nabla u|^2}$$

Conclude: when $\lambda, \mu > 0$, entropy per mass strictly increases along particle paths unless both $\frac{Dv}{Dt} = 0$ & $|\nabla u| = 0$.