SECTION-3
The Eulerian and Lagrangian Equations of Motion In One Space Dimension

……………………..

Math-280: A Mathematical Introduction to Shock Waves

Blake Temple, UC-Davis

①

- Euler equations as a 1-D system of conservation laws —

(MA) $\rho_t + (\rho u)_x = 0$

(MO) $(\rho u)_t + (\rho u^2 + p)_x = 0$

(En) $E_t + [(E+p)u]_x = 0$

(S) $S_t + (S u)_x = 0$

$E = \frac{1}{2}\rho u^2 + \rho e$

$S = \rho s$

e = specific energy

s = specific entropy

(E)

Can take either one on smooth solutions

$$\Longleftrightarrow \quad \underset{\sim}{u}_t + f(\underset{\sim}{u})_x = 0 \qquad \underset{\sim}{u} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad f = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}$$

In general: $\underset{\sim}{u} = (u_1, \dots, u_n)$

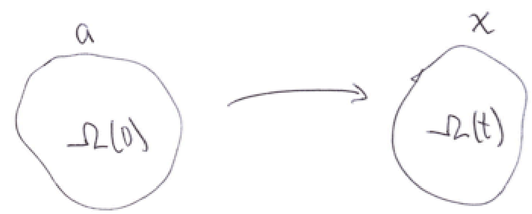
$$f = (f_1, \dots, f_n) = f(\underset{\sim}{u})$$

②

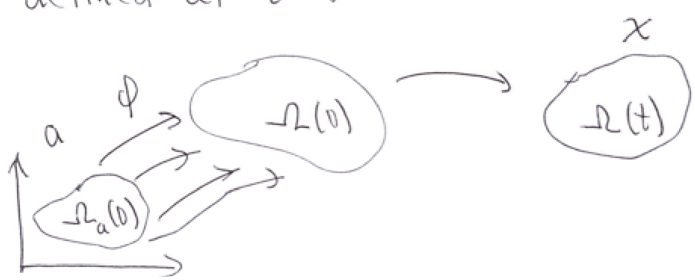
- Lagrangian Coordinates:

- Assume some reference coordinate a that names the fluid particles at time $t=0$:

$$a = (a_1, a_2, a_3)$$



Eg: we could choose a to be x-coord of particle at $t=0$, say $x(a,0) = a$. But we can choose a to be any smooth coord defined at $t=0$



• Then: we still have

$$\frac{\partial x}{\partial t}(a,t) = u$$

$$x(a,0) = \varphi(a)$$

defines the velocity field, and

$$\dot{f} = \frac{Df}{Dt} = \frac{\partial}{\partial t} f(x(a,t),t)\Big|_{a=\text{const}} = \nabla f \cdot u + f_t$$

(same formula).

It follows that if $J = \frac{\partial x}{\partial a}(a,t)$, $\dot{J}$ is the same as our old $\dot{J}$, and we still have

$$\frac{\dot{J}}{J} = \operatorname{div} u, \quad \text{indept of } \varphi(a)$$

$\Rightarrow$ all previous derivations go thru unchanged

• In particular, we can use (MA) to relate the evolution of the density ρ to the evolution of J:

$$(MA) \Rightarrow 0 = \rho_t + \operatorname{div} \rho u = \rho_t + \nabla\rho\cdot u + \rho \operatorname{div} u$$

$$= \frac{D\rho}{Dt} + \rho \operatorname{div} u$$

$$\Rightarrow \boxed{-\frac{1}{\rho}\frac{D\rho}{Dt} = \operatorname{div} u}$$

$$\overset{v=\frac{1}{\rho}}{\Longrightarrow} \quad -v\frac{D(\frac{1}{v})}{Dt} = \boxed{\frac{1}{v}\frac{Dv}{Dt} = \operatorname{div} u}$$

and we conclude that

$$\boxed{\frac{1}{v}\frac{Dv}{Dt} = \frac{1}{J}\frac{DJ}{Dt}}$$

which we can integrate as follows:

5

$$\frac{1}{v}\frac{Dv}{Dt} = \frac{D}{Dt}[\ln v] = \frac{\partial}{\partial t}[\ln v](a,t)$$

so

$$\frac{\partial}{\partial t}[\ln v](a,t) = \frac{\partial}{\partial t}[\ln J](a,t)$$

$$\Rightarrow \quad [\ln v](a,t) = [\ln J](a,t) + \Psi(a)$$

$$\boxed{v(a,t) = e^{\Psi(a)} J(a,t)} \quad \text{holds } \forall t$$

In particular,

$$\boxed{v(a,0) = e^{\Psi(a)} J(a,0)}$$

$$\Leftrightarrow \quad \boxed{\Psi(a) = \ln\left[\frac{v(a,0)}{J(a,0)}\right] = \ln\left[\frac{v(a,t)}{J(a,t)}\right]} \quad (*)_t$$

6

Lagrangian equations in 1-D:

In 1-space dimension we can define the Lagrangian variables a by choosing $\varphi(a) = x$ so that $\Psi(a) = 0 \overset{(*)}{\Rightarrow} v(a,t) = J(a,t)$, and this simplifies the equation when we take a as the space variable instead of x ...

- Restrict to 1-d so $a, x \in \mathbb{R}$. By $(*)$, $\Psi(a) \equiv 0$ if $v(a,0) = J(a,0)$ or

$$\frac{1}{\rho(a,0)} = \frac{\partial x}{\partial a}(a,0) = \varphi'(a)$$

$$\Leftrightarrow \quad \rho(a,0) = \frac{\partial a}{\partial x}(x,0) = [\varphi^{-1}]'(x)$$

so define $\boxed{a = \int_0^x \rho(z,0)\,dz} = [\varphi^{-1}](x)$

⑦

$(*)_t$ then implies that in general

$$\boxed{a = \int_0^{x(a,t)} \rho(z,t)\,dt} \quad \forall t$$

- Taking $\xi \equiv a$ in 1-d, define the Lagrangian coordinate

$$\xi = \int_0^{x(\xi,t)} \rho(z,t)\,dt$$

This defines a mapping $(\xi,t) \longleftrightarrow (x,t)$ satisfying

$$\frac{\partial x}{\partial t}(\xi,t) = u, \quad \frac{\partial x}{\partial \xi}(\xi,t) = \frac{1}{\rho}$$

and $(**)$

$$f_t(\xi,t) = \frac{Df}{Dt}, \quad f_x(x,t) = \frac{\partial}{\partial x} f(\xi(x,t),t)$$
$$= f_\xi \frac{\partial \xi}{\partial x} = f_\xi \rho$$

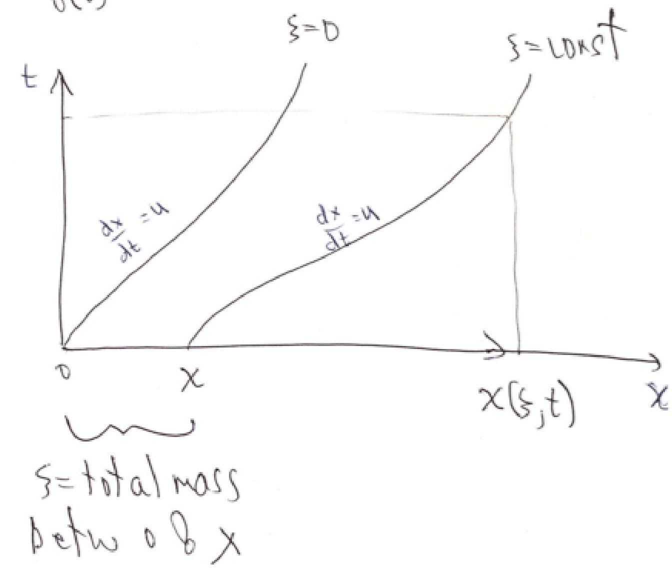
⑧

Lagrangian Coordinates: (Picture)

- For system (E), (x,t) are called Eulerian coordinate for fluid

- Define

$$\xi = \int_{x_0(t)}^{x_\xi(t)} \rho(z,t)\,dz \qquad (\xi)$$



- Using (**) in (MA) $\Rightarrow$

$$(MA)\quad 0=\rho_t+(\rho u)_x=\underbrace{\rho_t+\rho_x u}_{\frac{D\rho}{Dt}}+\rho u_x$$

$$=\frac{\partial}{\partial t}\rho(\xi,t)+\rho\frac{\partial u}{\partial \xi}(\xi,t)\cdot\rho$$

$$=\frac{\partial}{\partial t}\frac{1}{V}+\rho^2 u_\xi=-\rho^2 V_t+\rho^2 u_\xi$$

$$\Rightarrow\quad \boxed{V_t-u_\xi=0}$$

- Using (**) in (MO) $\Rightarrow$

$$(MO)\quad \rho\frac{Du}{Dt}=-\nabla p=-p_x \iff \rho\frac{\partial}{\partial t}u(\xi,t)+\rho p_\xi=0$$

$$\boxed{u_t+p_\xi=0}$$

- Using (**) in (En) $\Rightarrow$

$$E=\frac{1}{2}\rho u^2+\rho e$$

$$\mathcal{E}=\frac{1}{2}u^2+e$$

$$(En)\quad E_t+(Eu)_x+(pu)_x=0$$

$$(\rho\mathcal{E})_t+(\rho\mathcal{E}u)_x+(pu)_x=0$$

$$\rho\frac{D\mathcal{E}}{Dt}+(pu)_x=0$$

$$\rho\,\mathcal{E}_t(\xi,t)+\rho(pu)_\xi(\xi,t)=0$$

$$\boxed{\mathcal{E}_t+(pu)_\xi=0}$$

$$(S)\quad \mathcal{S}_t+(\mathcal{S}u)_x=0$$

$$(\rho s)_t+(\rho s u)_x=0$$

$$\rho\frac{Ds}{Dt}=0$$

$$\boxed{s_t=0}$$

Conclude: Euler Equations in Lagrangian Coordinates:

$(MA)_L$ $\quad V_t - u_x = 0$

$(MO)_L$ $\quad u_t + p_x = 0$

$(En)_L$ $\quad \mathcal{E}_t + (\mathcal{E}u)_x = 0$

$\qquad S_t = 0$

take either one on smooth solutions

$x \longleftrightarrow \xi$

$\mathcal{E}$ = specific total energy

S = specific entropy

- Note: We say fluid is barotropic if

$$p = P(V), \quad V = \frac{1}{\rho}$$

and we assume $P'(V) < 0$, $P''(V) > 0$. Then $(MA)_L$ & $(MO)_L$ uncouple from $(En)_L$ and reduce to the so-called p-system (name coined by Joel Smoller)

$$\boxed{\begin{aligned} &V_t - u_x = 0 \\ &u_t + P(V)_x = 0 \end{aligned}}$$