SECTION-9
Traveling Waves
for
nxn-systems

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Math-280: A Mathematical Introduction to Shock Waves

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Stability of traveling waves & Lax shock condt for n×n systems

Consider an n×n strictly hyperbolic system of conservation laws:

$$u_t + f(u)_x = 0 \qquad u=(u_1,\dots,u_n)$$

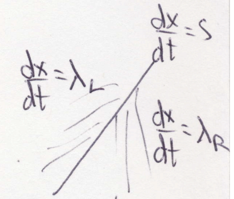
Assume the kth characteristic family (λ_k, R_k) is genuinely nonlinear:

$$\nabla\lambda_k\cdot R_k > 0$$

- By Lax's Thm, ∃ a shock curve that has C^2 contact with R_k at each u_L:

- Let $W_k(u_L) = S_k^- \cup R_k^+$ have arclength parameterization $u(\varepsilon)$, $u(0)=u_L$, and set $u_R = u(\varepsilon)$.

- We know that for ε suff small, $\varepsilon<0$, $[u_L,u_R]$ is a shock wave that satisfies the Lax entropy condt:



$$\lambda_k^R < s < \lambda_k^L$$

where s = shock speed comes from R-H.

$$s[u]=[f]$$

- On S_k^+, the reverse inequality holds — such a shock is an inadmissible rarefaction shock

$$\lambda_k^L < s < \lambda_k^R$$

• <u>Defn</u>: we say the shock has structure if $\exists$ traveling wave that connects u_L to u_R:

Ie. $u\left(\frac{x-st}{\varepsilon}\right)$, $u_t = -\frac{s}{\varepsilon}u'$, $u_x = \frac{1}{\varepsilon}u'$

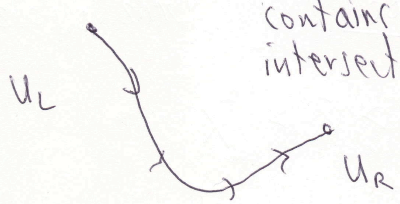
$$\Rightarrow \quad u_t + f(u)_x = \underbrace{\varepsilon u_{xx}}_{\text{artificial viscosity}}$$

Autonomous system so orbits that intersect are identical

$$\Rightarrow \quad u' = -s(u-u_L) + f(u) - f(u_L) = F(u) \quad \text{(ODE)}$$

• <u>Thus</u>: shock has structure if $\exists$ connecting orbit $u_L \to u_R$ of ODE $u' = F(u)$:

$\Rightarrow$ The unstable manifold of u_L contains a trajectory that intersects the stable manifold of u_R



• <u>Recall</u>: $u' = F(u)$ near rest pt u_0

Linearize: $F(u) = \cancel{F(u_0)}^{0} + dF(u_0)(u-u_0) + \text{h.o.t}$

$$\Rightarrow \quad (u-u_0)' = dF(u_0)(u-u_0) + \text{h.o.t}$$

linearized system: $y' = dF(u_0)\,y$

(For us assume $dF(u_0)$ has real distinct evals)

<u>Defn</u>: $N = \#$ pos evals, $M = \#$ neg evals

$\Rightarrow$ N = dimension of <u>unstable</u> manifold^n @ u_0

M = dimension of <u>stable</u> manifold^m @ u_0

Eg: (λ_i, R_i) e-pair of $dF(u_0)$ $\Rightarrow$

$$y = e^{\lambda_i t} R_i$$

solves $y' = dF(u_0)\,y$ ✓

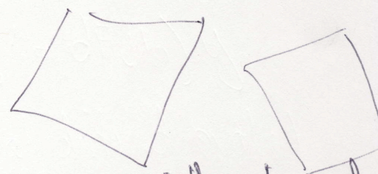
⑤

Conclude: $[u_L, u_R]$ has structure if $\mathcal{M} \cap \mathcal{N}$

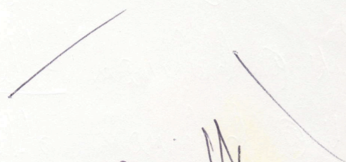
Note: two manifolds $\mathcal{M}, \mathcal{N}$ generically intersect in $\mathbb{R}^n$ if ~~dim~~ $N+M>n$, generically do not intersect if $N+M<n$

Eg: 2 planes in $\mathbb{R}^2$

$N+M=3$



Eg: 2 lines in $\mathbb{R}^3$



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Thm: If $u_R = u(\varepsilon) \in \mathcal{S}^-(u_L)$, then

$$N_L + M_R = n+1$$

If $u_R \in v\varepsilon \in \mathcal{S}^+(u_L)$, then

$$N_L + M_R = n-1$$

$\rightsquigarrow$ expect generically Lax shocks have structure, rarefaction shocks do not.

Proof: (ODE) $u' = -s(u-u_L) + (f-f_L) = F(u)$

- Linearize about u_0:

$$dF(u_0) = -sI + df(u_0)$$

$\Rightarrow$ linearized system $y = u - u_0$:

$$y' = (df - s)\,y = dF(u_0)\,y$$

- Evals of $dF(u_0)$:

$$|df - sI - \mu I| = 0$$

$$|df - (s+\mu)I| = 0$$

$\Rightarrow$ μ eval of $dF(u_0)$ iff λ is eval of $df(u_0)$

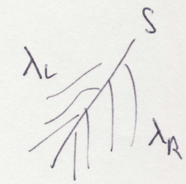
$$\lambda = s + \mu$$

$$\boxed{\mu = \lambda - s}$$

Assume $[u_L, u_R]$ a Lax admissible k-shock –

$$u_R = u(\varepsilon) \in \mathcal{S}_k^-(u_L)$$

$$\lambda_k^R < s < \lambda_k^L$$



- linearize about u_L, $s \approx \lambda_k(u_L)$

$$\lambda_1^L - s < \cdots < \underbrace{\lambda_{k-1} - s}_{\text{neg by Lax condt}} < 0 < \underbrace{\lambda_k - s}_{\text{pos by Lax condt}} < \cdots < \lambda_n - s$$

$$\mu_1 < \cdots < \mu_{k-1} < 0 < \mu_k < \cdots < \mu_n$$

$\Rightarrow$ $N_L = n - k + 1$

$M_L = k - 1$

For rarefaction shock:

$\bar{N}_L = n - k$

$\bar{M}_L = k$

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• linearize about u_R:

$$\lambda_1^R - s < \cdots < \underbrace{\lambda_k - s}_{\text{neg by Lax}} < 0 < \underbrace{\lambda_{k+1} - s}_{\text{pos by Lax}} < \cdots < \lambda_n - s$$

$$\mu_1^R < \cdots < \mu_k < 0 < \mu_{k+1} < \cdots < \mu_n$$

$\Rightarrow$ $N_L = n-k$, $M_L = k$ || For rarefaction shock: $\bar{N}_R = n-k+1$, $\bar{M}_R = k-1$

$\therefore$ $N_L + M_R = n-k+1+k = n+1$ ✓ For Lax Shock

Sim: $\bar{N}_L + \bar{M}_R = n-1$ for rarefaction Shock.

Expect: Lax shocks generically have structure, rarefaction shocks do not.

Thm: "weak Lax shocks have structure" (Foy, R. 1964)