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Introduction to Exactly Solvable Models in Statistical Mechanics

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1. Introduction. The purpose of these notes is to provide an introduction to exactly solvable models in lattice statistical mechanics. No prior knowledge of statistical mechanics is assumed. Hence these notes are not addressed to the experts in solvable models! They are not meant to be an exhaustive survey of solvable models, but rather by focusing on three examples—the two-dimensional Ising model, the symmetric eight-vertex model ("Baxter model"), and the hard hexagon model—it is hoped that some of the basic properties of all solvable models emerge. Even so we cannot give complete details for these three models.

For work on the 2-dimensional Ising model prior to 1973, the reader should consult the book by McCoy and Wu [**21**]. For details concerning the 8-vertex model and the hard hexagon model, the book by Baxter [**10**] is the basic reference. In addition to these books there are several reviews: Andrews [**4**] discusses the hard hexagon model from a q-series point of view; Baxter [**12**] gives an overview of solvable models as does Pearce [**25**], who stresses the physics associated with solvable models. Thacker [**33**] develops solvable models from the quantum inverse method (see also Takhtadzhan and Faddeev [**32**] and Kulish and Sklyanin [**19**]). Further references can be found in the recent review [**35**].

In §2 a brief introduction to statistical mechanics is presented. §3 surveys some properties of the two-dimensional Ising model, but the main theme of this paper begins in §4, where vertex models are introduced. §5 introduces corner transfer matrices and shows how they are applied to compute the order parameter for the Baxter model. §6 introduces the hard hexagon model, where we see that the Rogers-Ramanujan identities play a crucial role.

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2. Statistical mechanics. Statistical mechanics is the bridge between a microscopic description of matter and a macroscopic description of matter. At the microscopic level this means specifying the forces amongst the "particles." The particles may be atoms in a crystal, molecules in a gas or liquid, electrons in a plasma, or even amino acid units in a protein. For example, for molecules in a gas we might take that the interaction between the molecules is 2-body, and the Hamiltonian has the form

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j} V(q_i, q_j),$$

where p_i is the momentum of the ith molecule and q_i the position. Here $V(q, q')$ is the 2-body potential. A favorite choice of chemists is the so-called Lennard-Jones potential

$$v(r) = v_0 \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right]$$

with v_0, $\sigma \in \mathbf{R}$, and $V(q, q') = v(|q - q|)$. We may treat the Hamiltonian either as a classical Hamiltonian or a quantum Hamiltonian.

We know, of course, there is another description of a gas, and we remember from high school the formula for an ideal gas

$$pV = Nk_B T. \tag{2.1}$$

What this is telling us is that the macroscopic properties pressure, p, temperature, T, and volume, V, are related by the above "equation of state" where N is the number of particles and k_B is a constant, called the Boltzmann constant ($k_B = 1.36 \times 10^{-16}$erg/deg). A macroscopic description is in terms of thermodynamic functions and transport coefficients, including density, pressure, temperature, thermal and electrical conductivity, magnetization, tensile strength, fluid viscosity, specific heat, chemical reaction rates, etc.

A chemist or physicist postulates the form of the force laws (based upon experimental data, symmetry principles, or just plain physical intuition), and it is the task of statistical mechanicians to predict the bulk properties of matter with this force law. The "force law" may be classical (Hamilton's equations) or it may involve quantum mechanics (Schrödinger equation). In either case these force laws are for our purposes definitions.

From a practical point of view it is clear that a purely microscopic approach to bulk matter is hopeless. For example, at zero degrees Celsius and a pressure of one atmosphere, the number of molecules in one cubic centimeter of gas is 2.69×10^{19}. That is a lot of differential equations to solve! Recall that if N is the number of particles the dimension of phase space is $6N$. If we are interested, say, in the equation of state of a gas

$$p = p(V, T), \tag{2.2}$$

then there is a good deal less information here than is in the $6N$ differential equations (assuming a classical description). It is a good guess that statistical mechanics involves some sort of averaging procedure. By the way, it is

customary in thermodynamics to use the same letter for both the function and the variable as I have done above in the equation of state. In some instances pressure and temperature are the independent variables, and then the density is determined. This can get a little confusing at times, but it has the advantage of minimizing the number of symbols, p is pressure!

Before giving the basic definitions of classical statistical mechanics, I invite you to consider the following example, due to Arnold, which illustrates in a simple way the power of "statistical thinking." Our dynamical system is the sequence of digits formed by taking the first digit of 2^n, i.e., $2, 4, 8, 1, 3, 6, 1, \ldots$. The problem is to calculate the relative number of 7's to 8's in this sequence. (Answer: $(\log 8 - \log 7)/(\log 9 - \log 8) = 1.1337\ldots$.)

Implicitly in the discussion above, the particle position q was an element of $\mathbf{R}^3$. We make the simplifying assumption that the particles lie on a lattice $\mathbf{Z}^\nu$. We are now ready to give the basic definitions of classical (equilibrium) lattice statistical mechanics. Mathematically, this amounts to defining a probability space $(\Omega, \mathcal{F}, P)$.

Consider a "box" $\Lambda \subset \mathbf{Z}^\nu$, $|\Lambda| < \infty$, and take for $\Omega_{\Lambda,N}$ the set of microscopic states of the system with N particles confined to Λ. The "force law" is given by specifying an energy function

$$\mathcal{E}_{\Lambda,N}: \Omega_{\Lambda,N} \to \mathbf{R} \cup \{\pm\infty\}.$$

The probability measure for the system of N particles in equilibrium at temperature T in box Λ with energy function $\mathcal{E}_{\Lambda,N}$ is given by the *Gibbs measure*

$$P_{\Lambda,N}(\omega) = \exp(-\beta \mathcal{E}_{\Lambda,N}(\omega))/Z_{\Lambda,N}(\beta), \tag{2.3}$$

where $\beta = (k_B T)^{-1}$ and $Z_{\Lambda,N}(\beta)$, called the *partition function*, is the normalizing factor

$$Z_{\Lambda,N}(\beta) = \sum_{\omega \in \Omega_{\Lambda,N}} \exp(-\beta \mathcal{E}_{\Lambda,N}(\omega)). \tag{2.4}$$

Thermodynamics is obtained via

$$-\beta F_\Lambda = \log Z_{\Lambda,N}(\beta), \tag{2.5}$$

where F_Λ is the free energy at temperature T in box Λ. Averages are computed using the Gibbs measure. This probability space is called the *canonical ensemble.*

Sometimes it is convenient not to have the particle number N fixed. In this case we let Ω_Λ be the set of microscopic configurations with no restriction on the particle number, i.e., $\Omega_\Lambda = \bigcup_{N=0}^{\infty} \Omega_{\Lambda,N}$. For example, a simple lattice gas configuration $\omega \in \Omega_\Lambda$ is specified by giving the "occupation number" $\sigma_i(\omega)$, $i \in \Lambda$,

$$\sigma_i(\omega) = \begin{cases} 1 & \text{if particle present at site } i \in \Lambda, \\ 0 & \text{if no particle at site } i \in \Lambda. \end{cases}$$

That is, $\Omega_\Lambda = \{0,1\}^\Lambda$. The probability of configuration $\omega \in \Omega_\Lambda$ in box Λ at temperature T and chemical potential μ is given by

$$P_\Lambda(\omega) = \exp(-\beta \mathscr{E}_\Lambda(\omega))/Z_\Lambda(\mathfrak{z}, \beta), \tag{2.6}$$

where

$$\mathscr{E}_\Lambda(\omega) = \mathscr{E}_{\Lambda,N}(\omega) - \mu N(\omega),$$

$$N(\omega) = \text{number of particles in configuration } \omega,$$

and

$$Z_\Lambda(\mathfrak{z}, \beta) = \sum_{\omega \in \Omega_\Lambda} \exp(-\beta \mathscr{E}_\Lambda(\omega)) \tag{2.7a}$$

$$= \sum_{N=0}^{\infty} \mathfrak{z}^N \sum_{\omega \in \Omega_{\Lambda,N}} \exp(-\beta \mathscr{E}_{\Lambda,N}(\omega)), \qquad \mathfrak{z} = e^{\beta\mu},$$

$$= \sum_{N=0}^{\infty} \mathfrak{z}^N Z_{\Lambda,N}(\beta). \tag{2.7b}$$

The quantity $Z_\Lambda(\mathfrak{z}, \beta)$ is called the *grand partition function*, and the probability space so defined is called the *grand canonical ensemble.* The pressure in a volume V is given by

$$pV/k_B T = \log Z_V(\mathfrak{z}, \beta), \tag{2.8}$$

and the expected number of particles is

$$\overline{N} = \langle N \rangle = \sum_{\omega \in \Omega_\Lambda} N(\omega) P_\Lambda(\omega)$$

$$= \sum_{\omega \in \Omega_\Lambda}^{\infty} \mathfrak{z}^{N(\omega)} e^{-\beta \mathscr{E}_{\Lambda,N}(\omega)}/Z_\Lambda(\mathfrak{z}, \beta) \tag{2.9}$$

$$= \mathfrak{z} \frac{\partial}{\partial \mathfrak{z}} \log Z_\Lambda(\mathfrak{z}, \beta).$$

The reader can verify that

$$\langle (N - \overline{N})^2 \rangle = k_B T \rho \overline{N} K_T, \tag{2.10}$$

where $\rho = \overline{N}/V$ and

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

is the isothermal compressibility. Defining

$$\chi = k_B T \rho K_T, \tag{2.11}$$

the *reduced isothermal compressibility*, we see that $\chi = 1$ for the ideal gas (2.1).

For the lattice models we will consider, the above structure is not so interesting mathematically since the set Ω_Λ and $\Omega_{\Lambda,N}$ are finite sets. As indicated at the beginning, statistical mechanics is concerned with bulk properties of matter when Λ and N are quite large (the volume Λ is large compared

with microscopic length scales). In fact, statistical mechanics, as formulated above, cannot describe the macroscopic phenomenon of a phase transition. One characterization of a phase transition is nonanalytic behavior of various thermodynamic functions as a function of temperature. In the canonical ensemble, the bulk free energy $F_\Lambda(\beta)$ is a finite sum of terms, each term analytic in β. Thus $F_\Lambda(\beta)$ is analytic, and hence all thermodynamic quantities obtained from F_Λ (via differentiation) will be analytic in β.

To describe the phenomena of phase transitions, we must consider the *thermodynamic limit*

$$\Lambda \to \mathbf{Z}^\nu, \qquad N \to \infty \tag{2.12}$$

such that

$$N/|\Lambda| \to \rho, \qquad 0 < \rho < \infty,$$

where the sequence of "boxes" Λ approaching $\mathbf{Z}^\nu$ is "reasonable" (take actual boxes!). Thus we consider the free energy per lattice site

$$\begin{aligned} f(\beta, \rho) &= \lim_{\substack{\Lambda \to \mathbf{Z}^\nu,\ N \to \infty \\ N/|\Lambda| \to \rho}} \frac{1}{|\Lambda|} F_\Lambda(\beta) \\ &= -k_B T \lim_{\substack{\Lambda \to \mathbf{Z}^\nu,\ N \to \infty \\ N/|\Lambda| \to \rho}} \frac{1}{|\Lambda|} \log Z_{\Lambda,N}(\beta) \end{aligned} \tag{2.13}$$

in the canonical ensemble. In the grand canonical ensemble

$$p\beta = \lim_{\Lambda \to \mathbf{Z}^\nu} \frac{1}{|\Lambda|} \log Z_\Lambda(\mathfrak{z}, \beta), \tag{2.14a}$$

$$\rho = \lim_{\Lambda \to \mathbf{Z}^\nu} \frac{1}{|\Lambda|} \mathfrak{z} \frac{\partial}{\partial \mathfrak{z}} \log Z_\Lambda(\mathfrak{z}, \beta), \tag{2.14b}$$

where ρ is the density. The equation of state (2.2) is obtained by eliminating $\mathfrak{z}$ in (2.14a) and (2.14b).

In the above discussion we have not been careful about the boundary conditions at the walls of Λ. It is possible to discuss Gibbs states for infinite volume systems directly, as was first done by Dobrushin [**16**, **17**] and Lanford and Ruelle [**20**]. These infinite volume Gibbs states are obtained by passing to the thermodynamic limit with various boundary conditions (see discussion of the Ising model in §4). The reader is referred to Ruelle [**29**] for the theory of Gibbs states.

3. The two-dimensional Ising model. The Ising model is a simplified model of a magnet. At each site $i \in \Lambda \subset \mathbf{Z}^2$ there is a spin "pointing up or down," i.e.,

$$\sigma_i(\omega) = \begin{cases} +1 & \text{if spin is up at site } i, \\ -1 & \text{if spin is down at site } i. \end{cases}$$

The configuration space is $\Omega_\Lambda = \{-1, 1\}^\Lambda$, and the energy function is

$$\mathscr{E}_\Lambda(\omega) = -J \sum_{\substack{i,j\in\Lambda \\ |i-j|=1}} \sigma_i(\omega)\sigma_j(\omega), \tag{3.1}$$

where the sum is restricted to nearest neighbor pairs. For $J > 0$ the system is said to be *ferromagnetic* since like spins are more favorable.

A rigorous discussion requires a description of the boundary conditions at the walls of Λ. Theoretically, the two most obvious boundary conditions are to fix on and outside the walls all spins either $+1$ or -1. The resulting infinite volume Gibbs measures are called P_+ and P_-. It is a theorem due to Aizenman [1] that there is a unique temperature T_c (given by (3.2) below) such that for all $T < T_c$ any Gibbs state is a convex combination of the distinct measures P_+ and P_-, whereas for $T \geq T_c$ there is a unique Gibbs measure ($P_+ = P_-$), called the high-temperature measure. It should be noted that this holds only for the two-dimensional Ising model. For the three-dimensional Ising model, Dobrushin [18] has shown that for sufficiently low temperatures the number of extreme measures is infinite. There is a complete characterization of the measures P_+ and P_- for the two-dimensional Ising model (see Palmer and Tracy [23, 24] and references therein). From a computational point of view, the boundary conditions obtained by identifying opposite sides of the box Λ (periodic boundary conditions) are most convenient as they preserve the translational invariance.

In any case, the free energy per site in the thermodynamic limit is known exactly (and is independent of boundary conditions) and was first computed by Onsager [22] in his now-classic 1944 paper. The free energy per site, $f(\beta)$, has a singularity at a temperature T_c, called the *critical temperature*, defined by

$$\sinh(2\beta J) = 1. \tag{3.2}$$

Furthermore, the spontaneous magnetization (an example of an *order parameter*) is known exactly [39]

$$\langle\sigma_i\rangle_+ = \lim_{\Lambda\to\mathbf{Z}^2} E_{\Lambda,+}(\sigma_i) = \begin{cases} (1-k^2)^{1/8} & \text{for } T < T_c, \\ 0 & \text{for } T \geq T_c, \end{cases} \tag{3.3}$$

where $k = (\sinh 2\beta J)^{-2}$. A detailed discussion of these results can be found in McCoy and Wu [21]. One important aspect of (3.3) is that it illustrates the phenomenon of *spontaneous symmetry breaking*. The energy (3.1) is invariant under $\sigma_i \to -\sigma_i$, and one might naively conclude $\langle\sigma_i\rangle_\Lambda = 0$. However, the boundary conditions of all $+$ at the walls give $\langle\sigma_i\rangle_{\Lambda,+} \neq 0$ and these boundary conditions are still "felt" for low enough temperatures in the limit $\Lambda \to \mathbf{Z}^2$.

The reason exact calculations can be carried out in the nearest neighbor 2-dimensional Ising model is that the transfer matrix (see §4) is an element of the Clifford group. We will hear more about this in Professor Jimbo's lectures on holonomic quantum fields. Here it suffices to say that because of

this many exact calculations can be performed. To give one further example, it was shown in 1973 by Wu, McCoy, Tracy, and Barouch [**15**, **37**, **38**] that the two-spin correlation function

$$\langle \sigma_0 \sigma_j \rangle_+ = \lim_{\Lambda \to \mathbf{Z}^2} E_{\Lambda,+}(\sigma_0 \sigma_j), \qquad j \in \mathbf{Z}^2,$$

in the so-called *scaling limit* $|j| \to \infty$, $T \to T_c^-$ such that $|T - T_c|\,|j|$ is fixed is given by

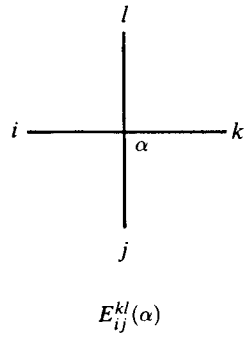
$$\lim \frac{\langle \sigma_0 \sigma_j \rangle}{\langle \sigma_0 \rangle^2} = \cosh\left(\frac{1}{2}\psi(r)\right) \exp\left\{\frac{1}{4}\int_r^\infty s\left[\sinh^2 \psi(s) - \left(\frac{d\psi}{ds}\right)^2\right] ds\right\},$$

where r is proportional to $|T - T_c|\,|j|$ and $\eta = \exp(-\psi)$ is a Painlevé function of the third kind. This result was reformulated and generalized by Sato, Miwa, and Jimbo [**30**] in the context of holonomic quantum fields. Finally, Palmer and Tracy [**23**, **24**] gave a rigorous account of this scaling limit and showed the existence of a generalized random field in this scaling limit.

4. Vertex models and the Yang-Baxter equations. Consider a square lattice Λ with M rows and N columns with periodic boundary conditions, i.e., opposite sides identified. Let X be a finite set. A configuration ω is defined by assigning to each bond of Λ an element of X. As above, we let Ω_Λ be the collection of all such configurations ω. To an individual vertex $\alpha \in \Lambda$ is assigned an energy $\varepsilon_{ij}^{kl}(\alpha)$, $i, j, k, l \in X$, and corresponding Boltzmann weight $S_{ij}^{kl}(\alpha) = \exp(-\beta \varepsilon_{ij}^{kl}(\alpha))$ (see Figure 1). The energy of a configuration, $\mathscr{E}_\Lambda(\omega)$, is defined to be the sum of the energies of the individual vertices. Then the partition function is

$$Z_\Lambda(\beta) = \sum_{\omega \in \Omega_\Lambda} \prod_{\alpha \in \Lambda} S_{ij}^{kl}(\alpha), \tag{4.1}$$

where the product is taken over all vertices of the lattice Λ. This defines an *X vertex model*.



$E_{ij}^{kl}(\alpha)$

FIGURE 1. Vertex configuration corresponding to energy $\varepsilon_{ij}^{kl}(\alpha)$.

The most studied vertex model is the case when X is a two-element set and the Boltzmann weight is independent of the site label α. The term *multistate vertex model* is used for cases $|X| > 2$. In the 2-state case it is conventional

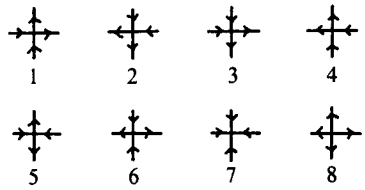


FIGURE 2. Eight elementary vertices in the eight-vertex model.

to indicate the bond state by drawing an arrow either up or down on vertical bonds and either right or left on horizontal bonds. Thus, there are sixteen elementary vertices, and the resulting model is called the 16-*vertex model*. The exact evaluation of the free energy per site in the thermodynamic limit is beyond present techniques. Thus, additional restrictions are imposed, the first being that only an even number of arrows is allowed into each vertex. This leaves eight elementary vertices (see Figure 2), and to each vertex we assign an energy ε_i, $i = 1, \ldots, 8$. Since we are assuming periodic boundary conditions, there must be as many sinks as sources in a configuration ω so it is no restriction to take $\varepsilon_7 = \varepsilon_8$. Likewise, vertical sinks and sources occur in equal number so we may take $\varepsilon_5 = \varepsilon_6$. Thus the resulting partition function depends upon six parameters. Even with these restrictions the exact evaluation of f as a function of these six parameters is not known.

We impose the second condition on the eight-vertex model; namely, $\varepsilon_1 = \varepsilon_2$ and $\varepsilon_3 = \varepsilon_4$, which amounts to the restriction that reversal of all arrows leaves the energy of a vertex unchanged. We introduce the four parameters

$$a = e^{-\beta\varepsilon_1}, \quad b = e^{-\beta\varepsilon_3}, \quad c = e^{-\beta\varepsilon_5}, \quad d = e^{-\beta\varepsilon_7}. \tag{4.2}$$

It is this symmetric eight-vertex model ("Baxter model") that Baxter [**6, 7**] computed the free energy per site as a function of the Boltzmann weights (4.2). Baxter's methods are presently the most powerful methods available for computing the free energy. We now give an introduction to these methods.

For a general X-vertex model we introduce the associated *transfer matrix*. Let $H_N = \mathbf{C}^q \otimes \cdots \otimes \mathbf{C}^q$ (N times), $q = |X|$, and let $e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N}$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_j = 1, \ldots, q$ be the standard basis for H_N, then we define the transfer matrix T by

$$T_{\alpha,\alpha'} = \sum_{\{\lambda_j\}} S^{\lambda_2\alpha_1'}_{\lambda_1\alpha_1} S^{\lambda_3\alpha_2'}_{\lambda_2\alpha_2} \cdots S^{\lambda_1\alpha_n'}_{\lambda_N\alpha_N}, \tag{4.3}$$

where the summation is over all horizontal bond states λ_j (see Figure 3). An easy calculation shows that

$$Z_\Lambda(\beta) = \mathrm{Tr}(T^M), \tag{4.4}$$

where the trace is on H_N. If we denote by x the vector of independent Boltzmann weights and D the number of components of x, then $T = T(\mathrm{x})$. For the Baxter model $\mathrm{x} = (a, b, c, d)$ and $D = 4$. Since changing all the Boltzmann weights by the same multiplicative factor does not change the

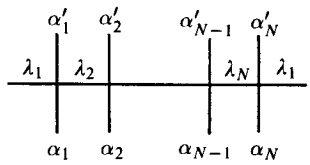


FIGURE 3. Configuration in a row.

free energy in a significant way, it is perhaps better to think of x as the homogeneous coordinates of a point in projective space (note that for physical values all the coordinates must be nonnegative).

It is useful to derive a more compact expression for T. Define $T^{\lambda\lambda'}: \mathbf{C}^q \to \mathbf{C}^q$ by

$$(T^{\lambda\lambda'})_{\alpha\alpha'} = S^{\lambda'\alpha'}_{\lambda\alpha} \tag{4.5}$$

and $T_j^{\lambda\lambda'}: \mathscr{H}_N \to \mathscr{H}_N$, $1 \le j \le N$, by

$$T_j^{\lambda\lambda'} = I \otimes \cdots \otimes \underset{\substack{\downarrow \\ j\text{th slot}}}{T^{\lambda\lambda'}} \otimes \cdots \otimes I. \tag{4.6}$$

Finally, if we denote by $\mathscr{H}_{N,q}$ the direct sum of $\mathscr{H}_N$ with itself q-times and by $L_j: \mathscr{H}_{N,q} \to \mathscr{H}_{N,q}$, $1 \le j \le N$, the $q \times q$ matrix whose entries are

$$L_j(\lambda, \lambda') = T_j^{\lambda\lambda'}, \qquad \lambda, \lambda' \in X, \tag{4.7}$$

then an elementary calculation gives

$$T = \mathrm{Tr}_q(L_1 L_2 \cdots L_N), \tag{4.8}$$

where $\mathrm{Tr}_q(\cdots)$ indicates the trace is only over the $q \times q$ matrix structure of the L_j's.

We now come to the fundamental question posed by Baxter [**6**, **10**]: namely, to ask whether there exist commuting families of such transfer matrices. That is, we ask for a deformation from x, to say, x′ such that the associated transfer matrices $T(\mathrm{x})$ and $T(\mathrm{x}')$ commute. From the representation (4.8) and the commutativity of L_j and L_k it follows that

$$T(\mathrm{x})T(\mathrm{x}') = \mathrm{Tr}_{q^2}[(L_1 \otimes L_1') \cdots (L_N \otimes L_N')], \tag{4.9}$$

where $L_j = L_j(\mathrm{x})$ and $L_j' = L_j(\mathrm{x}')$, and the subscript q^2 on the trace denotes the trace is only over the $q^2 \times q^2$ matrix structure of $L_j \otimes L_j'$. To compute $T(\mathrm{x}')T(\mathrm{x})$, one simply switches the primed and unprimed labels. From the invariance of the trace under similarity transformations, a sufficient condition for the commutativity of $T(\mathrm{x})$ and $T(\mathrm{x}')$ is the existence of a nonsingular matrix $R(\mathrm{x}, \mathrm{x}'): \mathbf{C}^{q^2} \to \mathbf{C}^{q^2}$ such that

$$(L_j(\mathrm{x}') \otimes L_j(\mathrm{x}))R(\mathrm{x}, \mathrm{x}') = R(\mathrm{x}, \mathrm{x}')(L_j(\mathrm{x}) \otimes L_j(\mathrm{x}')) \tag{4.10}$$

for all j. This equation is the *Yang-Baxter equation* (or *star-triangle equation*) for the vertex model. For a generic vertex model, the only solutions

to (4.10) will be the trivial solution $R(\mathrm{x}, \mathrm{x}') = I$ corresponding to the trivial deformation $\mathrm{x} \to \mathrm{x}' = \rho \mathrm{x}$, $\rho \in \mathbf{C}$.

For the Baxter model there exist nontrivial solutions to (4.10). Baxter proved **[6, 10]**

THEOREM 4.1. *Let* $\mathrm{x} = (x_1, x_2, x_3, x_4)$ *denote homogeneous coordinates of a point in* $\mathbf{CP}^3$. *Define the elliptic curve* M_{YB} *by the intersection of the two quadrics*

$$\frac{x_1 x_2}{x_3 x_4} = c_1, \qquad \frac{x_1^2 + x_2^2 - x_3^2 - x_4^2}{2 x_1 x_2} = c_2, \tag{4.11}$$

where $c_1, c_2 \in \mathbf{C}$. *If* $\mathrm{x} = (a, b, c, d)$ *and* $\mathrm{x}' = (a', b', c', d')$ *are two Boltzmann vectors for the symmetric eight-vertex model lying on* M_{YB}, *then the corresponding transfer matrices* $T(\mathrm{x})$ *and* $T(\mathrm{x}')$ *commute.*

It is essential for Baxter's computation of the free energy per site to parametrize the Boltzmann weights (4.2) in terms of Jacobi theta functions (or elliptic functions). If we write

$$\begin{aligned} a &= \rho\theta(2\eta)\theta(v - \eta)H(v + \eta), \\ b &= \rho\theta(2\eta)H(v - \eta)\theta(v + \eta), \\ c &= \rho H(2\eta)\theta(v - \eta)\theta(v + \eta), \\ d &= \rho H(2\eta)H(v - \eta)H(v + \eta), \end{aligned} \tag{4.12}$$

where $H(u) = -\mathcal{J}_{11}(u/2K, \tau)$, $\theta(u) = \mathcal{J}_{01}(u/2K, \tau)$, then the transfer matrices $T(v; \eta, \tau, \rho)$ for fixed η, τ, ρ form a family of commuting transfer matrices. The variable v is sometimes called the *spectral parameter*. A calculation shows that the parametrization (4.12) makes the combinations (4.11) independent of the parameter v. An explicit form for R can be found, for instance, in **[32]**. The equations (4.10) become, with parametrization (4.12), addition formulas for theta functions. In fact a slightly different parametrization of (a, b, c, d) makes (4.10) a special case of a Riemann quartic theta identity, and the quadrics in (4.11) follow from the Jacobi equations for the four theta functions $\mathcal{J}_{ab}(z, \tau)$, $a, b = 0, 1$. (See **[34]** for details.)

It seems to be the case (though not a theorem!) that once one has found a nontrivial solution to the Yang-Baxter equations, one can evaluate the partition function per site

$$\kappa = \lim_{N \to \infty} Z_N^{1/N}. \tag{4.13}$$

The transfer matrix for the symmetric eight-vertex model is a normal matrix. Elementary spectral arguments and equation (4.4) show that κ, as defined above in (4.13), can be obtained in the limit $M \to \infty$ by knowing only the largest eigenvalue of T. Baxter, in an intricate analysis **[6, 7]**, derives an equation for this eigenvalue, which can be solved exactly in the limit $N \to \infty$. The parametrization (4.12) plays a crucial role in this analysis.

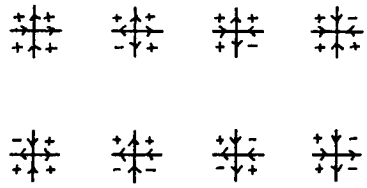
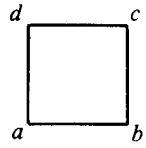


FIGURE 4. Eight spin configurations that correspond to the eight vertices. Reversal of all spins corresponds to the same vertex configuration.



$w(a, b, c, d)$

FIGURE 5. Boltzmann weight $w(a, b, c, d)$.

Somewhat later Baxter [**11**] showed that with certain analyticity assumptions, the partition function per site κ satisfied certain functional equations from which κ could then be derived. It is certainly fair to say that neither calculation of κ is completely rigorous, but that the functional equation method is certainly the least rigorous. It is also the method most frequently used in multistate models!

Rather than pursue the details of these calculations of κ, it is better for this audience to be introduced to the methods needed to compute the order parameters for various models. Here again, it is Baxter who pioneered the way by his introduction of *corner transfer matrices.* Before proceeding, however, it will be convenient to introduce a dual version of the equations (4.10).

Associated to any vertex model there is a spin model (see, e.g., Perk and Wu [**26**]). For the Baxter model, the spins σ_i are two-valued random variables, ± 1, and are site variables on a square lattice that is the dual lattice of the vertex lattice. The correspondence (2 to 1) from spin configurations to arrow configurations is shown in Figure 4. In terms of these spin variables we have Boltzmann factors corresponding to elementary squares (see Figure 5). For the Baxter model

$$w(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \exp(\beta J \sigma_1 \sigma_3 + \beta J' \sigma_2 \sigma_4 + \beta J_4 \sigma_1 \sigma_2 \sigma_3 \sigma_4) \tag{4.14}$$

with $\sigma_i = \pm 1$, and the mapping between the vertex energies and the spin coupling constants is

$$\varepsilon_1 = -J - J' - J_4, \qquad \varepsilon_3 = J + J' + J_4,$$
$$\varepsilon_5 = J - J' + J_4, \qquad \varepsilon_7 = J + J' + J_4.$$

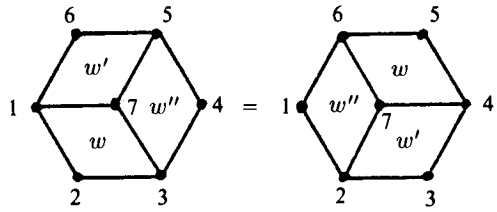


FIGURE 6. Graphical representation of the star-triangle equation (4.15)

We see that for the special case $J_4 = 0$, the Baxter model partition function is a product of two two-dimensional Ising model partition functions.

In spin language we can also define a transfer matrix $T = T[w]$ where we have indicated its dependence on the Boltzmann factors $w(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, $\sigma_i =$ spin variable. Two transfer matrices $T = T[w]$ and $T' = T[w']$ commute if there exists a third function w'' such that for all σ_i, $i = 1, \ldots, 6$,

$$\sum_{\sigma_7} w(\sigma_2, \sigma_3, \sigma_7, \sigma_1) w'(\sigma_1, \sigma_7, \sigma_5, \sigma_6) w''(\sigma_4, \sigma_5, \sigma_7, \sigma_3)$$

$$= \sum_{\sigma_7} w''(\sigma_7, \sigma_6, \sigma_1, \sigma_2) w'(\sigma_2, \sigma_3, \sigma_4, \sigma_7) w(\sigma_7, \sigma_4, \sigma_5, \sigma_6). \tag{4.15}$$

This is the *Yang-Baxter* (or *star-triangle*) *equation* in spin variables. The function w'' is the spin version of "R" in (4.10). (4.15) may be derived from (4.10) via the vertex $\leftrightarrow$ spin correspondence, or a derivation similar to that of (4.10) may be given directly (see Baxter **[10, 12]**). (4.15) has a nice graphical representation (see Figure 6). We can think of (4.15) as a partition function of a graph containing three quadrilateral faces, with a center spin σ_7 and boundary spins $\sigma_1, \ldots, \sigma_6$. The star-triangle relation says these two partition functions are equal.

Again, I wish to emphasize that once nontrivial solutions to (4.15) are found, then the model has turned out to be "exactly solvable." Until quite recently all solutions to (4.10) or (4.15) involved curves of either genus 0 or 1. As Professor McCoy will report on at this conference, solutions have been found by Au-Yang et. al. **[5]** where the genus of the curve is greater than one. This new development is extremely important for solvable models.

5. Corner transfer matrices and order parameters. The *corner transfer matrix* A associated with the spin configurations shown in Figure 7 is defined to be

$$A_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} = \begin{cases} \sum \prod w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) & \text{if } \sigma_1 = \sigma_1', \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$, $\boldsymbol{\sigma}' = (\sigma_1', \ldots, \sigma_m')$, and the sum is over all internal spins (depicted as open circles in Figure 7) and the product is over all the elementary squares in the corner depicted in Figure 7. The weight $w(a, b, c, d)$, in the case of the Baxter model, is the Boltzmann weight (4.14). The matrix A is

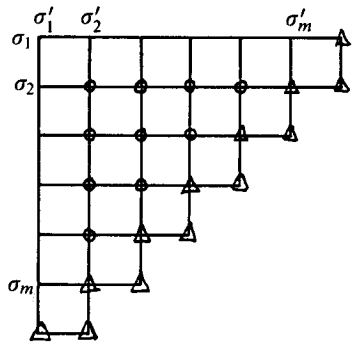


FIGURE 7. Corner transfer matrix A. The open circles correspond to sites summed over in sum (5.1). The triangular sites correspond to sites fixed at the ground state values. For the Baxter model order parameter $\langle\sigma_i\rangle$, the ground state corresponds to $\sigma_1 = +1$ for all $i \in \Lambda$.

for the SE corner; matrices B, C, and D for the NE, NW, and SW corners, respectively, are similarly defined. Then the partition function is given by

$$Z_\Lambda = \mathrm{tr}(ABCD), \tag{5.2}$$

and if we define the matrix $\hat{\sigma}_1$

$$(\hat{\sigma}_1)_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} = \sigma_1 \delta(\boldsymbol{\sigma}, \boldsymbol{\sigma}'),$$

then

$$\langle\sigma_1\rangle = \lim_{m\to\infty} \frac{\mathrm{Tr}(\hat{\sigma}_1 ABCD)}{\mathrm{Tr}(ABCD)}. \tag{5.3}$$

Fixing the edge spins (see Figure 7) to their ground state value $\sigma_1 = +1$, suitably normalized corner transfer matrices can be diagonalized in the limit $m \to \infty$. The spectrum remains discrete, which is in contrast to the row transfer matrix. The Yang-Baxter equations and the elliptic parametrization are crucial in this diagonalization (for details see Baxter [**10**, **12**]). Baxter thus arrives at formulas of the form

$$\langle\sigma_1\rangle = \lim_{m\to\infty} \sum_{\sigma_1\cdots\sigma_m} \sigma_1 M(\sigma_1,\ldots,\sigma_m) \Big/ \sum_{\sigma_1\cdots\sigma_m} M(\sigma_1,\ldots,\sigma_m), \tag{5.4}$$

where $M(\sigma_1,\ldots,\sigma_m)$ (for all values of $\sigma_1,\ldots,\sigma_m$) are the eigenvalues of $ABCD$. Explicitly for the Baxter model

$$M(\sigma_1,\ldots,\sigma_m) = \exp(-2\lambda(\sigma_1\sigma_3 + 2\sigma_2\sigma_4 + \cdots + m\sigma_m\sigma_{m+2}))$$

(λ is related to the parameter η in (4.12)). Making the change of variables $\mu_j = \sigma_j\sigma_{j+2}$ and $\sigma_1 = \mu_1\mu_3\cdots\mu_m$, Baxter finds

$$\begin{aligned}\langle\sigma_1\rangle &= \lim_{m\to\infty} \frac{\sum_{\mu_i=\pm1} \mu_1\mu_3\cdots\mu_m x^{\mu_1+2\mu_2+\cdots+m\mu_m}}{\sum_{\mu_i=\pm1} x^{\mu_1+2\mu_2+\cdots+m\mu_m}} \\ &= \prod_{j=1}^{\infty} \frac{1-q^{2j-1}}{1+q^{2j-1}}, \qquad q = e^{-4\lambda} = x^2,\end{aligned} \tag{5.5}$$

which is the order parameter for the Baxter model.

6. Hard hexagon model. The hard hexagon model is a lattice gas on a triangular lattice with the rule that if a site is occupied then the six neighboring sites are necessarily vacant. Such a rule can be visualized by placing "hard" hexagons on the lattice (see Figure 8). Since the energy of a configuration is either $+\infty$ (such a configuration violates the hard hexagon rule) or is some fixed constant, say 0, the partition function, $Z_{n,N}$ in (2.4), becomes simply a combinatoric factor $g(n, N)$, equal to the number of ways of placing n hexagons on a lattice of N sites (assuming periodic b.c.). Thus the grand partition function (2.7b) is

$$Z_N = \sum_{n=0}^{\infty} \mathfrak{z}^n g(n, N), \tag{6.1}$$

where $g(n, N)$ is zero for $n > [N/3]$ and $\mathfrak{z}$ is the activity (see (2.7)). For example, $g(1, N) = N$ and $g(2, N) = \frac{1}{2}N(N - 7)$.

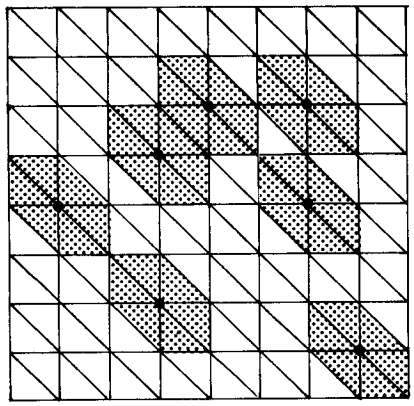


FIGURE 8. A configuration of hard hexagons

If we introduce an occupation variable $\sigma_i = 0, 1$ ($\sigma_i = 1$ means there is a particle at site i, etc.), then another quantity of interest is the local density at site i

$$\rho_i = \langle \sigma_i \rangle. \tag{6.3}$$

Intuitively we see that there are two extreme cases; namely, that of high density (close packing of the hexagons) and that of low density. In the close packing limit, one of three possible sublattices is completely occupied, say the "1" sublattice, and the other two, say "2" and "3" sublattices, are completely vacant. That is, in the close packing limit we expect

$$\rho_1 = 1, \qquad \rho_2 = \rho_3 = 0, \tag{6.4}$$

where the subscript i means $\rho_i = \langle \sigma_i \rangle$ is evaluated for a point in sublattice "i". In the low density limit we expect no preferential ordering on the sublattices,

i.e.,

$$\rho_1 = \rho_2 = \rho_3. \tag{6.5}$$

In a remarkable pair of papers Baxter [**8**, **9**] (see also Baxter and Pearce [**13**, **14**]) showed that the hard hexagon model is exactly solvable in the sense of commuting transfer matrices. To show this Baxter generalized the hard hexagon model to a square lattice gas model with diagonal interactions and nearest neighbor exclusion (this is necessary to obtain a spectral parameter, i.e., a family of commuting transfer matrices). Precisely, if $w(a,b,c,d)$ is the Boltzmann weight for the configuration of an elementary square, $a,b,c,d \in \{0,1\}$, then

$$w(a,b,c,d) = \begin{cases} \mathfrak{z}^{(a+b+c+d)/4} & \exp(Lac + Mbd) \\ & \text{if } ab = bc = cd = da = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6.6}$$

where L and M are diagonal interaction constants (see Figure 5) and $\mathfrak{z}$ is the activity. The hard hexagon model is regained by taking $L = 0$ and $M = -\infty$.

Using (6.6), Baxter showed that the star-triangle relations (4.15) have a nontrivial solution provided that $\mathfrak{z}, L$, and M satisfy

$$\mathfrak{z} = (1 - e^{-L})(1 - e^{-M})/(e^{L+M} - e^{L} - e^{M}). \tag{6.7}$$

In the hard hexagon limit, (6.7) puts no restriction on $\mathfrak{z}$. An explicit parametrization of the weights in (6.6) that satisfies (6.7) is [**8**, **10**]

$$\begin{aligned} &w(0,0,0,0) = \theta(2\lambda + u)/\theta(2\lambda), \\ &w(1,0,0,0) = w(0,0,1,0) = \theta(u)/[\theta(\lambda)\theta(2\lambda)]^{1/2}, \\ &w(0,1,0,0) = w(0,0,0,1) = \theta(\lambda - u)/\theta(\lambda), \\ &w(1,0,1,0) = \theta(2\lambda - u)/\theta(2\lambda), \\ &w(0,1,0,1) = \theta(2\lambda + u)/\theta(\lambda), \end{aligned} \tag{6.8}$$

where $\theta(u) = \mathcal{J}_{11}(u/\pi, \tau)$ and $\lambda = \pi/5$. The hard hexagon model is $u = -\pi/5$.

As in the Baxter model, Baxter introduces corner transfer matrices A, B, C, and D. Again suitably normalized corner transfer matrices have limits (this is not rigorous, though) in the thermodynamic limit $m \to \infty$ (see Figure 7). These limits are taken fixing the edge spins to their "ground state" values. The hard square lattice gas, as defined by (6.6) and (6.7), has four phases, so there are four "ground states" to consider. The simplest case is the disordered regime (called Regime I by Baxter), in which case Baxter shows that in the diagonal representation

$$(ABCD)_{\sigma,\sigma'} = r_0^{2\sigma_1} q^{\sigma_2 + 2\sigma_3 + 3\sigma_4 + \cdots} \delta_{\sigma,\sigma'}, \tag{6.9}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots)$, etc. The spin indices σ_i are subject to the *hard hexagon constraint*

$$\sigma_i \sigma_{i+1} = 0, \qquad i = 1, 2, \ldots, \tag{6.10}$$

and r_0^2 is given by

$$r_0^2 = -xG(x)/H(x), \qquad q = x^6, \tag{6.11}$$

where $G(x)$ and $H(x)$ are

$$G(x) = \prod_{n=1}^{\infty}[(1-x^{5n-4})(1-x^{5n-1})]^{-1}, \tag{6.12a}$$

$$H(x) = \prod_{n=1}^{\infty}[(1-x^{5n-3})(1-x^{5n-2})]^{-1}. \tag{6.12b}$$

Those familiar with the Rogers-Ramanujan identities (see, e.g., [2]) recognize (6.12) as the product side of these identities.

Thus, when computing the trace in (5.3), Baxter was led to sums of the form

$$F(\sigma_1) = \sum_{\sigma_2,\sigma_3,\ldots}{}' q^{\sigma_2+2\sigma_3+3\sigma_4+\cdots}, \tag{6.13}$$

where the sum $\sum'$ is subject to the constraint (6.10). Thus, in the disordered regime, Baxter finds

$$\rho = \langle\sigma_1\rangle = \frac{r_0^2 F(1)}{F(0)+r_0^2 F(1)}. \tag{6.14}$$

However, $F(0)$ and $F(1)$ are related to the sum side of the Rogers-Ramanujan identity (perhaps more clearly, $F(0)$ and $F(1)$ are related to one side of the *combinatoric* version of the Rogers-Ramanujan identities):

$$F(0) = G(q), \qquad F(1) = H(q). \tag{6.15}$$

Using (6.11) and (6.15) in (6.14) we have

$$\rho = -xG(x)H(x^6)/[H(x)G(x^6) - xG(x)H(x^6)]. \tag{6.16}$$

A further identity of Ramanujan is

$$H(x)G(x^6) - xG(x)H(x^6) = P(x)/P(x^3)$$

with $P(x) = \prod_{n=1}^{\infty}(1-x^{2n-1})$, which implies

$$\rho = -xG(x)H(x^6)P(x^3)/P(x), \tag{6.17}$$

where $\rho_1 = \rho_2 = \rho_3 = \rho$.

The analysis in the other three regimes is more complicated since new Rogers-Ramanujan type identities had to be discovered and proved by Baxter [9] and Andrews [3]. In the hard hexagon limit there are just two phases, a disordered phase and an ordered phase.

We give Baxter's complete results for the hard hexagon model. If $\mathscr{H} = \{z \in \mathbf{C} | \operatorname{Im} z > 0\}$ denotes the upper half-plane, then in the *disordered regime*

$$\kappa(\tau) = \frac{\eta^2(5\tau)\eta^2_{(1,0)}(5\tau;5)\eta_{(3,0)}(6\tau;6)\eta_{(2,0)}(6\tau;6)}{\eta^2(6\tau)\eta^3_{(2,0)}(5\tau;5)\eta_{(1,0)}(6\tau;6)}, \tag{6.18}$$

$$\mathfrak{z}(\tau) = -\left[\frac{\eta_{(1,0)}(5\tau;5)}{\eta_{(2,0)}(5\tau;5)}\right]^5, \tag{6.19}$$

$$\rho(\tau) = \rho_\tau = -\frac{\eta(2\tau)\eta(3\tau)}{\eta(\tau)\eta(6\tau)}\left[\eta_{(1,0)}(5\tau;5)\eta_{(2,0)}(30\tau;5)\right]^{-1}, \tag{6.20}$$

where the physical values of $\tau \in \mathscr{H}$ are $\operatorname{Re}\tau = \frac{1}{2}$, and in the *ordered regime*

$$\kappa(\tau) = \frac{\eta^2(5\tau)\eta^2_{(2,0)}(5\tau;5)\eta_{(1,0)}(3\tau;3)}{\eta^2(3\tau)\eta^2_{(1,0)}(5\tau;5)}, \tag{6.21}$$

$$\mathfrak{z}(\tau) = \left[\frac{\eta_{(2,0)}(5\tau;5)}{\eta_{(1,0)}(5\tau;5)}\right]^5, \tag{6.22}$$

$$R(\tau) = \rho_1 - \rho_2 = \frac{\eta(\tau)\eta(5\tau)}{\eta^2(3\tau)}, \tag{6.23}$$

and

$$\rho_2(\tau) = \rho_3(\tau) = \frac{\eta(\tau)\eta(9\tau)}{\eta^2(3\tau)}[\eta_{(2,0)}(5\tau;5)\eta_{(2,0)}(45\tau;5)]^{-1}, \tag{6.24}$$

and the physical values of $\tau \in \mathscr{H}$ in the ordered regime are $\operatorname{Re}\tau = 0$.

We have written Baxter's results in terms of the Dedekind eta function $\eta(\tau)$ and the generalized Dedekind eta function $\eta_{\mathbf{g}}(\tau;N)$ (see Schoeneberg [31] and Appendix A). The remarkable mathematical structure of Baxter's results (6.8)–(6.24) is summarized in the following theorem [36]. First define $\Gamma_1[N] = \{A \in \mathrm{SL}(2,\mathbf{Z}) | A \equiv \pm\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \bmod N\}$.

THEOREM 6.1. *In the disordered regime, $\mathfrak{z}(\tau)$ is a modular function with respect to the group $\Gamma_1[5]$, and $\kappa(\tau)$ and $\rho(\tau)$ are modular functions with respect to the group $\Gamma_1[30]$. In the ordered regime, $\mathfrak{z}(\tau)$ is a modular function with respect to the group $\Gamma_1[5]$, $\kappa^3(\tau)$ and $R^3(\tau)$ are modular functions with respect to the group $\Gamma_1[15]$, and $\rho_2(\tau) = \rho_3(\tau)$ are modular functions with respect to the group $\Gamma_1[45]$.*

Baxter's corner transfer matrix has produced meromorphic functions on various compact (number-theoretical) Riemann surfaces in uniformized form! The only way we understand this result is to do the calculation. That is, there is no proof before these calculations that these physical quantities are such beautiful functions.

In the grand canonical ensemble one has, see (2.14), $\kappa = \kappa(\mathfrak{z})$ and $\rho = \rho(\mathfrak{z})$. Using the modular properties one can show [36] that the only branch points of κ and ρ occur at $\mathfrak{z} = 0, \mathfrak{z}_c, \mathfrak{z}_{NP}, \infty$, where

$$\mathfrak{z}_c = \frac{11 + 5\sqrt{5}}{2} = \left(\frac{1+5\sqrt{5}}{2}\right)^5$$

is the critical activity, and

$$\mathfrak{z}_{\mathrm{NP}} = \frac{11 - 5\sqrt{5}}{2} = -\left(\frac{\sqrt{5}-1}{2}\right)^5$$

is a nonphysical activity. In the disordered regime κ is a 24-sheeted function of $\mathfrak{z}$, and on the physical sheet the point $\mathfrak{z} = 0$ is a holomorphic point. The function $\zeta(\tau) = \eta_{(1,0)}(5\tau; 5)/\eta_{(2,0)}(5\tau; 5)$ is the famous *Klein icosahedron function.*

In the disordered regime $\kappa(\tau)$ has valence 22 and $\rho(\tau)$ has valence 8, so there exists a polynomial of degree less than or equal to 22 in ρ and degree less than or equal to 8 in κ:

$$P(\kappa, \rho) = 0. \tag{6.19}$$

A complete cusp analysis for $\kappa(\tau)$ and $\rho(\tau)$ can be given (the transformation properties of $\eta(\tau)$ and $\eta_{\mathfrak{g}}(\tau; N)$ under $\mathrm{SL}(2, \mathbf{Z})$ are well known, see Schoeneberg [31] or Appendix A). This makes an explicit determination of P possible [27]: Let

$$\begin{aligned}
y &= \rho^{-1},\\
g_0 &= 432^2,\\
g_1 &= 432y^{10}[1 + 2y + 48y^2 + 56y^3 - 42y^4 - 12y^5\\
&\qquad + 100y^6 - 132y^7 - 625y^{12}],\\
g_2 &= 16y^4 + 192y^5 + 645y^6 - 516y^7 - 5826y^8 - 4116y^9\\
&\quad + 9349y^{10} - 11400y^{11} - 42672y^{12} - 9800y^{13}\\
&\quad + 7350y^{14} - 4500y^{15} + 1750y^{16} + 3125y^{22},\\
g_3 &= -1 - 12y - 48y^2 - 56y^3 + 42y^4 + 12y^5\\
&\quad - 100y^6 + 132y^7 + 625y^{12},\\
g_4 &= y^2;
\end{aligned}$$

then

$$g_0 + g_1\kappa^2 + g_2\kappa^4 + g_3\kappa^6 + g_4\kappa^8 = 0. \tag{6.20}$$

This is the equation of state for the hard hexagon model in the disordered regime. The critical point (κ_c, ρ_c),

$$\kappa_c = \left[\frac{27}{250}(25 + 11\sqrt{5})\right]^{1/2}, \qquad \rho_c = \frac{5 - \sqrt{5}}{10},$$

is a cusp with tangent line $\kappa - \kappa_c = 0$ of the algebraic curve (6.20).

The reduced compressibility χ (see (2.10) and (2.11)) in terms of κ and ρ is

$$\chi = \kappa \Big/ \frac{d\kappa}{d\rho} = -\kappa \frac{\partial P}{\partial \kappa} \Big/ \frac{\partial P}{\partial \rho}, \tag{6.21}$$

i.e., χ is a rational function of κ and ρ. Similar polynomial relations can be derived in the ordered regime including a polynomial relation between κ in the disorderd phase and κ in the ordered phase [28]. For $\rho \to \rho_c^-$ we have

$$\chi(\rho) = \frac{5+\sqrt{5}}{75} t^{-1/2} \left[1 - 2t^{1/2} + \frac{1}{8}(1+4\sqrt{5})t + O(t^{3/2}) \right] \tag{6.22}$$

with $t = \sqrt{5}(\rho_c - \rho)$. This divergence of the compressibility (see (2.10) and (2.11)) is characteristic of second-order phase transitions.

Appendix A: Dedekind functions. For $\tau \in \mathscr{H}$ the Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau}). \tag{A.1}$$

For $\mathbf{g} = (g,h) \in \mathbf{Z}_N \times \mathbf{Z}_N$ we have the generalized Dedekind functions [31]

$$\eta_{\mathbf{g}}(\tau; N) = \alpha(\mathbf{g}) e^{\pi i P_2(g/N)\tau} \prod_{\substack{m>0 \\ m \equiv g \bmod N}} (1 - \omega_N^h e^{(2\pi i/N)\tau m}) \times \prod_{\substack{m>0 \\ m \equiv -g \bmod N}} (1 - \omega_N^{-h} e^{(2\pi i/N)\tau m}) \tag{A.2}$$

with

$$\alpha(\mathbf{g}) = \begin{cases} (1 - \omega_N^{-h}) e^{\pi i P_1(h/N)} & \text{if } g \equiv 0 \text{ and } h \not\equiv 0 \bmod N, \\ 1 & \text{otherwise,} \end{cases}$$

$\omega_N = \exp(2\pi i/N)$, $P_1(x) = x - [x] - \frac{1}{2}$, and $P_2(x) = (x-[x])^2 - (x-[x]) + \frac{1}{6}$.

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbf{Z})$, $\eta(\tau)$ and $\eta_{\mathbf{g}}(\tau; N)$ transform as follows:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \exp\left(\frac{\pi i}{12}\Phi(A)\right) \sqrt{\frac{c\tau+d}{i}}\, \eta(\tau), \tag{A.3}$$

and for $\mathbf{g} \not\equiv (0,0) \bmod N$,

$$\eta_{\mathbf{g}}\left(\frac{a\tau+b}{c\tau+d}; N\right) = \exp(\pi_{\mathbf{g}}(A)) \eta_{\mathbf{g}'}(\tau; N), \tag{A.4}$$

where $\mathbf{g}' = (g',h') = (g,h)A$,

$$\Phi(A) = \begin{cases} (a+d)/c - 12\,\mathrm{sgn}(c) S(d,|c|) & \text{for } c \neq 0, \\ b/d & \text{for } c = 0, \end{cases} \tag{A.5}$$

$$\pi_{\mathbf{g}}(A) = \begin{cases} \pi i \left[\frac{a}{c} P_2\left(\frac{g'}{n}\right) + \frac{d}{c} P_2\left(\frac{g}{N}\right) - 2\,\mathrm{sgn}(c) S_{\mathbf{g}}^N(a,c) \right] & \text{for } c \neq 0, \\ \pi i \frac{b}{d} P_2\left(\frac{g}{n}\right) & \text{if } c = 0, \end{cases} \tag{A.6}$$

and $S(a,c)$ is the Dedekind sum

$$S(a,c) = \sum_{\nu \bmod c} \left(\left(\frac{\nu}{c}\right)\right) \left(\left(\frac{a\nu}{c}\right)\right) \tag{A.7}$$

and $S^N_{g,h}(a,c)$ is the generalized Dedekind sum

$$S^N_{g,h}(a,c) = \sum_{\nu \bmod c} \left(\left(\frac{g+\nu N}{cN}\right)\right)\left(\left(\frac{g'+a\nu N}{cN}\right)\right), \tag{A.8}$$

where $((x)) = x - [x] - \frac{1}{2} + \frac{1}{2}\delta(x)$ with $\delta(x) = 1$ if $x \in \mathbf{Z}$ and 0 otherwise.

In particular, for $A = \Gamma[N]$ and $\mathbf{g} \not\equiv (0,0) \bmod N$

$$\eta_{\mathbf{g}}(A\tau; N) = e^{\pi_{\mathbf{g}}(A)}\eta_{\mathbf{g}}(\tau; N). \tag{A.9}$$

Since $\pi_{\mathbf{g}}(A) \in \mathbf{Q}$, there exists an integer N_1 such that $(\eta_{\mathbf{g}}(\tau; N))^{N_1}$ is a *modular function of level* N, $\mathbf{g} \not\equiv (0,0)$. According to Schoeneberg, the choice $N_1 = 12N/(6,N)$ works. The periods $\pi_{\mathbf{g}}(A)$ satisfy

$$\pi_{\mathbf{g}}(AB) = \pi_{\mathbf{g}}(A) + \pi_{t_{Ag}}(B), \tag{A.10}$$

so that if $A \in \Gamma[N]$ we have

$$\pi_{\mathbf{g}}(AB) = \pi_{\mathbf{g}}(A) + \pi_{\mathbf{g}}(B). \tag{A.11}$$

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