2002 Spring Math Contest

1. Prove that

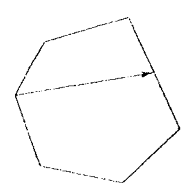
$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

2. Evaluate the integral

$$\int_0^\infty e^{-\sqrt{x}}\,dx.$$

3 3. Suppose that for some n the numbers 2^n and 5^n begin with the same digit. What are all of the possibilities for this digit?

4. Given a hexagon, consider the vector from a vertex to the midpoint of the 3rd edge clockwise from that vertex.



Show that the sum of these vectors over all vertices is 0.

5. 4. Show that

$$1 - \left(\frac{2x}{\pi}\right)^2 \geq \cos x$$

when $|x| \leq \frac{\pi}{2}$.

6. Consider a one-player game in which you move from 0 to other numbers in a sequence of steps. At each step you can either double your number or add 1 to it. How many steps do you need to get to 100?

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No books or calculators are allowed. Pens, pencils, and erasers are okay.

Write your name and student ID in the upper right hand corner of this sheet. Write your initials on every page of work you wish graded. when you are finished, staple all of your work together behind this sheeet.

Every problem is worth the same amount. You must completely justify every answer; partial credit may be given for significant progress towards a solution. The proctor may not answer any questions about individual problems.

1. Suppose you have a painting with a string attached and two nails in a wall. How can you hang the painting winding the string about the two nails in such a way that the painting will fall to the ground if *either* nail is removed?
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with period 2. Prove that there exists an $x \in \mathbb{R}$ such that $f(x+1) = f(x)$.
3. Let $a_1, \ldots, a_m$ be m numbers, each equal to ± 1. Prove that if
$$a_1a_2 + a_2a_3 + \cdots + a_{m-1}a_m + a_ma_1 = 0,$$
then m is divisible by 4.
4. Let k, n be integers such that $1 < k < n$. Prove that the system of n equations in the variables $x_1, x_2, \ldots, x_n$
$$x_i + x_{i+1} + \cdots + x_{i+k-1} = 0 \qquad \text{for } i = 1, 2, \ldots, n$$
has a non-trivial solution if and only if $\gcd(n, k) > 1$. Here we identify x_{i+n} with x_i.
5. Let C_1 and C_2 be two circles of equal radius. On C_1, 100 points are marked. On C_2, finitely many arcs are marked such that the total length of all arcs is less than 1% of the circumference of the circle. Show that one can put C_1 on top of C_2 in such a way that no marked point of C_1 will be inside a marked arc of C_2.