1. (14 Points) In parts (a)-(e), give a careful definition of the term(s) in bold.

(a) The **kernel** of a group homomorphism $\varphi : G \to H$.

(b) A **normal subgroup** of a group G.

(c) A **field** F.

(d) A **vector space** over a field F.

(e) An **eigenvector** for a linear operator $T : V \to V$.

2. (14 Points) In parts (a)-(e), decide if the given statement is **True** or **False**. If it is true, give a *brief* explanation. If it is false, explain why or give a counter-example.

(a) **True** or **False**: The index of $\langle \overline{8} \rangle$ in $\mathbb{Z}_{20}$ is 4.

(b) **True** or **False**: If N is a normal subgroup of a group G and G/N is abelian, then G is abelian.

(c) **True** or **False**: There is an isomorphism $\varphi : S_5 \to \mathbb{Z}_{120}$.

(d) **True** or **False**: The set of real numbers $\mathbb{R}$ is a vector space over the set of rational numbers $\mathbb{Q}$ under the usual addition and multiplication.

(e) **True** or **False**: $|\,\mathrm{GL}_3(\mathbb{Z}_5)| = (124)(120)(100)$.

3. (12 Points) Suppose that $T : \mathbb{Z}_5^3 \to \mathbb{Z}_5^2$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ 0 \end{bmatrix}.$$

Compute the matrix of T with respect to the standard bases in $\mathbb{Z}_5^j$, $j = 2, 3$.

4. (12 Points) (a) Compute the order of $\overline{9}$ in $\mathbb{Z}_{48}$.

(b) Compute the index $[\mathbb{Z}_{48} : \langle \overline{9} \rangle]$.

(c) Identify the quotient $\mathbb{Z}_{48}/\langle \overline{9} \rangle$.

5. (12 Points) Show that every group of prime order is cyclic.

6. (12 Points) Let H be the subset of $\mathrm{GL}_2(\mathbb{R})$ defined by

$$H = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, x \neq 0 \right\}.$$

(a) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(b) Show that the map $\varphi : H \to \mathbb{R}^\times$ defined by

$$\varphi\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \right) = x$$

is a group homomorphism.

(c) Identify the quotient group $H/\operatorname{Ker}\varphi$.

7. (12 Points) Let $P_3 = P_3(\mathbb{R})$ be the vector space of polynomials over $\mathbb{R}$ of degree less than or equal to 3 and let $\frac{d}{dx} : P_3 \to P_3$ be given by

$$\frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

It is well known that $\frac{d}{dx}$ is linear.

(a) Compute the matrix of $\frac{d}{dx}$ with respect to the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for P_3.

(b) Compute the characteristic polynomial $p(\lambda)$ for $\frac{d}{dx}$.

(c) Is $\frac{d}{dx}$ diagonalizable? Explain.

8. (12 Points) Let V be a vector space over a field F, and let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis for V. Recall that F is a one dimensional vector space over itself and define

$$L(V, F) = \{T : V \to F \mid T \text{ is a linear transformation}\}.$$

For each $j = 1, \ldots, n$, let $T_j \in L(V, F)$ be defined by

$$T_j(v_k) = \delta_{jk} = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{if } k \neq j. \end{cases}$$

Show that $\mathcal{C} = (T_1, \ldots, T_n)$ is a basis for the vector space $L(V, F)$. Is $L(V, F)$ isomorphic to V? Prove your answer.