Homework 2
due October 16, 2001

(1) (a) Let σ be the m-cycle $(a_1 a_2 \ldots a_m)$ in S_n. Show that $|\sigma| = m$.
(b) Show that the order of an element in S_n is the least common multiple of the lengths of the cycles in its cycle decomposition.

(2) Let $\phi : G \to H$ be a homomorphism of groups, A a subgroup of G, and B a subgroup of H. Show that
(a) $\ker\phi$ and $\phi^{-1}(B) = \{a \in G \mid \phi(a) \in B\}$ are subgroups of G.
(b) $\phi(A)$ is a subgroup of H.

(3) Dummit, Foote I.1.7 Exercise 18 (page 45)

(4) Dummit, Foote I.1.7 Exercise 19 (page 45)

(5) Let G and H be groups. Define the direct product of G and H to be the set $G \times H$ with binary operation
$$(a, b)(a', b') = (aa', bb') \qquad \text{where } a, a' \in G \text{ and } b, b' \in H.$$
(a) Show that $G \times H$ is a group.
(b) Let $\langle a \rangle$ and $\langle b \rangle$ be finite cyclic groups of orders m and n, respectively, which are relatively prime. Prove that $\langle a \rangle \times \langle b \rangle$ is cyclic.
(c) What about the converse?

(6) Dummit, Foote I.2.2 Exercise 10 (page 54)

(7) Dummit, Foote I.2.3 Exercise 26 (page 62)

Extra problem: Two elements $a, b \in G$ of the group G are called conjugate if there is a $c \in G$ such that $a = cbc^{-1}$.

(1) Prove that the cycles of maximal length in S_n are conjugate. How many cycles of maximal length are there in S_n?

(2) Consider the cycle $\xi = (12 \cdots n) \in S_n$. What is the centralizer of ξ?

(3) We showed in class that every permutation $\xi \in S_n$ can be written as

$$\xi = \xi_1 \circ \cdots \circ \xi_r$$

where the ξ_i are cycles and each number $1, 2, \dots, n$ occurs in precisely one cycle. Let n_i denote the length of cycle ξ_i. We may assume without loss of generality that

$$n_1 + \cdots + n_r = n$$
$$n_1 \geq n_2 \geq \cdots \geq n_r.$$

A tuple $(n_1, \dots, n_r)$ with these properties is called a partition of n. To each $\xi \in S_n$ we have associated a unique partition of n. Prove the following: Two permutations $\xi_1, \xi_2 \in S_n$ are conjugate if and only if the corresponding partitions of n agree.