Homework 8
due December 4, 2001

(1) A ring R such that $a^2 = a$ for all $a \in R$ is called a Boolean ring. Prove that every Boolean ring R is commutative and $a + a = 0$ for all $a \in R$.

(2) (a) Let R be a ring. An element $x \in R$ is called nilpotent if $x^n = 0$ for some n. Prove that the set of all nilpotent elements in a commutative ring is an ideal I. Besides the zero element, the ring R/I has no nilpotent elements.
(b) Give an example of a (noncommutative) ring, in which the set of nilpotent elements is not an ideal.

(3) Let $\mathbb{Q}$ be the field of rational numbers and R any ring. If $f, g : \mathbb{Q} \to R$ are homomorphisms of rings such that $f|\mathbb{Z} = g|\mathbb{Z}$, then $f = g$.
[Hint: Show that for $n \in \mathbb{Z}, n \neq 0$, $f(\frac{1}{n})g(n) = g(1)$].

(4) Let I be an ideal in a commutative ring R and let $\mathrm{Rad}I = \{r \in R \mid r^n \in I \text{ for some } n\}$. Show that $\mathrm{Rad}I$ is an ideal.

(5) For each part give an example of a ring R with the specified properties:
(a) R has a left ideal, that is not a right ideal.
(b) R has zero divisors and an ideal I such that R/I has no zero divisors and R/I has at least 17 elements.

(6) (a) Show that $\mathbb{Z}$ is a principal ideal ring.
(b) Every homomorphic image of a principal ideal ring is also a principal ideal ring.
(c) $\mathbb{Z}_m$ is a principal ideal ring for every $m > 0$.

Extra Problem: Prove using Zorn's lemma:
Let R be a commutative ring. Define

$$S = \bigcap_{P \text{ prime ideal of } R} P.$$

Then S is the set of all nilpotent elements of R.
[Hint: If $r \in R$ is not nilpotent, consider the set of all ideals in R which do not contain any power of r.]