Homework Set Four: Matrices and Linear Maps

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises **separately** at the **beginning** of lecture on **Friday, October 26, 2007**. The two problems sets will be graded by different persons.

Calculational Exercises

Submit solutions to Exercises 1, 2(i, m, r), 3(a), and 4(a).

1. Suppose that A, B, C, D, and E are matrices over $\mathbb{F}$ having the following sizes:

$$A \text{ is } 4\times 5, \quad B \text{ is } 4\times 5, \quad C \text{ is } 5\times 2, \quad D \text{ is } 4\times 2, \quad E \text{ is } 5\times 4.$$

   Determine whether the following matrix expressions are defined, and, for those that are defined, determine the size of the resulting matrix.

   (a) BA (b) $AC+D$ (c) $AE+B$ (d) $AB+B$ (e) $E(A+B)$ (f) $E(AC)$

2. Suppose that A, B, C, D, and E are the following matrices:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \text{ and } E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

   Determine whether the following matrix expressions are defined, and, for those that are defined, compute the resulting matrix.

   (a) $D+E$ (b) $D-E$ (c) $5A$ (d) $-7C$ (e) $2B-C$
   (f) $2E-2D$ (g) $-3(D+2E)$ (h) $A-A$ (i) AB (j) BA
   (k) $(3E)D$ (l) $(AB)C$ (m) $A(BC)$ (n) $(4B)C+2B$ (o) $D-3E$
   (p) $CA+2E$ (q) $4E-D$ (r) DD

3. In each of the following, find matrices A, x, and b such that the given system of linear equations can be expressed as the single matrix equation $Ax=b$.

$$\text{(a)}\ \left.\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 5x_3 & = & 7 \\ 9x_1 & - & x_2 & + & x_3 & = & -1 \\ x_1 & + & 5x_2 & + & 4x_3 & = & 0 \end{array}\right\} \qquad \text{(b)}\ \left.\begin{array}{rcrcrcrcr} 4x_1 & & & - & 3x_3 & + & x_4 & = & 1 \\ 5x_1 & + & x_2 & & & - & 8x_4 & = & 3 \\ 2x_1 & - & 5x_2 & + & 9x_3 & - & x_4 & = & 0 \\ & & 3x_2 & - & x_3 & + & 7x_4 & = & 2 \end{array}\right\}$$

4. In each of the following, express the matrix equation as a system of linear equations.

$$\text{(a)} \begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Proof-Writing Exercises

1. Let U, V, and W be vector spaces over $\mathbb{F}$, and suppose that the linear maps $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$ are both injective. Prove that the composition map $T \circ S$ is injective.

2. Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is surjective. Given a spanning list $(v_1, \ldots, v_n)$ for V, prove that $\text{span}(T(v_1), \ldots, T(v_n)) = W$.