Sample Final Exam

The real final exam will consist of 8 questions, one of which is a question where you will need to state definitions or theorems from class, and one of which is a true or false question. One of the problems below will appear on the final exam.

1. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be given by $v_1 = (1, 2, 1)$, $v_2 = (1, -2, 1)$, and $v_3 = (1, 2, -1)$. Apply the Gram-Schmidt procedure to the basis (v_1, v_2, v_3) of $\mathbb{R}^3$, and call the resulting orthonormal basis (u_1, u_2, u_3).

2. Let $A \in \mathbb{C}^{3\times 3}$ be given by
$$A = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -1 \end{bmatrix}.$$
   (a) Calculate $\det(A)$.
   (b) Find $\det(A^4)$.

3. Let $P \subset \mathbb{R}^3$ be the plane containing 0 perpendicular to the vector $(1, 1, 1)$. Using the standard norm, calculate the distance of the point $(1, 2, 3)$ to P.

4. Let $V = \mathbb{C}^4$ with its standard inner product. For $\theta \in \mathbb{R}$, let
$$v_\theta = \begin{pmatrix} 1 \\ e^{i\theta} \\ e^{2i\theta} \\ e^{3i\theta} \end{pmatrix} \in \mathbb{C}^4.$$
   Find the canonical matrix of the orthogonal projection onto the subspace $\{v_\theta\}^\perp$.

5. Let $r \in \mathbb{R}$ and let $T \in \mathcal{L}(\mathbb{C}^2)$ be the linear map with canonical matrix
$$T = \begin{pmatrix} 1 & -1 \\ -1 & r \end{pmatrix}.$$
   (a) Find the eigenvalues of T.
   (b) Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of T.
   (c) Find a unitary matrix U such that UTU^* is diagonal.

6. Let A be the complex matrix given by:

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$$

   (a) Find the eigenvalues of A.
   (b) Find an orthonormal basis of eigenvectors of A.
   (c) Calculate $|A| = \sqrt{A^*A}$.
   (d) Calculate e^A.

7. Give an orthonormal basis for $\operatorname{null}(T)$, where $T \in \mathcal{L}(\mathbb{C}^4)$ is the map with canonical matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

8. Describe the set of solutions $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ of the system of equations

$$\left.\begin{array}{rl} x_1 - x_2 + x_3 & = 0 \\ x_1 + 2x_2 + x_3 & = 0 \\ 2x_1 + x_2 + 2x_3 & = 0 \end{array}\right\}.$$

9. Prove or give a counterexample: For any $n \geq 1$ and $A, B \in \mathbb{R}^{n\times n}$, one has

$$\det(A + B) = \det(A) + \det(B).$$

10. Prove or give a counterexample: For any $r \in \mathbb{R}$, $n \geq 1$ and $A \in \mathbb{R}^{n\times n}$, one has

$$\det(rA) = r\det(A).$$

11. Prove or give a counterexample: For any $n \geq 1$ and $A \in \mathbb{C}^{n\times n}$, one has

$$\operatorname{null}(A) = (\operatorname{range}(A))^\perp.$$

12. Prove or give a counterexample: The Gram-Schmidt process applied to an an orthonormal list of vectors reproduces that list unchanged.

13. Prove or give a counterexample: Every unitary matrix is invertible.