# Homework Set Nine: Orthogonality and Diagonalization

**Directions**: Submit your solutions at the **beginning** of lecture on **Friday, December 4, 2009**.

## Calculational Exercises

1. Consider $\mathbb{R}^3$ with two orthonormal bases: the canonical basis $e = (e_1, e_2, e_3)$ and the basis $f = (f_1, f_2, f_3)$, where

$$f_1 = \frac{1}{\sqrt{3}}(1,1,1),\ f_2 = \frac{1}{\sqrt{6}}(1,-2,1),\ f_3 = \frac{1}{\sqrt{2}}(1,0,-1)\,.$$

   Find the matrix, $S$, of the change of basis transformation such that

$$[v]_f = S[v]_e, \quad \text{for all } v \in \mathbb{R}^3\,,$$

   where $[v]_b$ denotes the column vector of $v$ with respect to the basis $b$.

2. Consider $\mathbb{R}^3$ with two orthonormal bases: the canonical basis $e = (e_1, e_2, e_3)$ and the basis $f = (f_1, f_2, f_3)$, where

$$f_1 = \frac{1}{\sqrt{3}}(1,1,1),\ f_2 = \frac{1}{\sqrt{6}}(1,-2,1),\ f_3 = \frac{1}{\sqrt{2}}(1,0,-1)\,.$$

   Find the canonical matrix, $A$, of the linear map $T \in \mathcal{L}(\mathbb{R}^3)$ with eigenvectors $f_1, f_2, f_3$ and eigenvalues $1, 1/2, -1/2$, respectively.

3. Let $U$ be the subspace of $\mathbb{R}^3$ that coincides with the plane through the origin that is perpendicular to the vector $n = (1,1,1) \in \mathbb{R}^3$.

   (a) Find an orthonormal basis for $U$.

   (b) Find the matrix (with respect to the canonical basis on $\mathbb{R}^3$) of the orthogonal projection $P \in \mathcal{L}(\mathbb{R}^3)$ onto $U$, i.e., such that range$(P) = U$.

# Proof-Writing Exercises

1. Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V)$ satisfies $T^2 = T$. Prove that $T$ is an orthogonal projection if and only if $T$ is self-adjoint.

2. Let $V$ be a finite-dimensional inner product space over $\mathbb{C}$, and suppose that $T \in \mathcal{L}(V)$ has the property that $T^* = -T$. (We call $T$ a **skew Hermitian** operator on $V$.)

   (a) Prove that the operator $iT \in \mathcal{L}(V)$ defined by $(iT)(v) = i(T(v))$, for each $v \in V$, is Hermitian.

   (b) Prove that the canonical matrix for $T$ can be unitarily diagonalized.

   (c) Prove that $T$ has purely imaginary eigenvalues.