LECTURE 3: k-CONJUGATES AND THE PIERI RULE

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1. k-Schur Functions

Recall the map P from $(k+1)$-cores to k-bounded partitions and its inverse map c.

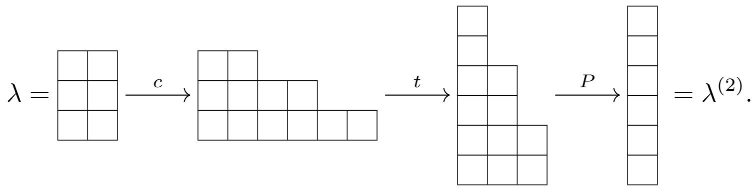
Definition 1.1. Let λ be a k-bounded partition, the *k-conjugate* is $\lambda^{(k)} := P(c(\lambda)^t)$.

Definition 1.2. Let $r \leq k$ and $s_{\emptyset}^{(k)} = 1$. The *k-Pieri rule* is:

$$h_r s_{\lambda}^{(k)} = \sum_{\mu} s_{\mu}^{(k)} \tag{1.1}$$

where the sum is over all k-bounded partitions μ such that μ/λ is a horizontal r-strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r-strip.

Example 1.3. Let $k = 2$, then we have

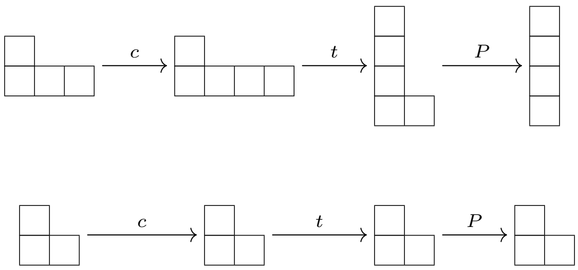


Let $\Lambda_{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k]$ and $\Lambda^{(k)} = \Lambda/\langle m_\lambda \mid \lambda_1 > k \rangle$. We note that they are dual with respect to the Hall inner product $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$.

Definition 1.4. A *k-Schur* function $s_{\lambda}^{(k)} \in \Lambda_{(k)}$, labeled by k-bounded partitions, are defined by the *k-Pieri rule*.

Remark 1.5. Note that this is different than the usual Pieri rule for Schur functions since μ/λ implies that μ^t/λ^t.

Example 1.6. Let $k = 3$, $\mu =$  and $\lambda =$ . We first note that $\mu/\lambda = \square$. Next we have



and so $\mu^{(3)}/\lambda^{(3)}$ is not well-defined since $\lambda^{(3)} \not\subseteq \mu^{(3)}$.

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Example 1.7. Consider $h_1 s^{(3)}_{(3,1)}$. Thus the possible shapes are $(4,1)$, $(3,2)$ and $(3,1,1)$, however the first shape is not 3 bounded, and a simple check will show that the last two satisfy all conditions. Thus we have

$$h_1 s^{(3)}_{(3,1)} = s^{(3)}_{(3,2)} + s^{(3)}_{(3,1,1)}.$$

Remark 1.8. From this point onwards we will have $n = k + 1$.

2. Affine Symmetric Group And Affine Grassmannian Elements

Definition 2.1. The *affine symmetric group* is the group with the following presentation:

$$\left\langle s_0, s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{cc} s_i^2 = 1 & \text{for all } i \\ s_i s_j = s_j - s_i & 1 < |i-j| < n-1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{indices considered mod } n \end{array} \right\rangle$$

Remark 2.2. The symmetric group S_n is a finite group, however $\widetilde{S}_n$ is an infinite group. Consider the infinite reduced word:

$$\cdots s_0 s_{n-1} \cdots s_1 s_0 s_{n-1} \cdots s_1 s_0$$

and it we truncate this at finite length k, it is an element of length k in $\widetilde{S}_n$.

Looking now at $\widetilde{S}_n / S_n$, the left cosets are called *affine Grassmannian elements* and are identified with a minimal length coset representative $w \in \widetilde{S}_n / S_n$

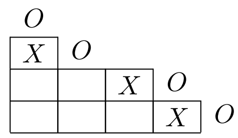
$$\cdots s_0 = w.$$

Such elements are affine Grassmannian if $w = 1$ or s_0 is the only generator such that $\ell(ws_0) < \ell(w)$.

Proposition 2.3. *There exists a bijection between affine Grassmannian elements in $\widetilde{S}_n$ of length m and $(k+1)$-cores of length m.*

To begin, we must define an action of $\widetilde{S}_n / S_n$ on $(k-1)$-cores. Let μ be a partition and the content of a cell $c = (i, j)$ is defined at $j - i$. The *reside* is the content modulo n. A cell c is called an *addable corner* if $\mu \cup \{c\}$ is a partition and c is called a *removable corner* if $\mu - \{c\}$ is a partition.

Example 2.4. Consider the partition $(4, 3, 1)$.



The boxes marked with an X are removable corners, and those with an O are addable.

Define the action of s_i on a $(k+1)$-core κ as the partition where you either:

(1) Add all possible corners of reside i.
(2) Remove all possible corners of reside i.
(3) Do nothing.

We note that these actions are mutually exclusive since if there exists an addable corner of reside i and a removable corner of reside i, then κ would not be a $(k+1)$-core (i.e. there exists a $(k+1)$-ribbon).

Proof sketch. We being by having $s_{\emptyset}^{(k)} \mapsto 1$. Then we note that adding corners of reside i corresponds to multiplying by s_i and increasing the length of the Grassmannian element. In particular, the only thing we can do is multiply by s_0 and we get $\square$. Next we have a choice of either s_{n-1} which yields $\ydiagram{1,1}$ or s_1 which yields $\ydiagram{2}$. Then proceed in this fashion. $\square$

3. Pieri Rule

Recall the usual Pieri rule is $h_r s_\lambda = \sum_\mu s_\mu$ where we sum over all partitions μ such that μ/λ is a horizontal r-strip. For example

$$h_2 s_{\ydiagram{2,1}} = s_{\ydiagram{3,1}} + s_{\ydiagram{3,2}} + s_{\ydiagram{2,1,1}} + s_{\ydiagram{2,2,1}}.$$

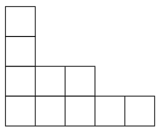
Definition 3.1. The *hook length* of a cell $(i, j) \in \lambda$, where i is the row index and j is the column index, is defined as $\lambda_i + \lambda_j^t - i - j + 1$ where λ^t is the transpose or conjugate partition.

Heuristically this is the number of cells above and to the right of cell $c = (i, j)$ plus 1 (or one can think of also counting c). For example, consider the partition $(4, 2, 1)$ or $\ydiagram{4,2,1}$, the hook length of $(1, 2)$ is $2 + 1 + 1 = 4$.

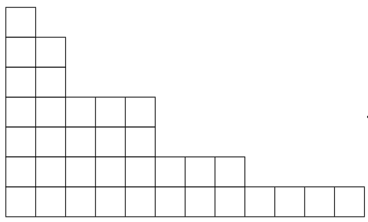
Definition 3.2. An r-core is a partition λ such that no cell has a hook length of r.

Example 3.3. The partition $\ydiagram{2}$ is note a 2-core. In fact, it is easy to see the only 2-cores are staircase partitions $(n, n-1, n-2, \ldots, 2, 1)$.

Example 3.4. The partition $(5, 3, 1, 1)$ is a 3-core.



Example 3.5. The partition $(12, 8, 5, 5, 2, 2, 1)$ is a 5-core.



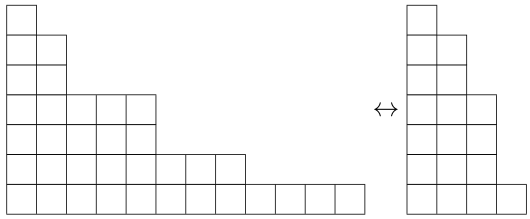
Remark 3.6. We can push the hook to the boundary, and so we get a ribbon whose size is the hook length. An ribbon is skew-shape which does not contain any 2×2 shape. Thus a partition is an r-core if there does not exist an r-ribbon which can be removed.

The *size* of an r-core λ is the number of cells of λ and denoted by $|\lambda|$. The *r-length* of an r-core λ is the number of cells with hook length less than r and denoted by $|\lambda|_r$. We say a partition λ is k-bounded if $\lambda_1 < k$.

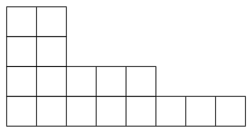
Proposition 3.7 (Lapointe & Morse)**.** *There exists a bijection between $(k+1)$-cores of $(k+1)$-length m and k-bounded partitions of size m.*

Proof. The bijection is described by removing all cells with hook length greater than k and sliding the rows to the right to obtain a partition. The inverse map is starting from the top, slide rows to the right until all cells in the top row have hook length less than k, then add cells to obtain a partition. $\square$

Example 3.8. Let $k = 4$, then the partition $(12, 8, 5, 5, 2, 2, 1)$ under the bijection becomes $(4, 3, 3, 3, 2, 2, 1)$ by removing the shape $(8, 5, 2, 2)$. In terms of Young diagrams, we have:



by removing



To go back, we have: