MAT 115A University of California Fall 2015

Homework 3

due October 16, 2015

1. Rosen 3.3 #22, pg. 100

Let $a_1, a_2, \ldots, a_n$ be integers not all equal to zero. Is it true that the greatest common divisor of these integers $(a_1, \ldots, a_n)$ is the least positive integer of the form $m_1a_1+m_2a_2+\cdots+m_na_n$ where $m_1, \ldots, m_n \in \mathbb{Z}$? If so, prove it. If not, give a counterexample.

2. Rosen 3.5 #8, pg. 120

Show that every positive integer can be written as the product of a square (possibly 1) and a square-free integer. A *square-free* integer is an integer that is not divisible by any perfect squares other than 1.

3. Rosen 3.5 #17, 20, 21, pg. 121

Let $\alpha = a + b\sqrt{-5}$ where $a, b \in \mathbb{Z}$. Define the *norm* of α, denoted $N(\alpha)$, as $N(\alpha) = a^2 + 5b^2$.

(a) Show that if $\alpha = a+b\sqrt{-5}$ and $\beta = c+d\sqrt{-5}$, where $a, b, c, d \in \mathbb{Z}$, then $N(\alpha\beta) = N(\alpha)N(\beta)$.

(b) Show that the numbers $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are prime numbers, that is, there are no numbers $\alpha = a+b\sqrt{-5}$ and $\beta = c+d\sqrt{-5}$ different from ±1 such that $1 \pm \sqrt{-5} = \alpha\beta$.

(Hint: Use part (a)).

(c) Find two different factorizations of the number 21 into primes of the form $a + b\sqrt{-5}$, where a and b are integers.

4. Rosen 3.5 #43, pg. 123

Show that $\sqrt{2} + \sqrt{3}$ is irrational.