Homework 1
due October 2, 2015 in class

We will use Artin's numbering system so that "Artin 1.4.2" means Chapter 1, Section 4, Problem 2.

You are expected to hand in all problems. The reader will pick several problems at random for grading. To ensure credit for your homework you have to solve all of the problems!

Make sure your homework paper is legible, stapled and has your name clearly showing on each page. The homework is due on October 2 in class. **No late homeworks will be accepted!**

Read: Artin Chapters 1.4, 1.5, 2.1, 2.2, 3.2

1. Heisenberg group: Let

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in F \right\}$$

where F is a field. We will prove in this problem that $H(F)$ actually forms a group under matrix mulitplication (called the **Heisenberg group** over the field F). Let

$$X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}$$

be two elements in $H(F)$.

(1) Compute the matrix product XY and deduce that $H(F)$ is closed under matrix multiplication. Exhibit explicit matrices such that $XY \neq YX$ (which shows that $H(F)$ is always non-abelian).
(2) Find an explicit formula for the matrix inverse X^{-1} and deduce that $H(F)$ is closed under inverse.
(3) Prove the associative law for $H(F)$ and deduce that $H(F)$ is a group of order $|F|^3$ (do not assume that matrix multiplication is associative).
(4) Find the order of each element of the finite group $H(\mathbb{F}_2)$.

(5) Prove that every nonidentity element of the group $H(\mathbb{R})$ has infinite order.

2. Artin 1.4.3 (page 34)
Compute the determinant of the following $n \times n$ matrices by induction on n:

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

3. Artin 1.5.2 (page 34)
Let p be the permutation $(1, 3, 4, 2)$ in cycle notation.

(a) Find the associated permutation matrix P.
(b) Write p as a product of transpositions and evaluate the corresponding matrix product.
(c) Determine the sign of p.

4. Artin 1.5.3 (page 34)
Prove that the transpose of a permutation matrix P is its inverse.