MAT 146 University of California Spring 2014

Homework 4

due Friday May 2, 2014 in class

1. Stanley, Chapter 4.4
Let q be a prime power, and let $\mathbb{F}_q$ denote the finite field with q elements. Let $V = V_n(q) = \mathbb{F}_q^n$, the n-dimensional vector space over $\mathbb{F}_q$ of n-tuples of elements in $\mathbb{F}_q$. Let $B_n(q)$ denote the poset of all subspaces of V, ordered by inclusion. It is easy to see that $B_n(q)$ is graded of rank n, the rank of the subspace of V being its dimension.

(a) Draw the Hasse diagram of $B_3(2)$. (It has 16 elements).

(b) Show that the number of elements in $B_n(q)$ of rank k is given by the q-binomial coefficient

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}.$$

(c) Show that $B_n(q)$ is rank-symmetric.

(d) Show that every element $x \in B_n(q)_k$ covers $(k)_q = 1 + q + \cdots + q^{k-1}$ elements and is covered by $(n-k)_q = 1 + q + \cdots + q^{n-k-1}$ elements.

(e) Define operators $U_i \colon \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i+1}$ and $D_i \colon \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i-1}$ by

$$U_i(x) = \sum_{\substack{y \in B_n(q)_{i+1} \\ y > x}} y,$$
$$D_i(x) = \sum_{\substack{z \in B_n(q)_{i-1} \\ z < x}} z.$$

Show that

$$D_{i+1}U_i - U_{i-1}D_i = ((n-i)_q - (i)_q)I_i.$$

(f) Deduce that $B_n(q)$ is rank-unimodal and Sperner.