Homework 1

due April 11, 2014 in class

We will use Artin's numbering system so that "Artin 12.2.4" means Chapter 12, Section 2, Problem 4.

1. (Artin 12.2.3) How many roots does the polynomial $x^2 - 2$ have, modulo 8?

2. (Artin 12.2.4) Euclid proved that there are infinitely many prime integers in the following way: If $p_1, \ldots, p_k$ are primes, then any prime factor p of $p_1 \cdots p_k + 1$ must be different from all of the p_i. Adapt this argument to prove that for any field F there are infinitely many monic irreducible polynomials in $F[x]$.

3. Prove that the greatest common divisor of two polynomials f and g in $\mathbb{Q}[x]$ is also their greatest common divisor in $\mathbb{C}[x]$.

4. Prove or disprove the following.

(a) The polynomial ring $\mathbb{R}[x, y]$ in two variables is a Euclidean domain.

(b) The ring $\mathbb{Z}[x]$ is a principal ideal domain.

5. Let R be a principal ideal domain.

(a) Prove that there is a *least common multiple* $[a, b] = m$ of two elements which are not both zero such that a and b divide m, and that if a, b divide an element $r \in R$, then m divides r. Prove that m is unique up to unit factor.

(b) Denote the greatest common divisor of a and b by (a, b). Prove that $(a, b)[a, b]$ is an associate of ab.

6. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$. Show that R is not a principal ideal domain by explicitly checking that the ideal $I = (3, 2 + \sqrt{-5})$ is not principal.