MAT 150C University of California Spring 2014

Homework 2

due April 18, 2014 in class

1. Give an example showing that division with remainder need not be unique in a Euclidean domain.

2. Prove that in a principle ideal domain R, every pair a, b of elements, not both zero, has a greatest common divisor d, with these properties:

(i) $d = ar + bs$ for some $r, s \in R$;
(ii) d divides a and b;
(iii) if $e \in R$ divides a and b, it also divides d.

Moreover, d is determined up to unit factor.
(This problem basically asks for a detailed proof of Proposition 12.2.8 in Artin).

3. If a, b are integers and a divides b in the ring of Gauss integers, then a divides b in $\mathbb{Z}$.

4. Prove that the factorizations $2 = (1+i)(1-i)$ and $5 = (2+i)(2-i)$ are prime factorizations of 2 and 5 in $\mathbb{Z}[i]$, respectively.

5. (Artin 12.3.2) Prove that two integer polynomials are relatively prime elements of $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains a nonzero integer.

6. We showed already that $(2, x)$ is not a principal ideal in $\mathbb{Z}[x]$. This shows that $\mathbb{Z}[x]$ is not a PID.

(i) Show that $(2, x)$ is principal in $\mathbb{Q}[x]$. Which element generates $(2, x)$ in $\mathbb{Q}[x]$?
(ii) What is $(2, x)$ in $(\mathbb{Z}/p\mathbb{Z})[x]$ where p is prime? For which p is $(2, x)$ maximal?

7.

(i) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. Suppose $r/s \in \mathbb{Q}$ is a root of $p(x)$ where r and s are coprime. Then $r \mid a_0$ and $s \mid a_n$.
(ii) Use part (i) to show that $x^3 - 3x - 1$ is irreducible in $\mathbb{Z}[x]$.