Homework 3

due April 25, 2014 in class

(1) Let a, b be elements of a field F, with $a \neq 0$. Prove that a polynomial $f(x) \in F[x]$ is irreducible if and only if $f(ax+b)$ is irreducible.
(2) Factor 30 into primes in $\mathbb{Z}[i]$.
(3) (Artin 12.5.5) Let π be a Gauß prime. Prove that π and $\overline{\pi}$ are associate if and only if either π is associate to an integer prime or $\pi\overline{\pi} = 2$.
(4) (Artin 12.5.6) Let R be the ring $\mathbb{Z}[\sqrt{3}]$. Prove that a prime integer p is a prime element of R if and only if the polynomial $x^2 - 3$ is irreducible in $\mathbb{F}_p[x]$.
(5) For the proof of Theorem 12.3.8 of Artin it is assumed that factorization exists in the polynomial ring $\mathbb{Z}[x]$. Explain why this is true.
(6) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ and let $p \in \mathbb{Z}$ be prime. Suppose that the coefficients of f satisfy the following conditions:
  (a) p does not divide a_n;
  (b) p divides $a_{n-1}, \cdots, a_0$;
  (c) p^2 does not divide a_0.

  Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$. If f is primitive, it is irreducible in $\mathbb{Z}[x]$.
(7) Use Problem 6 to show that $x^4 + 10x + 5$ is irreducible in $\mathbb{Z}[x]$. Show that $x^n - p$ is irreducible in $\mathbb{Z}[x]$ for $n \geq 2$ and p a prime integer. Is it possible to use Problem 6 to show that $x^4 + 1$ is irreducible? (Hint: Combine Problem 6 with Problem 1 with $a = b = 1$).