Homework 1

due January 21, 2004

Question 1. In class, we defined the complex numbers $\mathbb{C}$ as the set $\mathbb{R}^2$ with addition $z + z' = (x, y) + (x', y') = (x + x', y + y')$ and multiplication $z \cdot z' = (x, y)(x', y') = (xx' - yy', xy' + x'y)$. Show that $\mathbb{C}$ satisfies the field axioms.

Question 2. Verify the following properties for $z, w \in \mathbb{C}$:

(1) $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$
(2) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$
(3) $z = \overline{z}$ if and only if $z \in \mathbb{R}$ is purely real.
$z = -\overline{z}$ if and only if $z \in i\mathbb{R}$ is purely imaginary.
Find the maximal subset of $\mathbb{C}$ which is left fixed by complex conjugation.

Question 3. Prove the following identities:

(1) $\langle z, w\rangle^2 + \langle iz, w\rangle^2 = |z|^2|w|^2$
(2) $|\langle z, w\rangle| \leq |z| \cdot |w|$ Cauchy-Schwartz inequality
(3) $|z + w|^2 = |z|^2 + |w|^2 + 2\langle z, w\rangle$ Cosine law
(4) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$
(5) $|z|^2 = \langle z, z\rangle$
(6) $|zw| = |z| \cdot |w|$
(7) $|z + w| \leq |z| + |w|$ triangle inequality
(8) $|\frac{z}{w}| = \frac{|z|}{|w|}$.

Question 4. In class we defined the function $\tilde{e}(\theta) = \cos(\theta) + i\sin(\theta)$. Show that

$$\frac{1}{\tilde{e}(\theta)} = \tilde{e}(-\theta)$$

and use this fact to extend de Moivre's theorem to all $n \in \mathbb{Z}$.

Question 5. Prove Lagrange's trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$

where we assume that $\sin\theta/2 \neq 0$. (*Hint:* Transform the left-hand side into the real part of sums of exponentials and use $1+x+x^2+\cdots+x^n = \frac{x^{n+1}-1}{x-1}$.)

Question 6. Another representation of complex numbers! Let us call

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then we have

$$z = \mathbf{1}\,x + \mathbf{i}\,y = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \tag{1}$$

(i) Show that the complex number algebra $\mathbf{1}\cdot\mathbf{1} = -\mathbf{i}\cdot\mathbf{i} = \mathbf{1}\cdot\mathbf{i} - \mathbf{i}\cdot\mathbf{1} + \mathbf{1} = \mathbf{1}$ holds, where "$\cdot$" denotes matrix multiplication.

(ii) Deduce that matrices of the form (1) form a field isomorphic to complex numbers under matrix multiplication and addition.

(iii) Verify that the matrix exponential satisfies

$$\exp(\mathbf{i}\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \mathbf{1}\cos\theta + \mathbf{i}\sin\theta\,.$$

(iv) How does the matrix in (iii) act on vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane $\mathbb{R}^2$?