Homework 6

due February 25, 2004

Question 1. Use Liouville's theorem to prove that there is no entire function mapping the entire complex plane to the inside of the unit disc.

Question 2. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be a circle of radius r around 0, that is, $\gamma(t) = re^{it}$. Calculate the winding number of the curve $f \circ \gamma$ with respect to 0, where $f(z)$ is a polynomial with complex coefficients which does not have any zeroes on the circle $|z| = r$.
[Hint: You may use that $f(z)$ is a product of linear polynomials.]

Question 3. Using the Hessian, show that a non-constant harmonic function can have neither maxima nor minima, but rather, only saddle points.

Question 4. The Maximum Modulus Principle: We saw that harmonic functions could not take local maxima or minima. A similar statement holds for the modulus of an analytic function $f(z) \neq$ constant. In this exercise we show that the modulus of an analytic function cannot take a maximum within its domain of analyticity D.

(i) Proceed by contradiction and assume $|f(z_0)|$ is a local maximum for $z_0 \in D$. Write a formula for $f(z_0)$ in terms of an integral around a small circle, center z_0. Rewrite this integral as a line integral with parameter t by calling $z = z_0 + \varepsilon e^{it}$.

(ii) Obtain an inequality by studying the modulus of both sides of your result in (i).

(iii) Employ continuity of $f(z)$ and the assumption that $|f(z_0)|$ was a maximum to show that $|f(z_0)| = |f(z_0 + \varepsilon e^{it})|$.

(iv) Show that $|f(z_0)| = |f(z_0 + \varepsilon e^{it})|$ in fact implies that $f(z_0) = f(z_0 + \varepsilon e^{it})$ contradicting the assumption that f is not constant.
[Hint: Write $f(z_0) = e^{i\theta}|f(z_0)|$ and calculate $|f(z_0)| = \mathrm{Re}|f(z_0)|$ using the integral in (i).]